Staggered fermions with tasty mass terms

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Workshop on novel lattice fermions …
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Hans Böckler Stiftung

BERGISCHE UNIVERSITÄT WUPPERTAL

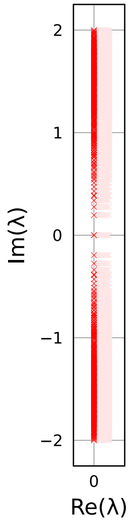
Lattice fermions

How to cope with doublers?

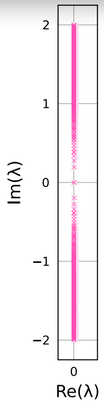
naive fermions

$D_{\mathrm{N}} = \gamma_\mu D_\mu$

16 species



remove 4-fold degeneracy



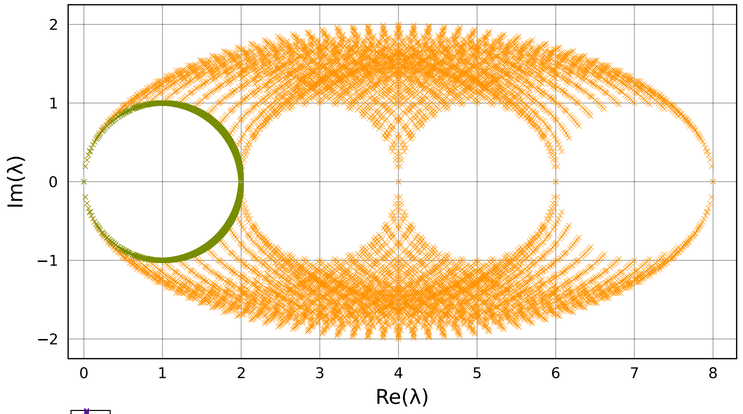
staggered fermions

$D_{\mathrm{st}} = \eta_\mu D_\mu$

4 species

real

add flavor dependent mass term



e.g. Wilson fermions

$D_{\mathrm{W}} = \gamma_\mu D_\mu + rW$

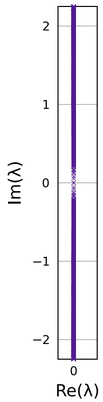
1+4+6+4+1 species

restore χ symmetry

e.g. overlap fermions

1+15 species

imaginary



e.g. minimally doubled

$D_{\mathrm{MD}} = \gamma_\mu D_\mu + i\zeta\gamma_4 W_{123}$

2 species

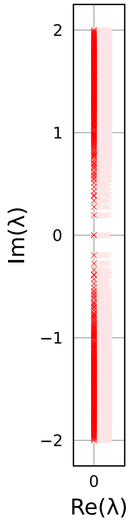
Lattice fermions

How to cope with doublers?

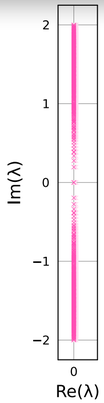
naive fermions

$D_{\mathrm{N}} = \gamma_\mu D_\mu$

16 species

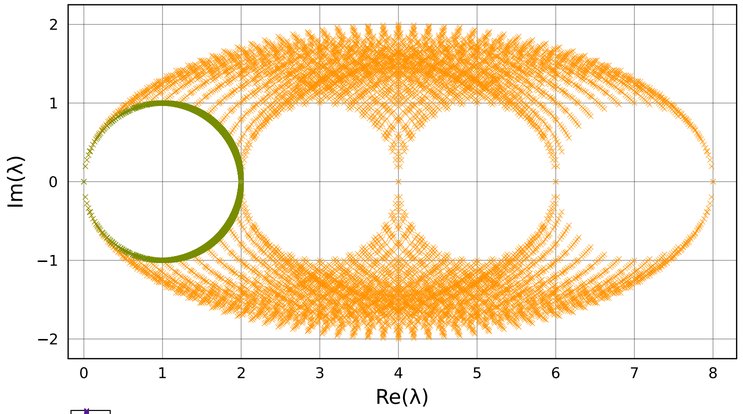


remove 4-fold degeneracy

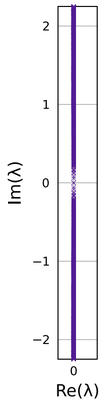


real

add flavor dependent mass term



imaginary



Are these options mutually exclusive?

Basics: staggered construction

Covariant hop: $(V_\mu)_{xy} = U_\mu(x)\delta_{x+\hat{\mu},y}$

Covariant Derivative: $D_\mu = (V_\mu - V_\mu^\dagger)/2$

Covariant second derivative (minus constant): $C_\mu = (V_\mu + V_\mu^\dagger)/2$

$V\mu$

naive action: $\bar{\psi} D_{\mathrm{N}} = \gamma_\mu D_\mu \psi$ $\xrightarrow{\text{staggered diagonalization:}}$ staggered action: $D_{\mathrm{st}} = \bar{\chi}\eta_\mu D_\mu \chi$

$$\psi(x) = \Gamma(x)\chi(x)$$

where $\Gamma(x) = \prod_\mu \gamma_\mu^{x_\mu}$ with $V_\mu \Gamma = \gamma_\mu \Gamma V_\mu$

staggered phase factors:

$$\eta_\mu = (-1)^{\sum_{\nu<\mu} x_\nu}$$

$$\zeta_\mu = (-1)^{\sum_{\nu>\mu} x_\nu}$$

- ϵ-hermiticity: $D_{st}\epsilon = \epsilon D_{st}^\dagger$ with $\epsilon = (-1)^{\sum_\mu x_\mu}$
- antihermiticity: $D_{st} = -D_{st}^\dagger$

$$\prod_\mu \eta_\mu \neq \epsilon!$$

Basics: staggered taste basis

spin-taste structure encoded in geometry: $(\gamma_S \otimes \xi_F)_{xy} = \frac{1}{2^{D/2}} \mathrm{tr}(\Gamma^\dagger(x)\gamma_S\Gamma(y)\gamma_F^\dagger) \qquad + O(a)$

- Γ periodic in elementary hypercubes → restrict x - y distance to hypercube
- local only for $\gamma_S = \xi_F$, spin-taste mismatch in μ direction implies hop in that direction
- use covariant, symmetrized averages to construct bilinears:

$(-1)^{\sum_\mu x_\mu}$

Levi-Civita

$$(\gamma_{1\ldots D} \otimes \xi_{1\ldots D}) \sim \epsilon$$

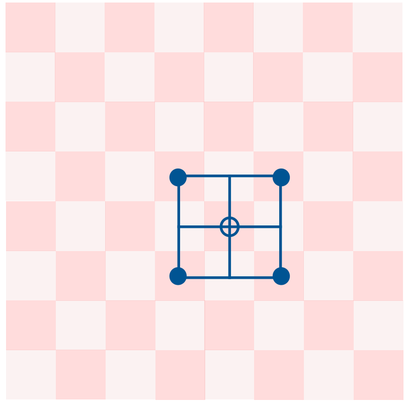
$$(1 \otimes \xi_{\mu_1\ldots\mu_{2n}}) \sim i^n \varepsilon_{\mu_1\ldots\mu_{2n}} \zeta_{\mu_1}\ldots\zeta_{\mu_{2n}} (C_{\mu_1}\ldots C_{\mu_{2n}})_{\mathrm{sym}} =: A_{\mu_1\ldots\mu_{2n}}$$

where we have defined $\gamma_{\mu_1\ldots\mu_n} = \gamma_{\mu_1}\ldots\gamma_{\mu_n}$

- $A_{\mu_1\ldots\mu_{2n}}$ is diagonal in spinor space, but distinguishes tastes

Taste dependent mass terms

(Golterman, Smit '84)



Adding taste dependent mass term $M_{\mu_1 \ldots \mu_{2n}}$ to the staggered operator:

$$M_{\mu_1 \ldots \mu_{2n}} = A_{\mu_1 \ldots \mu_{2n}} + n$$

- even number of hops → commutes with ϵ
- hermiticity: $A_{\mu_1 \ldots \mu_{2n}} = A^\dagger_{\mu_1 \ldots \mu_{2n}}$

→ ϵ-hermiticity: $A_{\mu_1 \ldots \mu_{2n}} \epsilon = \epsilon A^\dagger_{\mu_1 \ldots \mu_{2n}}$

hermiticity:

$$A_{\mu_1 \ldots \mu_{2n}} = i^n \varepsilon_{\mu_1 \ldots \mu_{2n}} \zeta_{\mu_1} \ldots \zeta_{\mu_{2n}} (C_{\mu_1} \ldots C_{\mu_{2n}})_{\text{sym}}$$

$$A^\dagger_{\mu_1 \ldots \mu_{2n}} = (-i)^n \varepsilon_{\mu_1 \ldots \mu_{2n}} (C_{\mu_1} \ldots C_{\mu_{2n}})_{\text{sym}} \zeta_{\mu_1} \ldots \zeta_{\mu_{2n}}$$

$(-1)^{k-1}$ for k^{th} highest μ_k

$$[C_\mu, C_\nu] = 0$$

$$C_\mu \zeta_\nu = \begin{cases} \zeta_\nu C_\mu & \mu \leq \nu \\ -\zeta_\nu C_\mu & \mu > \nu \end{cases}$$

→ total sign flips $(-1)^{\sum_{k=1}^{2n} (k-1)} = (-1)^{n(2n-1)} = (-1)^n$

Staggered Wilson fermions

(Adams 2010; C.H. 2010; deForcrand et. al. 2010, 2012; Durr 2012)

- adding $A_{\mu_1 \ldots \mu_{2n}}$ terms to the staggered operator (partially) lifts taste degeneracy
- ϵ-hermiticity of resulting operator → EVs real or in complex conjugate pairs, real determinant
- similar to Wilson term for low (staggered) momentum modes (i.e. there are $O(a)$ corrections)

Issues:

- Breaks $U_\epsilon(1)$ remnant chiral symmetry of staggered fermions
- Non-nearest neighbor interaction: $2n$ hops inside elementary hypercube for $A_{\mu_1 \ldots \mu_{2n}}$
- Breaks discrete symmetries

Discrete symmetries

(Adams 2010, Sharpe 2012, Misumi et. al. 2012)

shift (translational) symmetry:
- single hop symmetry S_μ broken
- subgroup remains, including hypercubic diagonal and 2-hop shifts

axis flip (parity):
- simple flip symmetry I_μ broken if mass term includes μ-direction
- flip+shift $I_\mu S_\mu$ unbroken

charge conjugation:
- unbroken for maximal mass term $A_{1\ldots D}$
- replaced by $C_T = R_{21}R_{13}C$ for $A_{12} + A_{34}$ in $4D$ (Misumi et. al. 2012)

Hypercubic rotational symmetry

(Sharpe 2012, Misumi et. al. 2012)

- $R_{\mu\nu}$ unbroken for maximal mass term $A_{1\ldots D}$

- broken to subgroups by other mass terms

  e.g.: mass term $A_{12} + A_{34}$ in 4D: R_{12}, R_{34} and $R_{24}R_{31}$ remaining

  $\rightarrow F_{12}^2 + F_{34}^2$ renormalize differently from other $F_{\mu\nu}^2$ components

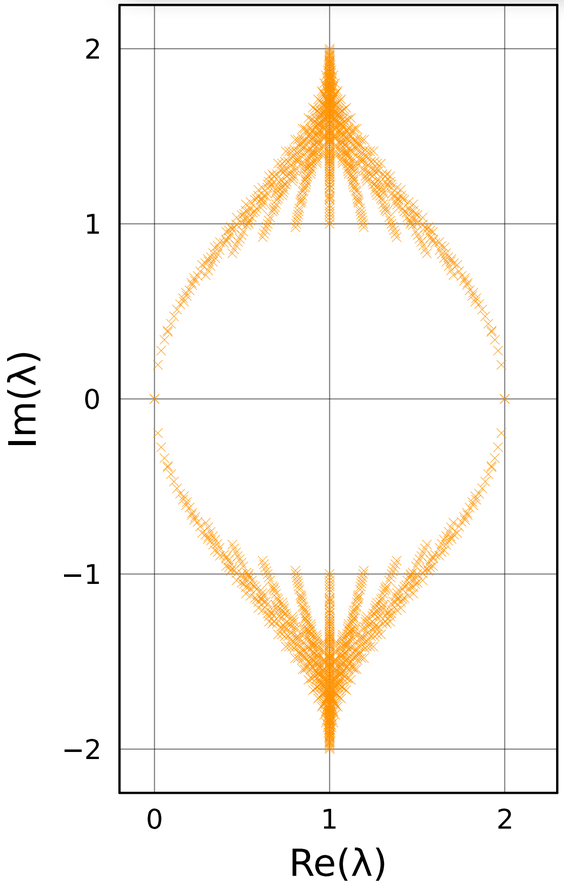
  gluonic counter terms

How big a problem is this in practice? We largely don't know!

