An Overview of Brillouin Fermions

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with
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Outline

An overview of *Brillouin fermions*

- S. Durr & G. K., `arXiv:1012.3615`
- S. Durr, G. K., & T. Lippert, `arXiv:1208.6270`
- S. Durr & G. K., `arXiv:1701.00726`

Outline

- Motivation
- Brillouin operator construction — 2D case
- Eigenvalue spectra in 2D
- Proof-of-concept investigation in quenched QCD
- Overlap construction
- Summary, conclusions, outlook

Motivation

$$D(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \frac{1}{2}\Delta(x,y) + m_0\delta(x,y)$$

Consider the first derivative and the laplacian term

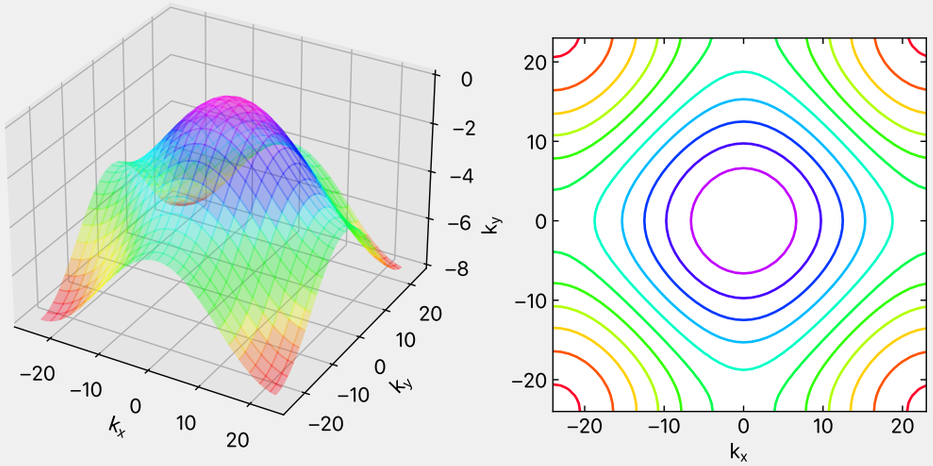
- Perfect actions
  - Extended operators, $\mathrm{O}(100)$ times more expensive
- Truncation
  - Parameter tuning for each set of new action parameters
- Here:
  - Consider more "geometric improvements", e.g. rotational symmetry
  - Consider effect on eigenvalue spectrum

Construction – Laplacian

"Standard" laplacian stencil

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} /1$$