- problem will appear when unquenching, with all flavors implemented identically

- would need further investigation (e.g. degenerate flavors related by $R_{\mu\nu}$ and back to rooting 🙂)

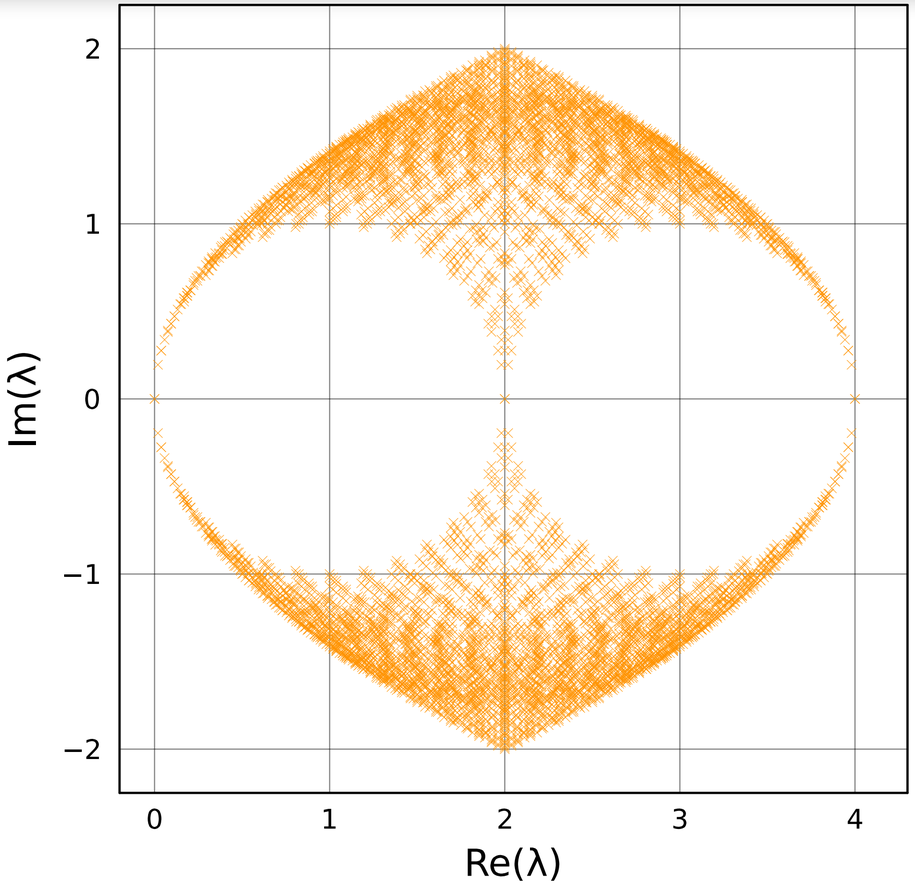
Free eigenvalue spectrum



$D_{\rm st} + M_{1234}$

split $\sim (1 \otimes \xi_5)$

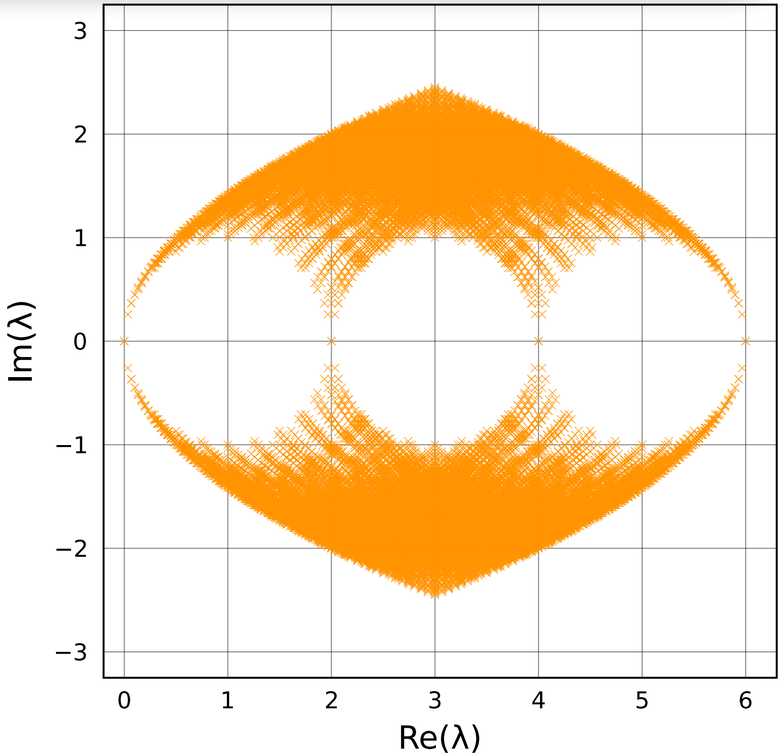
2+2 flavors (Adams 2010)



$D_{\rm st} + M_{12} + M_{34}$

split $\sim (1 \otimes (\xi_{12} + \xi_{34}))$

1+2+1 flavors



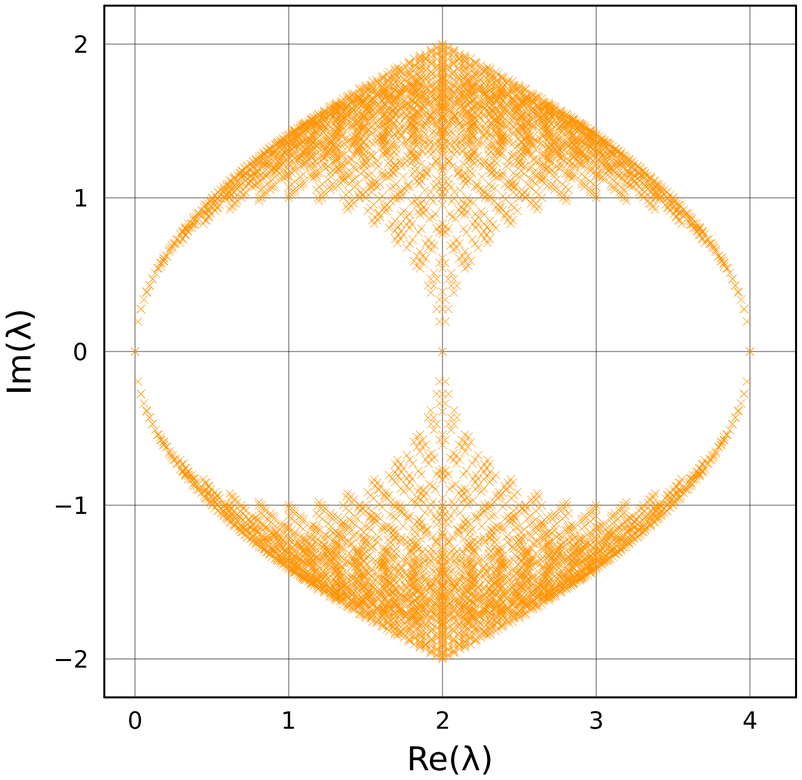
just for fun: 6D

$D_{\rm st} + M_{12} + M_{34} + M_{56}$

split $\sim (1 \otimes (\xi_{12} + \xi_{34} + \xi_{56}))$

1+3+3+1 flavors

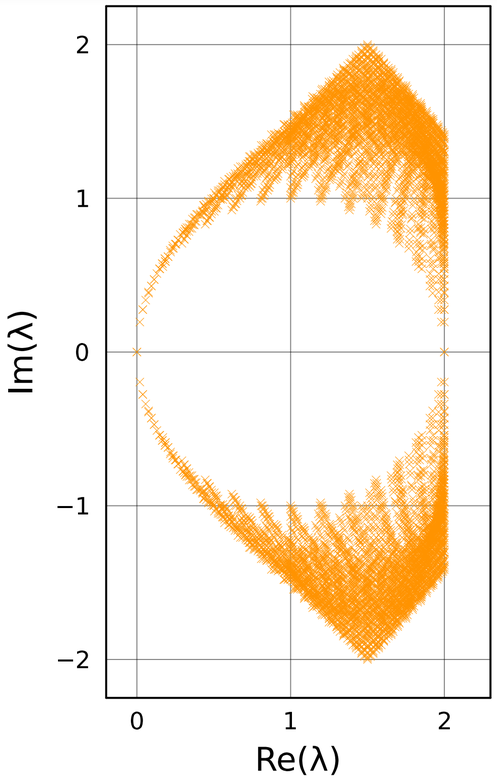
Some 4D single flavor options



$D_{\rm st} + M_{12} + M_{34}$

split $\sim (1 \otimes (\xi_{12} + \xi_{34}))$

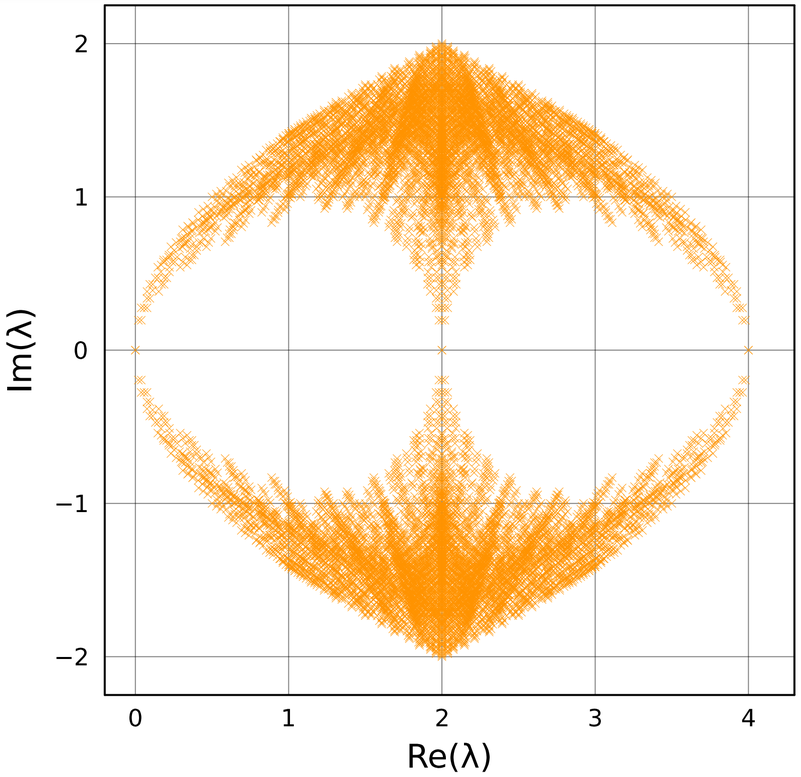
1+2+1 flavors



$D_{\rm st} + (M_{1234} + M_{12} + M_{34})/2$

split $\sim (1 \otimes (\xi_5 + \xi_{12} + \xi_{34})/2)$

1+3 flavors (Durr 2012)



$D_{\rm st} + M_{1234} + M_{12}$

split $\sim (1 \otimes (\xi_5 + \xi_{34}))$

1+2+1 flavors

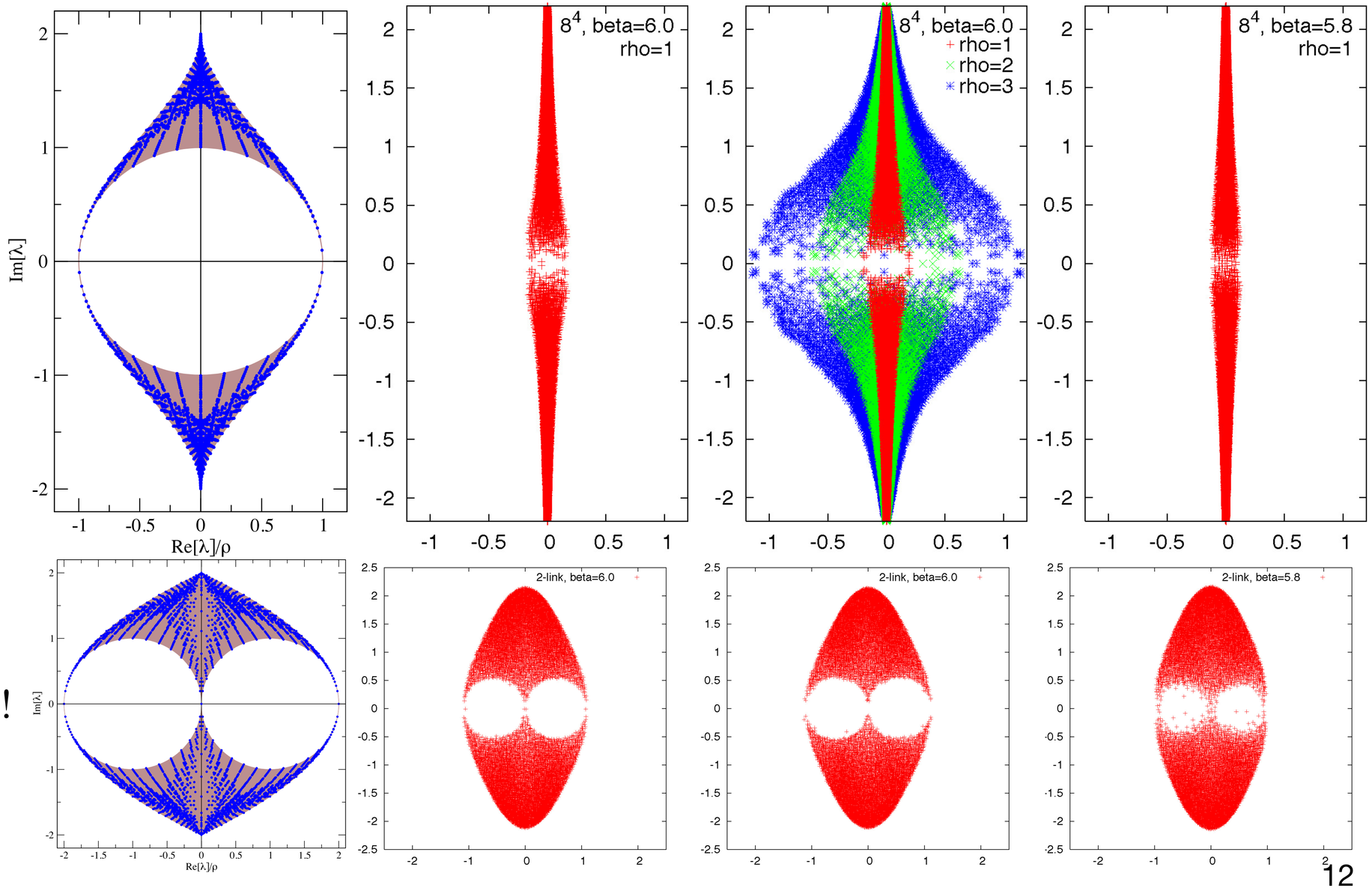
$\xi_5 = \mathrm{diag}(1,1,-1,-1)$ $\xi_{12} = \mathrm{diag}(-1,1,-1,1)$ $\xi_{34} = \mathrm{diag}(1,-1,-1,1)$

Eigenvalue spectra on dynamical configs

(deForcrand et. al. 2012)

$$M_A = M_{1234}$$

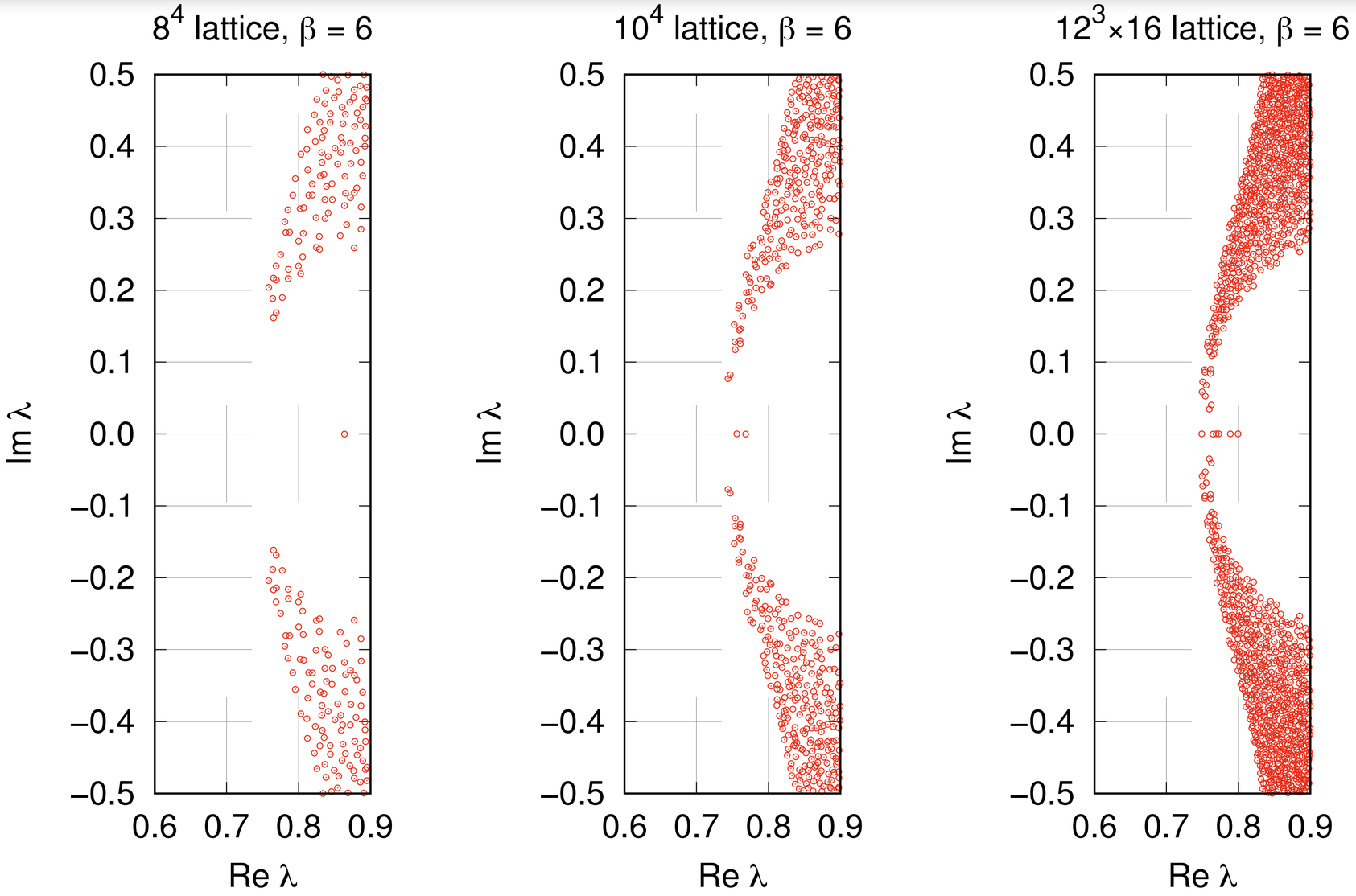
$$M_s = \sqrt{3}\varepsilon_{\mu\nu\alpha\beta}(M_{\mu\nu} + M_{\alpha\beta})/4!$$



Indications for continuum behavior

(Adams et.al. 2010-14, Zielinski 2016)

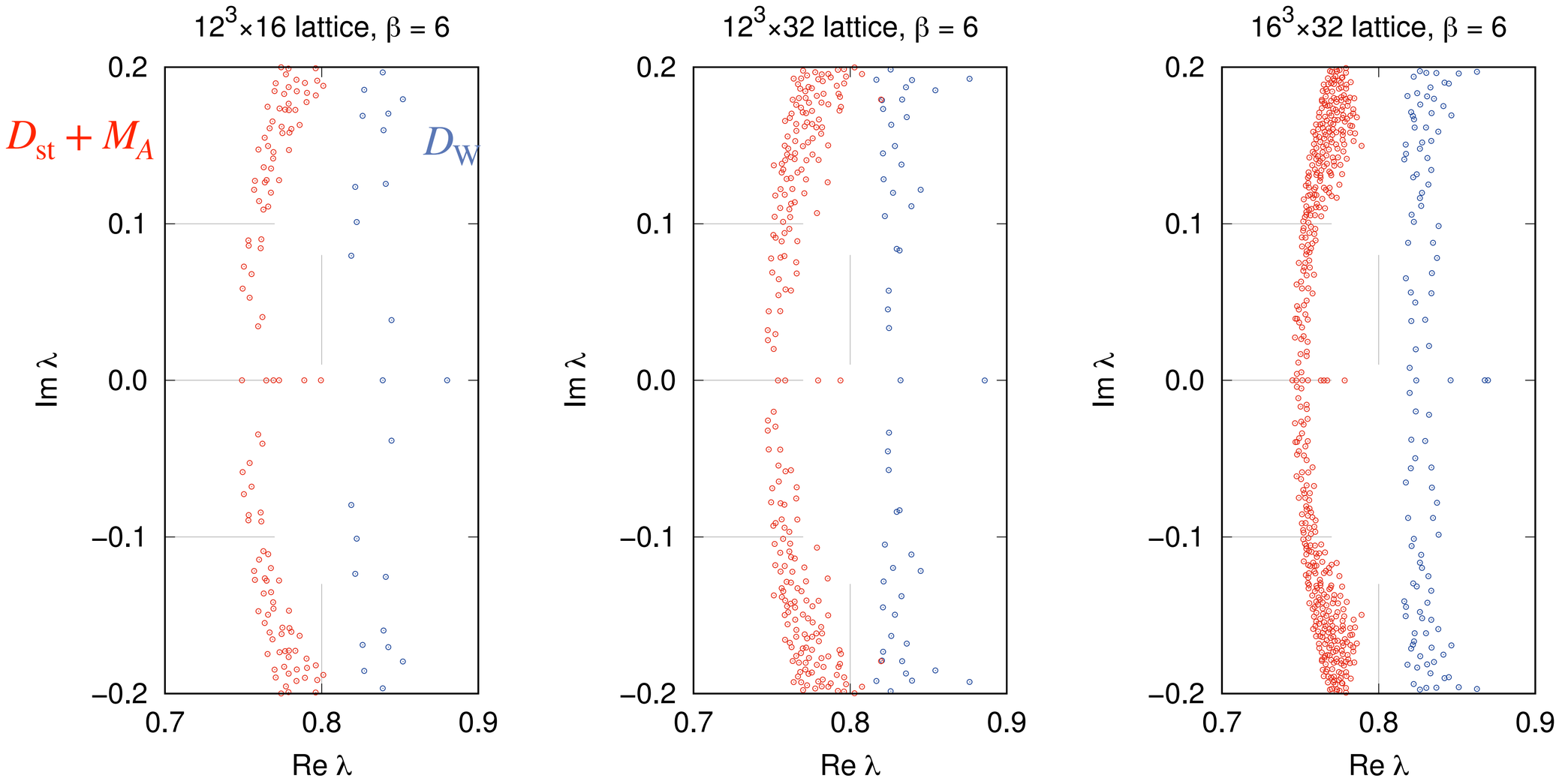
$D_{\mathrm{st}} + M_A$



(a) 8^4 lattice at $\beta = 6$ (b) 10^4 lattice at $\beta = 6$ (c) $12^3 \times 16$ lattice at $\beta = 6$

Comparison to Wilson fermions

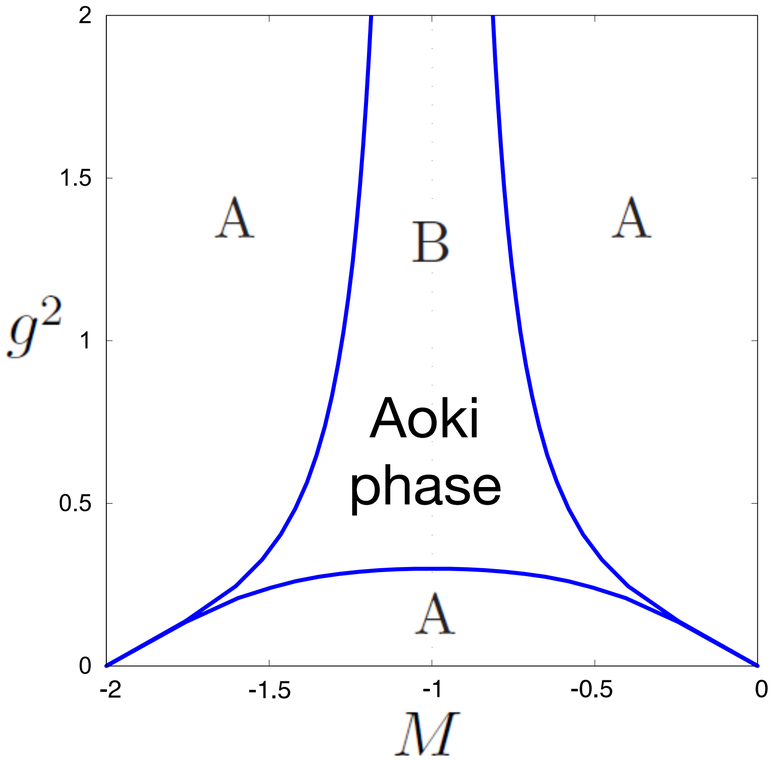
(Adams et.al. 2010-14, Zielinski 2016)



Strong coupling

(Creutz et. al. 2011, Misumi et. al. 2012)

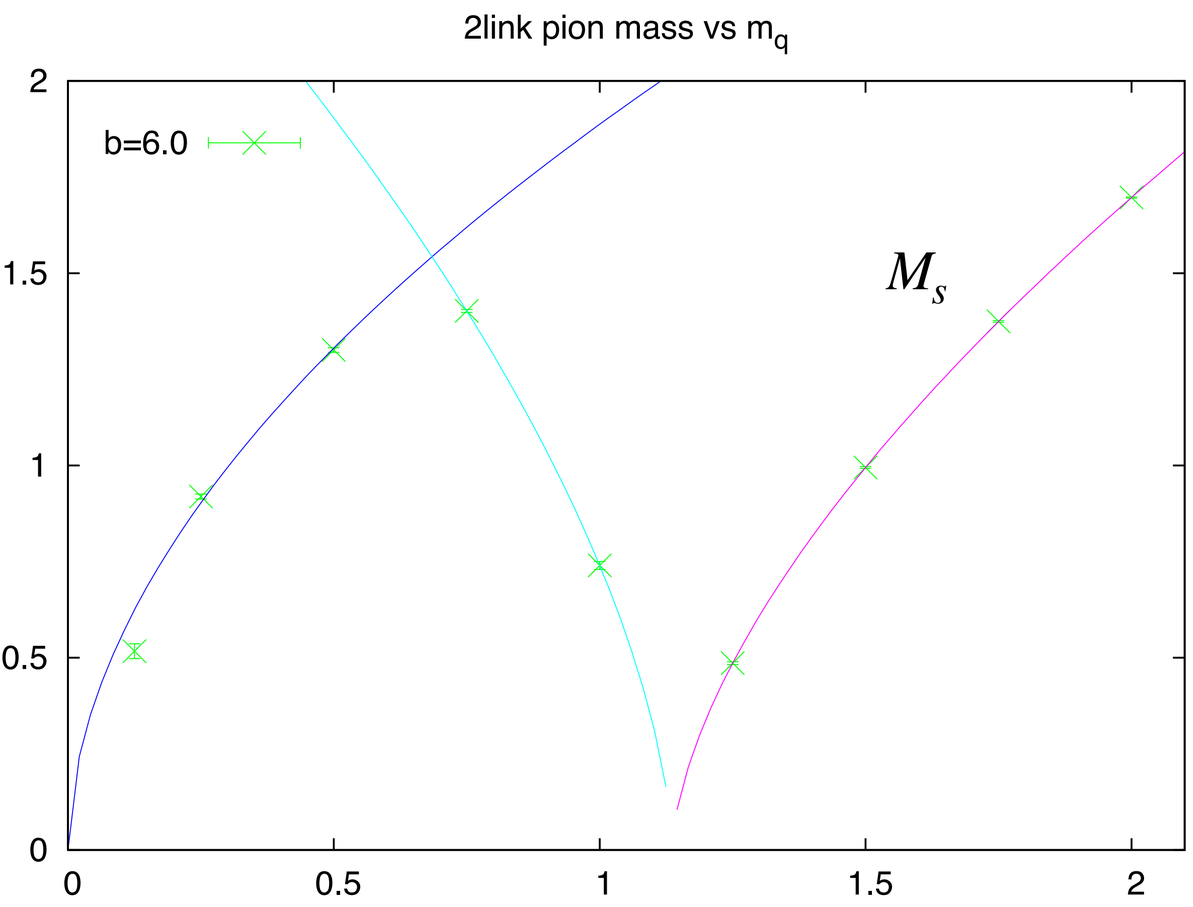
- Aoki phase could be established
- second order PT at boundary in strong coupling
- massless pions and PCAC relation
- continuum limit as for Wilson fermions

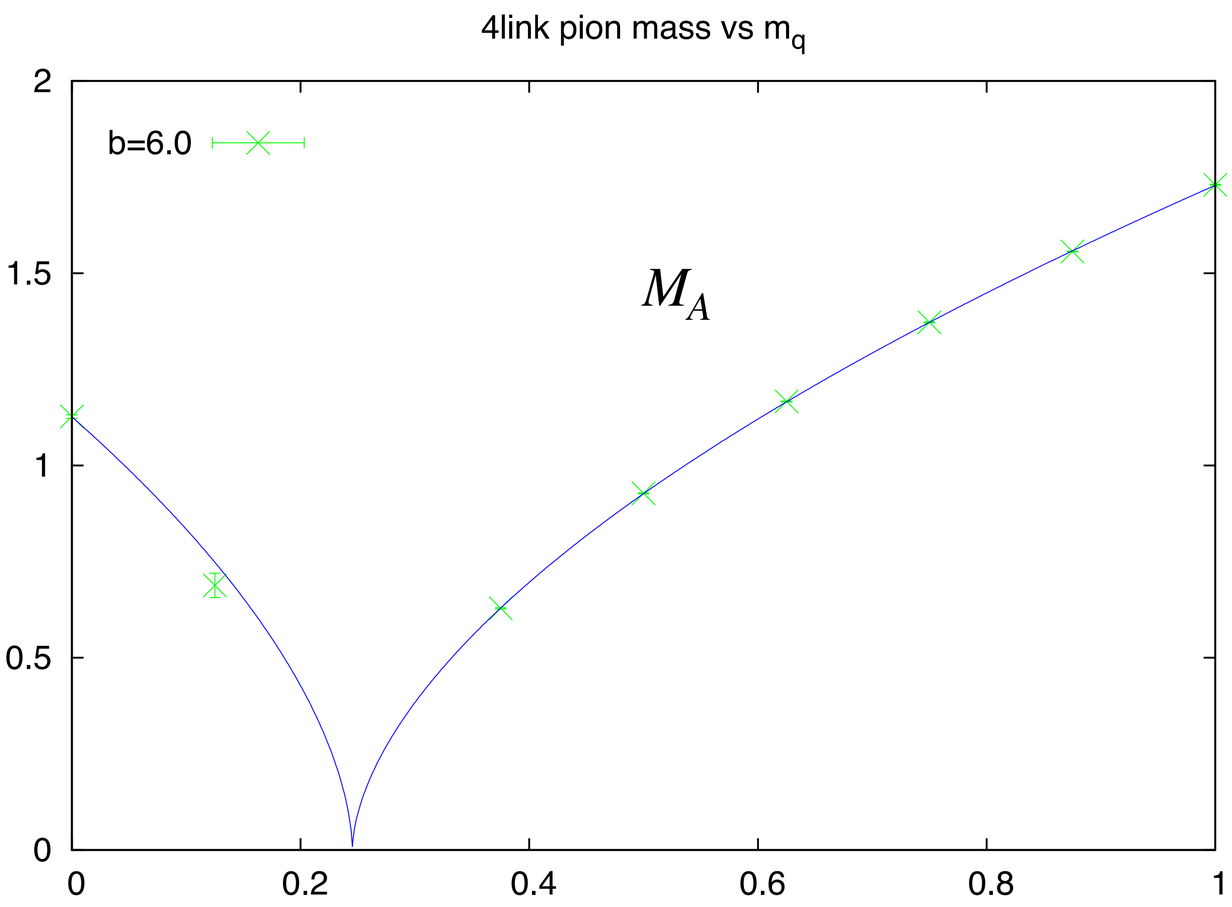


A: parity symmetric phase

Additive mass renormalization

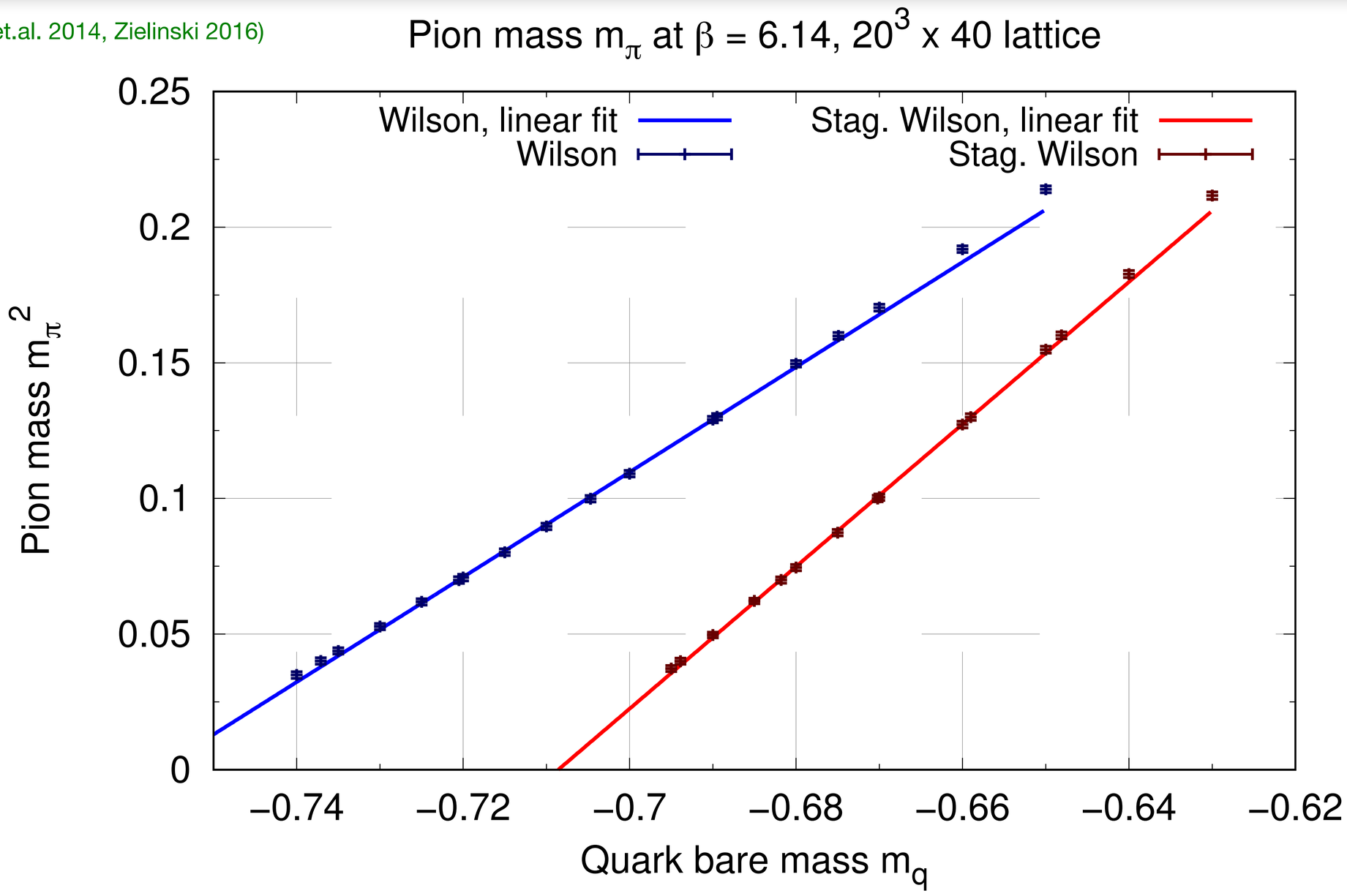
(deForcrand et. al. 2012)





Additive mass renormalization

(Adams et.al. 2014, Zielinski 2016)



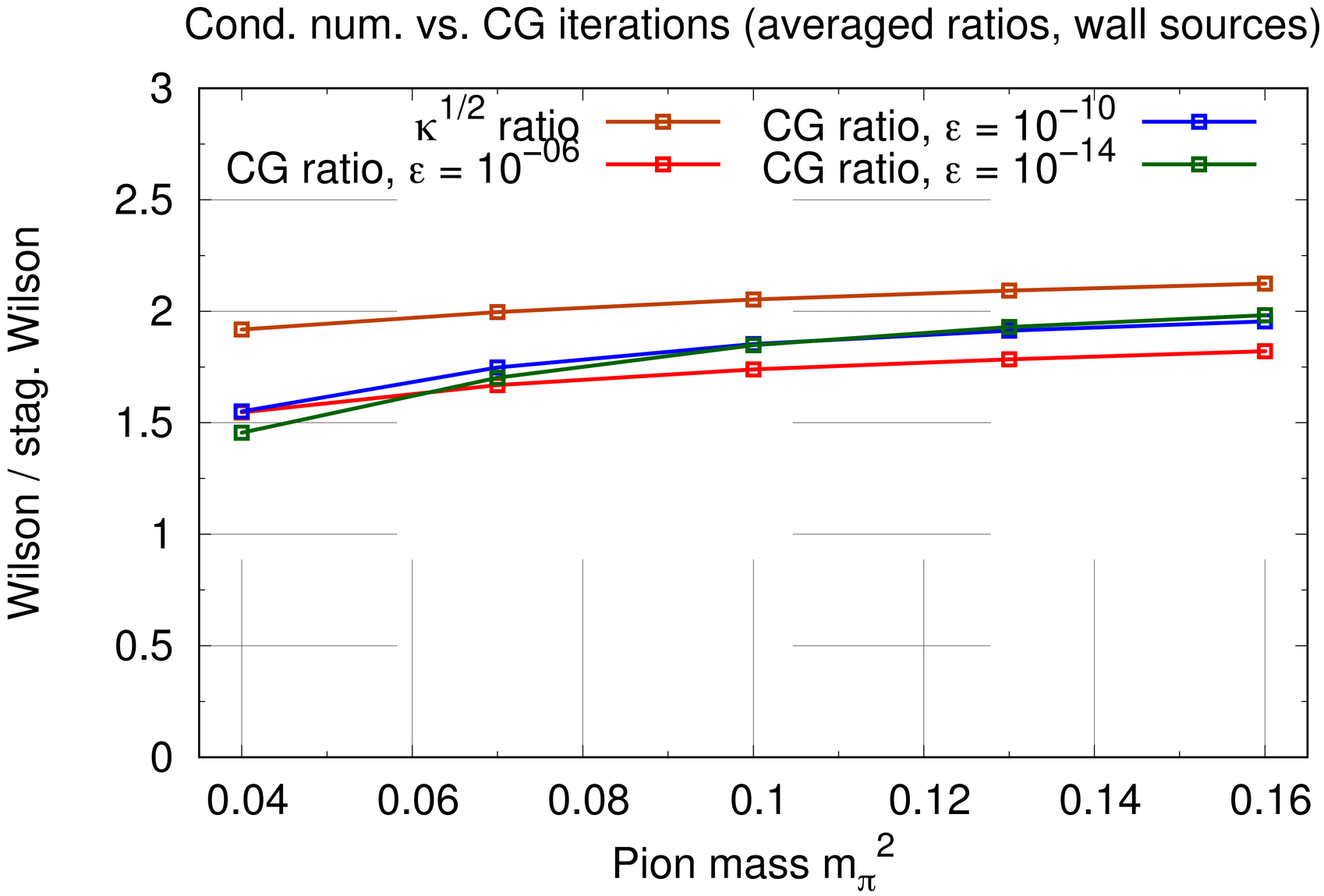
Computational cost estimates

(Adams et.al. 2014, Zielinski 2016)

Moderate improvements

But comparison is to unimproved Wilson!

Can one improve staggered Wilson?



Symanzik improvement

Clover improvement: similar to Wilson case

needs taste structure $(\sigma_{\mu\nu} \otimes 1) = i\eta_\mu \eta_\nu (C_\mu C_\nu)_{\mathrm{sym}}$

Suggestion: $D_i = D - \frac{c_{\mathrm{SW}}}{4} \sum_{\mu<\nu} \left\{ F_{\mu\nu}, i\eta_\mu \eta_\nu (C_\mu C_\nu)_{\mathrm{sym}} \right\}$

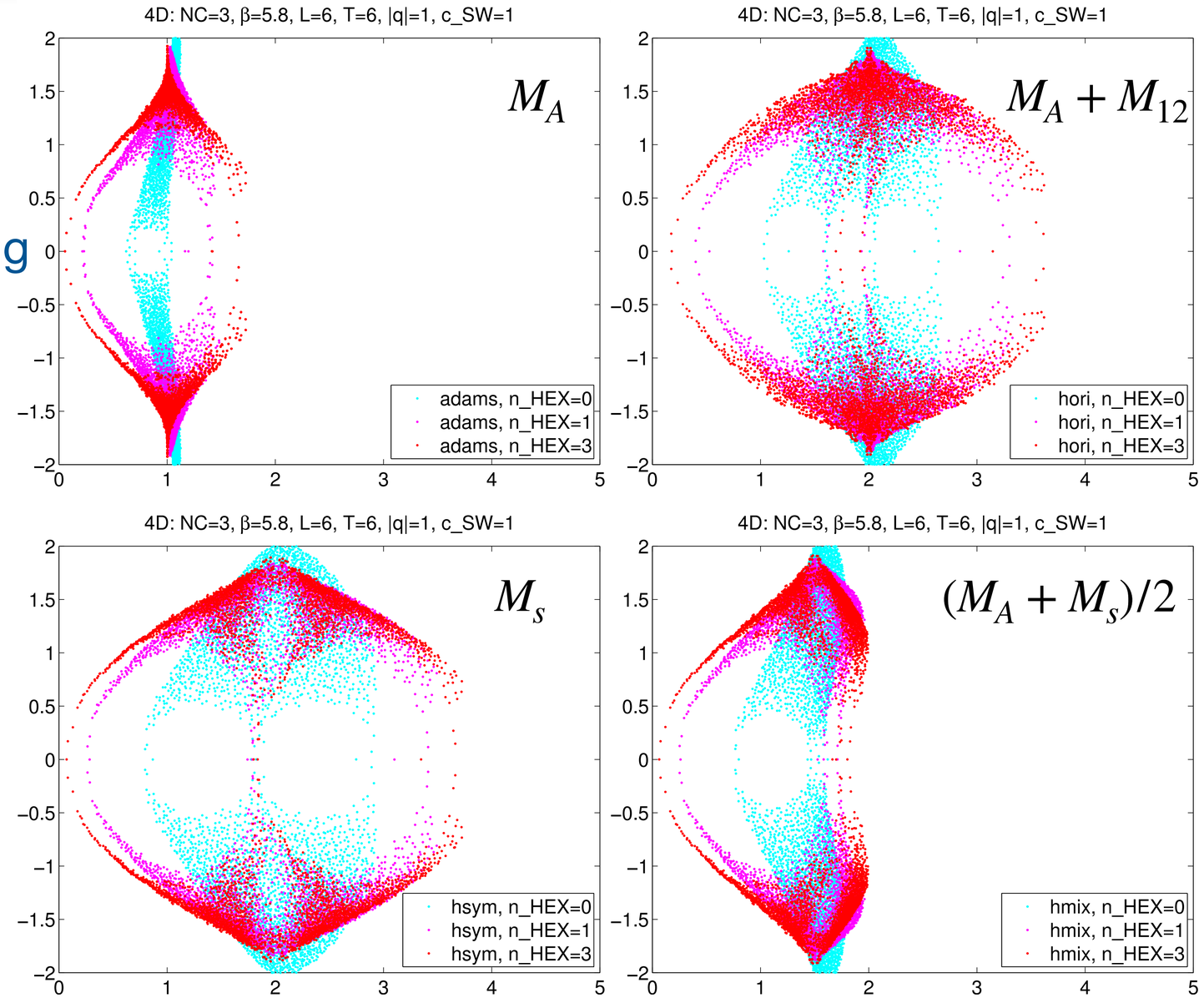
(Durr 2012)

Open question: is this unique?

Effects of clover term

(Durr 2012)

- clover improvement works
- large effect of coupled with smearing (just like in Wilson)



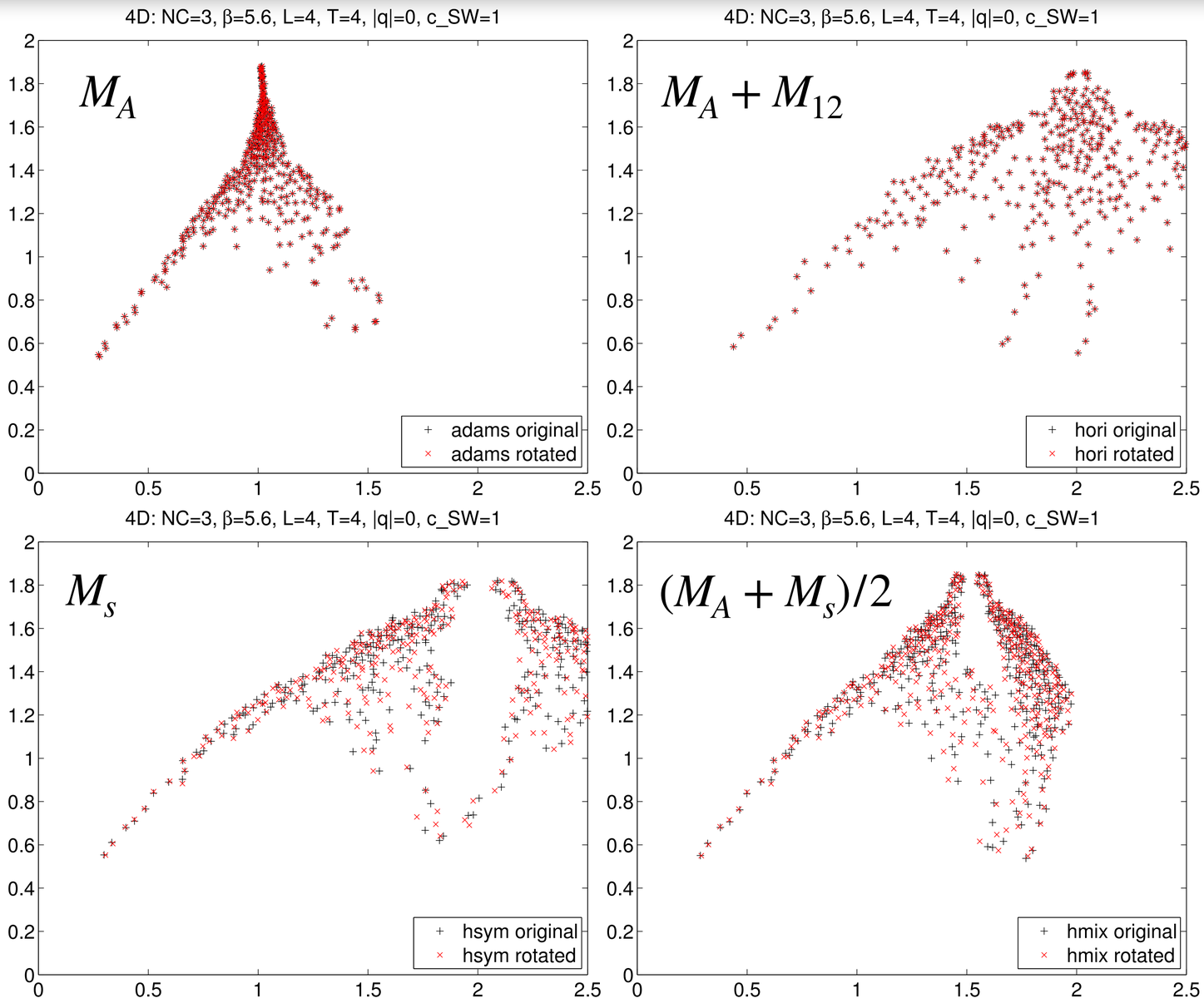
$$M_A = M_{1234}$$

$$M_s = \sqrt{3}\varepsilon_{\mu\nu\alpha\beta}(M_{\mu\nu} + M_{\alpha\beta})/4!$$

Rotational symmetry breaking in EV spectrum

(Dürr 2012)

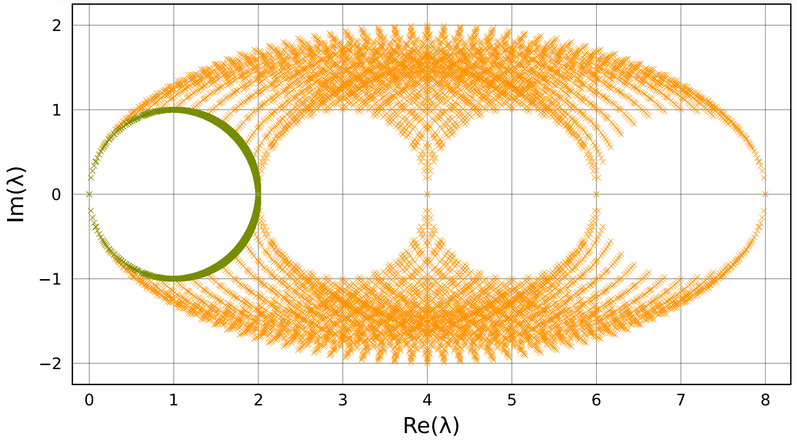
seems to mostly affect UV modes

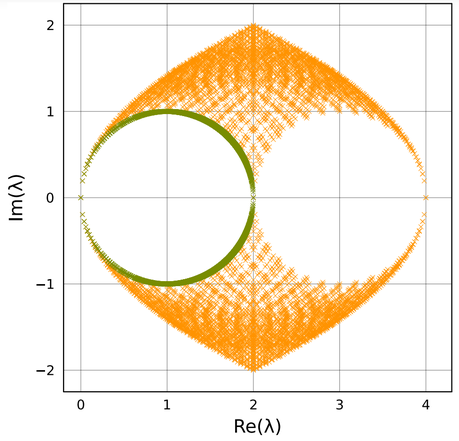


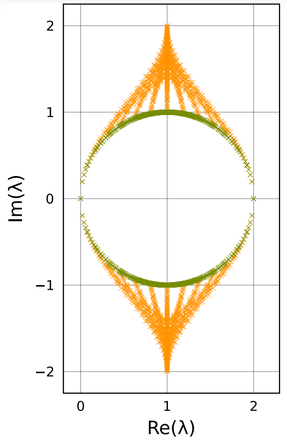
$$M_A = M_{1234}$$

$$M_s = \sqrt{3}\varepsilon_{\mu\nu\alpha\beta}(M_{\mu\nu} + M_{\alpha\beta})/4!$$

Chirally symmetric formulation







construction of overlap is straightforward with one key insight: (Adams 2010)

replace $\gamma_{1\ldots D}$ with $\epsilon = (-1)^{\sum_\mu x_\mu} \sim (\gamma_{1\ldots D} \otimes \xi_{1\ldots D})$

4D: replace γ_5 with $\epsilon = (-1)^{\sum_\mu x_\mu} \sim (\gamma_5 \otimes \xi_5)$

chiral operator is nontrivial in taste space!

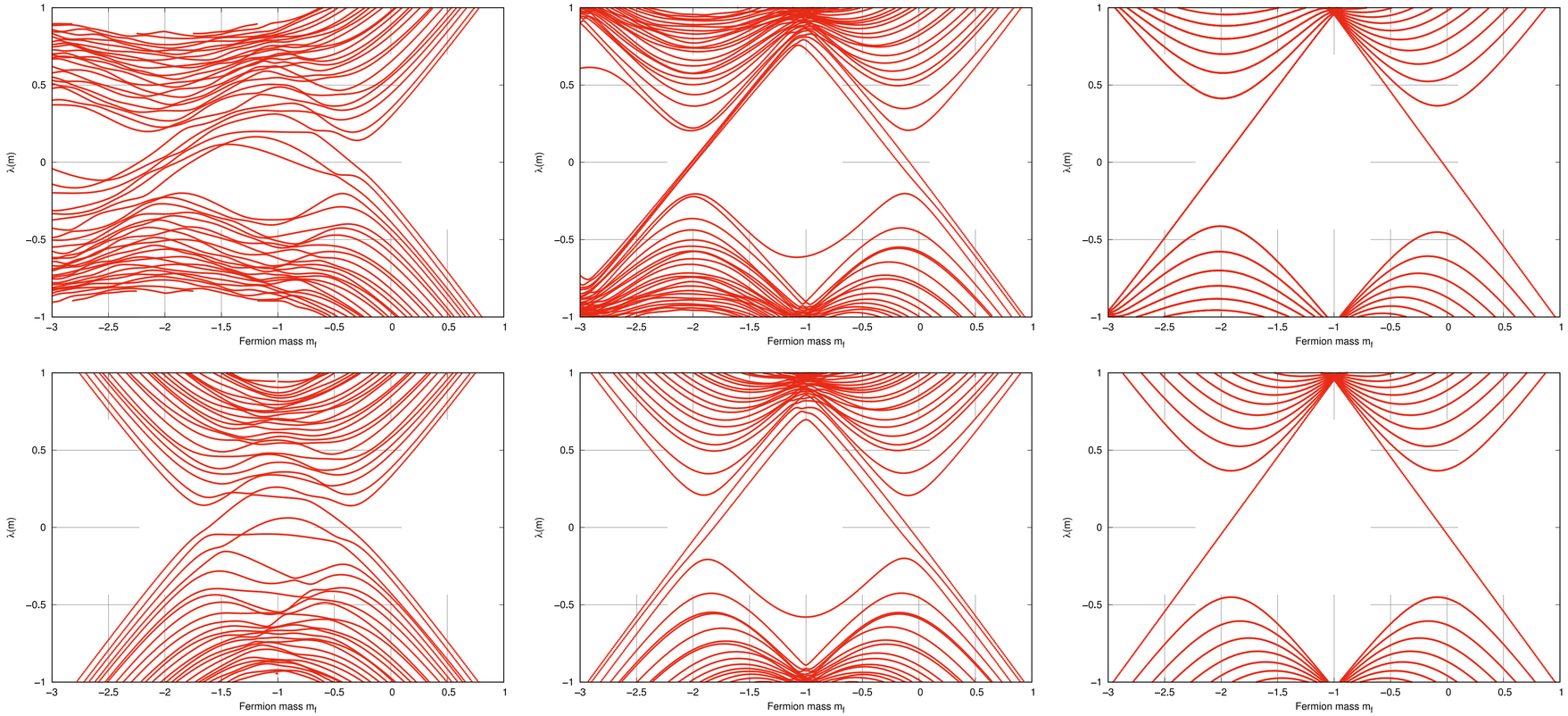
consistent with intuitive $\epsilon = \eta_{D+1} = (-1)^{\sum_{\nu<D+1} x_\nu}$, but remember $\prod_\mu \eta_\mu \neq \epsilon$!

4D: $\epsilon = (-1)^{x_1+x_2+x_3+x_4}$ but $\eta_1\eta_2\eta_3\eta_4 = 1 \times (-1)^{x_1} \times (-1)^{x_1+x_2} \times (-1)^{x_1+x_2+x_3} = (-1)^{x_1+x_3}$

Spectral flow

define hermitian kernel operator $H_{\mathrm{SW}}(m) = \epsilon D_{\mathrm{SW}}(m)$ with $D_{\mathrm{SW}} = D_{\mathrm{stag}} + M + m$

topology is evident in spectral flow (eigenvalues of $H_{\mathrm{SW}}(m)$ as m is varied)



(a) Unsmeared configuration (b) Smeared configuration (c) Topological configuration

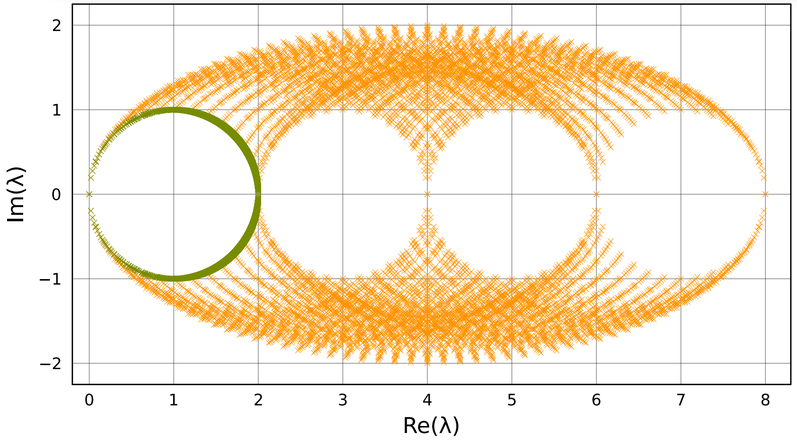
example: Schwinger model
Q=2

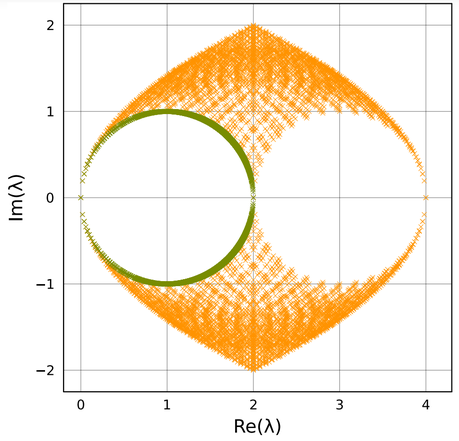
Wilson

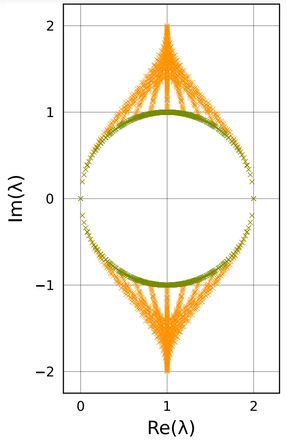
staggered Wilson

(deForcrand et. al. 2012; Azcoiti et.al. 2014; CH and Zielinski 2016, Dürr and Weber 2022)

Staggered overlap







staggered overlap operator: $D_{\mathrm{SO}} = 1 + \epsilon \dfrac{H_{\mathrm{SW}}(-\rho)}{\sqrt{H_{\mathrm{SW}}^2(-\rho)}}$

ρ … negative mass parameter
determines number of flavors
just as for standard overlap

$$D_{\mathrm{SO}} = 1 + D_{\mathrm{SW}}(-\rho)/\sqrt{D_{\mathrm{SW}}^{\dagger}(-\rho)D_{\mathrm{SW}}(-\rho)}$$

is it local?

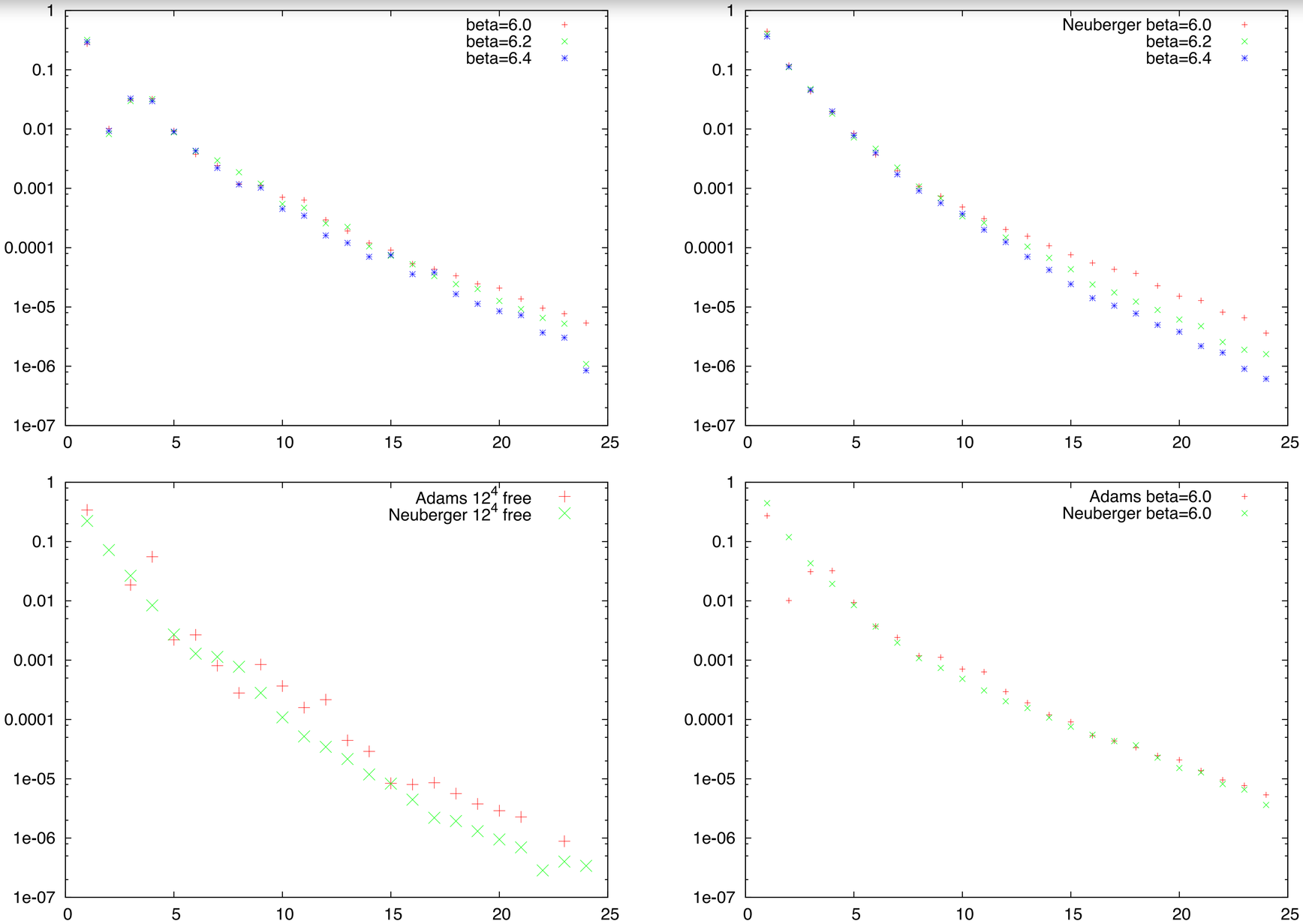
Locality: numerical evidence

(deForcrand et. al. 2012)

Numerical check:

2-flavor operator decays exponentially in lattice distance

practically indistinguishable from standard overlap



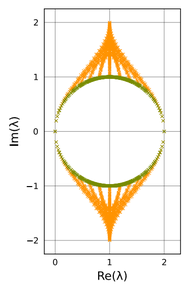
Locality: proof

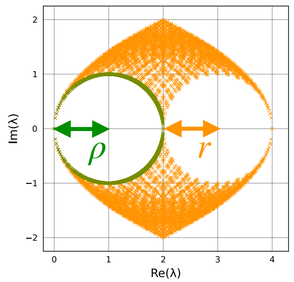
(Chreim et. al. 2022)

- established in 4D for kernel operator mass terms M_{1234} and $M_{12} + M_{34}$
- dependent on "admissibility condition" for plaquette:

$$\delta < \frac{r^2(1-|1-\rho|)^2}{6+12r+9r^2} \xrightarrow[r,\rho\to 1]{} \frac{1}{27} \text{ for } M_{1234}$$

$$\delta < \frac{r^2(1-|1-\rho|)^2}{6+4r+6r^2} \xrightarrow[r,\rho\to 1]{} \frac{1}{16} \text{ for } M_{12} + M_{34}$$





$$D_{SO} = 1 + A/\sqrt{A^\dagger A}$$

$$A = D_{st} + r(M-\rho)$$

- technique: expand $(A^\dagger A)^{1/2}$ in Legendre polynomials (similar to Hernandez et. al. 1999)

→ relates decay radius ξ to condition number C of $A^\dagger A$: $\xi^{-1} = \frac{1}{2la}\ln\frac{1+\sqrt{C}}{1-\sqrt{C}} \propto \frac{1}{a}$

hop distance in kernel

bound on C from bound on plaquette

Staggered domain wall

(Adams 2010; CH, Zielinski 2016)

standard domain wall operator: (Kaplan 1992; Shamir 1993; Furman and Shamir 1994)

$$\bar{\psi} D_{\mathrm{DW}} \psi = \sum_{s=1}^{N_s} \bar{\psi}_s (D_W^+ \psi_s - P_- \psi_{s+1} - P_+ \psi_{s-1}) \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5) \qquad D_W^\pm = D_W(-M_0) \pm 1$$

boundary conditions with mass term:

$$P_+(\psi_0 - m\psi_{N_s}) = 0 \qquad P_-(\psi_{N_s+1} - m\psi_1) = 0$$

Boriçi modification: (Boriçi 1999) $P_\pm \psi_{s\mp 1} \to -D_W^- P_\pm \psi_{s\mp 1}$

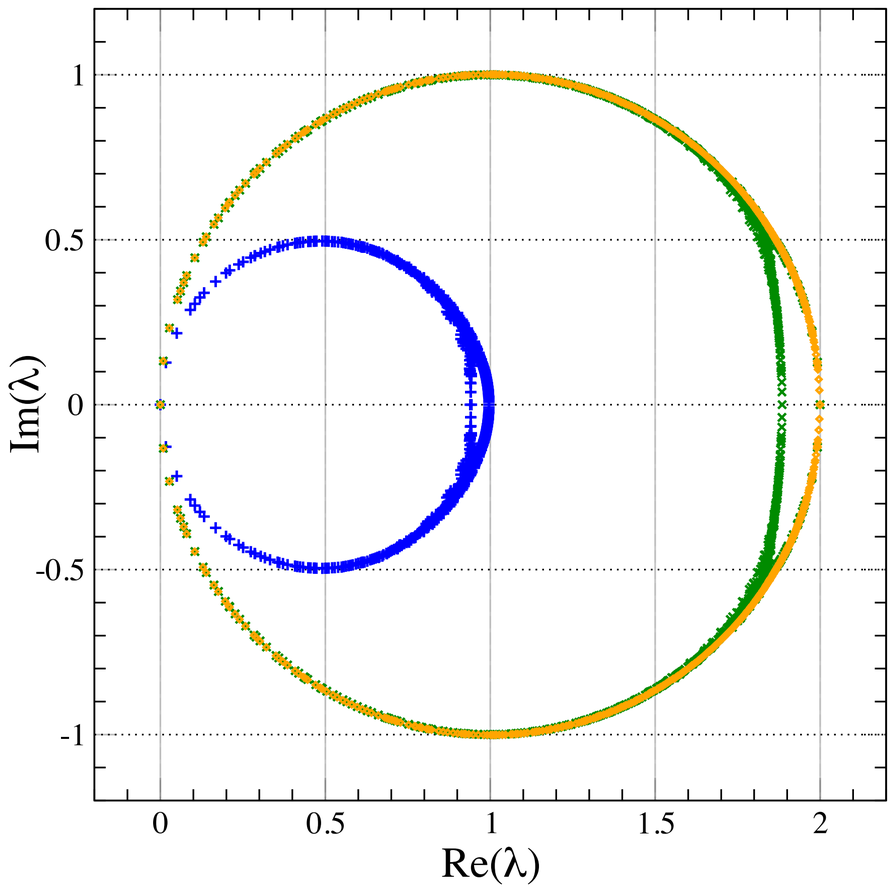
optimal DWF: (Chiu 2002) $D_W^\pm \to D_W^\pm(s) = \omega_s D_W(-M_0) \pm 1$ constructed as approximation to overlap

staggered version: (Adams 2010) $\gamma_5 \to \epsilon = (-1)^{\sum_\mu x_\mu} \sim (\gamma_5 \otimes \xi_5)$

$$D_W \to D_{SW}$$

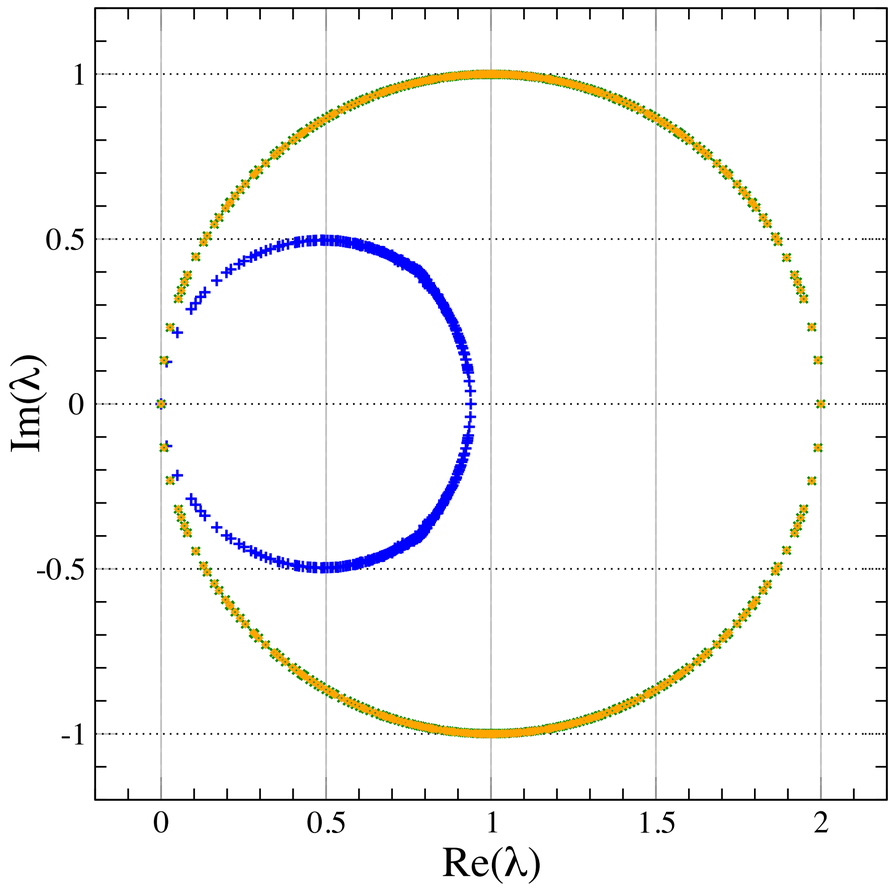
Schwinger model study

	m_{eff}	Δ_N	Δ_{GW}
std	5.22×10^{-5}	2.41×10^{-2}	2.27×10^{-1}
Bor	4.59×10^{-6}	2.96×10^{-2}	2.78×10^{-1}
opt	1.42×10^{-5}	1.67×10^{-2}	3.04×10^{-2}



Smeared Wilson kernel

	m_{eff}	Δ_N	Δ_{GW}
std	5.19×10^{-5}	2.29×10^{-2}	1.97×10^{-1}
Bor	1.16×10^{-6}	1.12×10^{-3}	5.97×10^{-3}
opt	1.47×10^{-5}	4.40×10^{-4}	6.06×10^{-4}



Smeared staggered Wilson kernel

normality violation:

$$\Delta_N = ||[D, D^\dagger]||$$

GW violation:

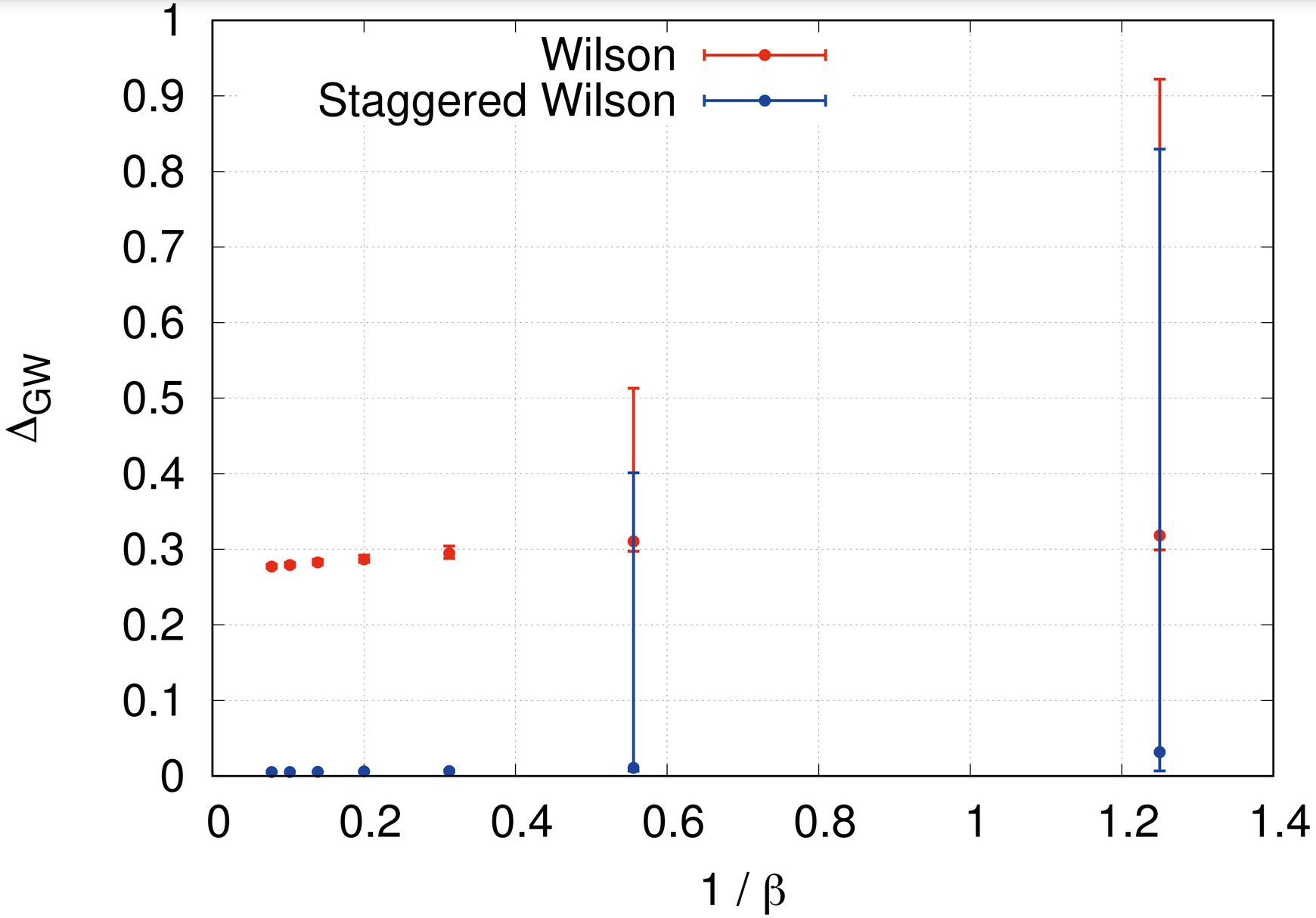
$$\Delta_{GW} = ||\gamma_5 D + D\hat{\gamma}_5||$$

$$\Delta_{GW} = ||\epsilon D + D\hat{\epsilon}||$$

residual mass:

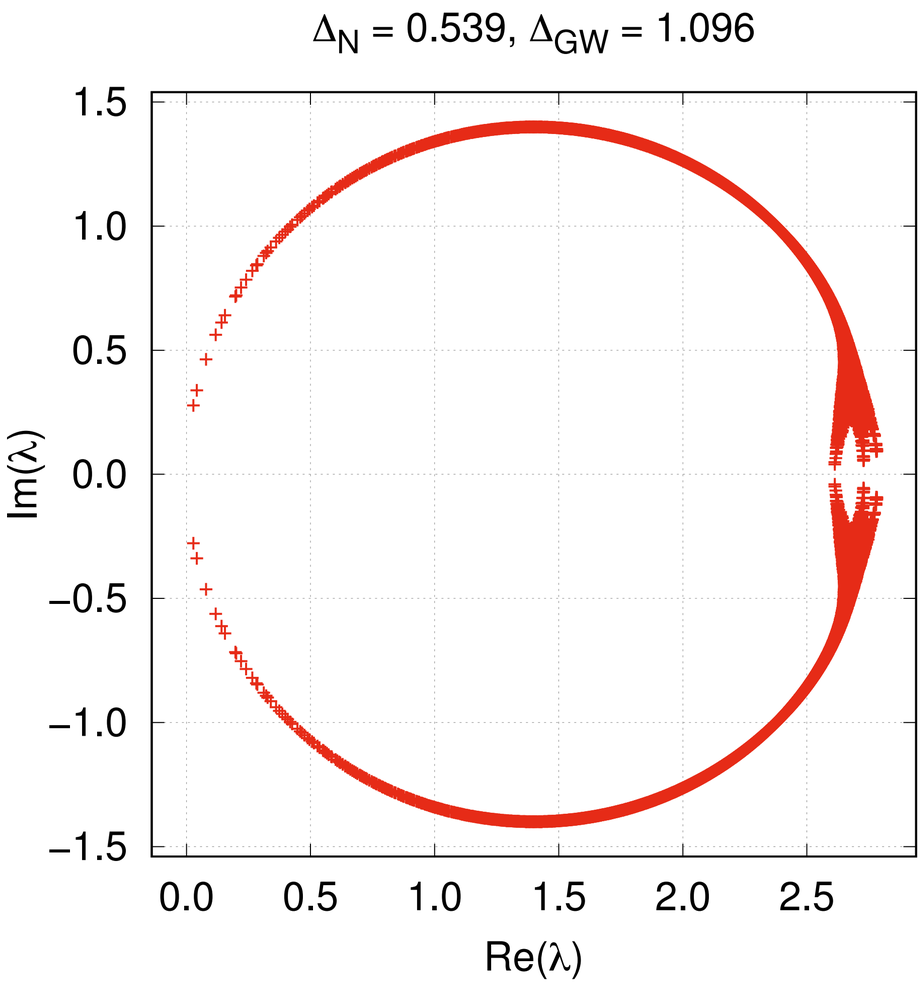
$$m_{\text{eff}} = |\lambda_{\min}|$$

Continuum behavior

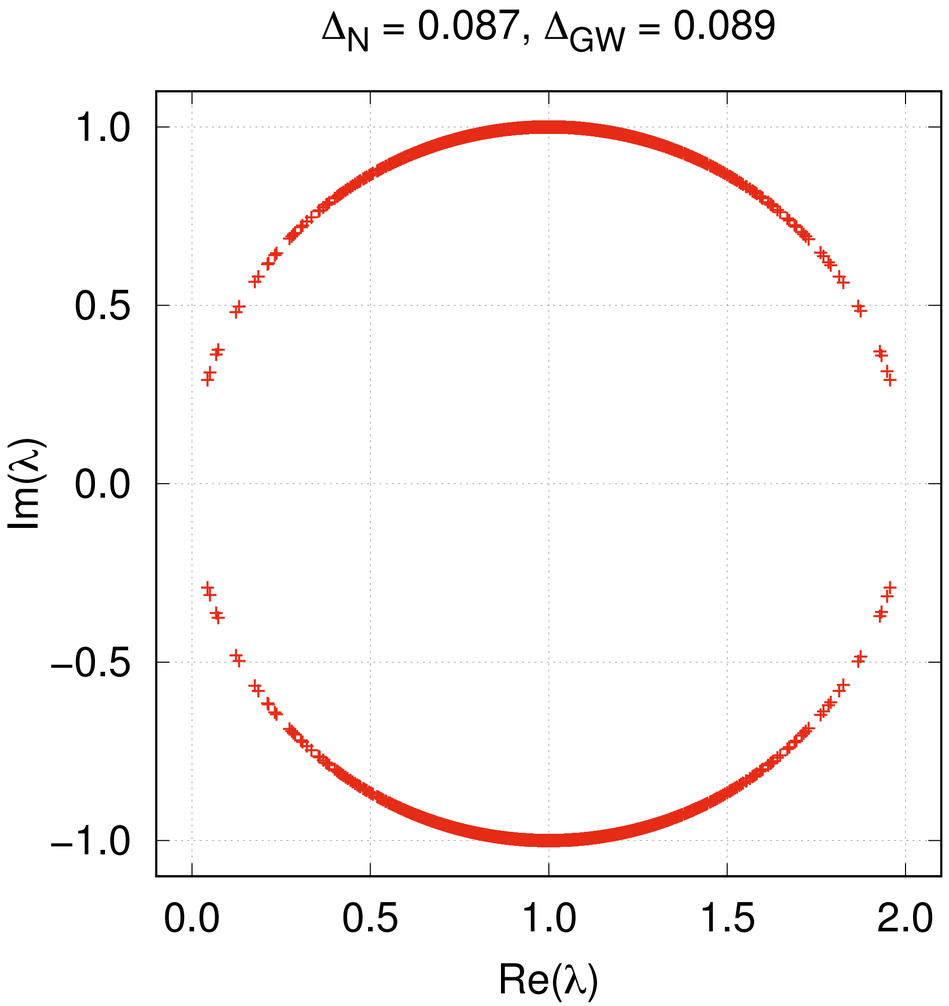


Peek at QCD

$6^4 \times 8$, $\beta = 6$, APE smeared SU(3) plaquette action

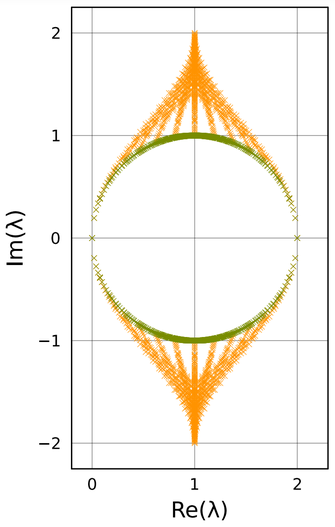


Boriçi Wilson

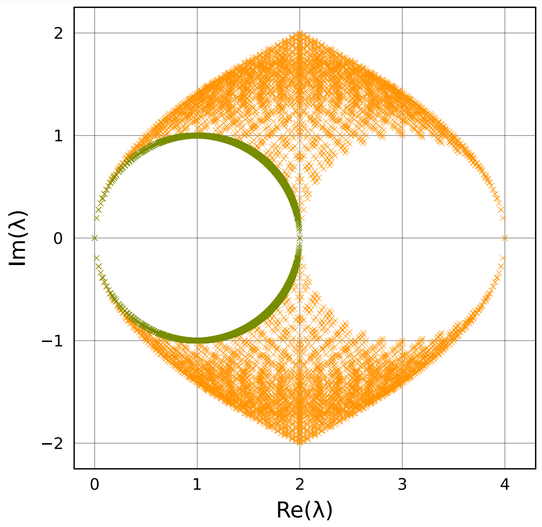


Boriçi staggered

Prospects



2 flavor



1 flavor

✔ smaller vectors

✔ better condition number

✔ somewhat improved chirality

✔ no additional counterterms

✘ gluonic CT

✘ 4-hop

✘ 2-hop

many open questions:

- symmetries
- mixing
- observables
- improvement

my take:

can potentially speed up one calculations by 1/2 to 1 order of magnitude

but

- competition is not plain Wilson
- lots of “tricks”, some of them based on genuine physical insight
- all of this will need to work for new formulations
- today, two degenerate light flavors is not good enough!

Minimally doubled staggered?

Old suggestion: (van den Doel, Smit '83)

- χ on even and $\bar{\chi}$ on odd sites only
- breaks ϵ-hermiticity → determinant not real
  (→ Catterall 2021+this workshop)

In 2D staggered is minimally doubled!

→construction needs to be impossible in arbitrary dimensions!

What about Karsten-Wilczek or Boriçi-Creutz like pole coalescence?

Start with free theory:

- staggered momentum eigenstates: $P_s(x) = e^{i(p_\mu + \pi s_\mu)x_\mu} = (-1)^{s_\mu x_\mu} e^{ip_\mu x_\mu}$

$s_\mu \in \{0,1\} \quad |p_\mu| < \pi/2$

- eigenvalues: $D_{\mathrm{st}} P_s = i\eta_\mu \hat{p}_\mu (-1)^{s_\mu} P_s$

$$\Downarrow$$

$$\lambda_{p\pm} = \pm i\sqrt{\hat{p}_\mu \hat{p}_\mu}$$

$\hat{p}_\mu = \sin(p_\mu)$

$\lambda_{p\pm}$ each 8-fold degenerate, with eigenvectors of opposite ϵ chirality

Minimally doubled staggered?

pole condition: $\lambda_p = 0 \rightarrow \hat{p}_\mu = 0 \rightarrow p_\mu = 0$

$|p_\mu| < \pi/2$

$\lambda_{p\pm} = \pm i\sqrt{\hat{p}_\mu \hat{p}_\mu}$

16 poles, 4 tastes

Try to coalesce poles:

- modify $\hat{p}_\mu \rightarrow \hat{p}'_\mu$ so that $\hat{p}'_\mu \hat{p}'_\mu = 0$ has only 8 solutions (4 per ϵ-chirality)
- results in new operator: $D_{\rm st} = i\eta_\mu \hat{p}_\mu (-1)^{s_\mu} \rightarrow D = i\eta_\mu \hat{p}'_\mu (-1)^{s_\mu}$
- with $\hat{p}'_\mu = \hat{p}_\mu + f_\mu(p, s)$, where $f_\mu(p, s)$
  - is smooth in full Brillouin zone
  - is real
  - commutes with ϵ

locality → is smooth in full Brillouin zone

antihermiticity → is real

ϵ-hermiticity → commutes with ϵ

→ obvious candidate: taste dependent mass term

Candidate operator

$D = i\eta_\mu \hat{p}'_\mu (-1)^{s_\mu}$ $\quad \hat{p}'_\mu = \hat{p}_\mu + f_\mu(p, s)$

- on momentum modes: $C_\mu P_s = \hat{c}^s_\mu P_s$ with $\hat{c}^s_\mu = (-1)^{s_\mu}\cos(p_\mu)$

$P_s(x) = (-1)^{s_\mu x_\mu} e^{ip_\mu x_\mu}$

- simplest guess: $f_i = 0$ and $f_4 = \zeta(1 - \hat{c}_1\hat{c}_2)$

- would give: $\hat{p}'_\mu \hat{p}'_\mu = 0 \quad \rightarrow \quad \hat{p}_i = 0 \quad \hat{p}_4 + \zeta(1 - \hat{c}_1\hat{c}_2) = 0$

note: $\hat{p}_i = 0 \rightarrow \hat{c}_i = (-1)^{s_i}$

solutions for $|p_\mu| < \pi/2$: $\quad p_i = 0 \quad \sin(ap_4) = \zeta((-1)^{s_1+s_2} - 1)$

for $\zeta > 1$ only even $s_1 + s_2$ ✔

- full operator: $D = D_{\text{st}} + i\eta_4\zeta(1 - (C_1C_2)_{\text{sym}})(-1)^{s_4}$

discontinuous! try $\hat{c}^s_4 = (-1)^{s_4}\cos(p_4)$

- candidate: $D = D_{\text{st}} + i\eta_4\zeta(1 - (C_1C_2)_{\text{sym}})C_4$

Candidate operator

- candidate: $D = D_{\rm st} + i\eta_4\zeta(1-(C_1C_2)_{\rm sym})C_4$
- free case: $D = i\eta_\mu\hat{p}_\mu(-1)^{s_\mu} + i\eta_4\zeta(1-\hat{c}_1\hat{c}_2)\hat{c}_4$

→ pole conditions: $\hat{p}_i = 0 \quad \hat{p}_4 + \zeta(1-\hat{c}_1\hat{c}_2)\hat{c}_4 = 0$

note: $\hat{p}_i = 0$ → $\hat{c}_i = (-1)^{s_i}$

→solutions for $|p_\mu| < \pi/2$: $\quad p_i = 0 \quad \hat{p}_4/\hat{c}_4 = \tan(ap_4) = \zeta((-1)^{s_1+s_2} - 1)$

16 solutions for arbitrary ζ !

note: actual p_4 of doubler poles need not be realized exactly on the lattice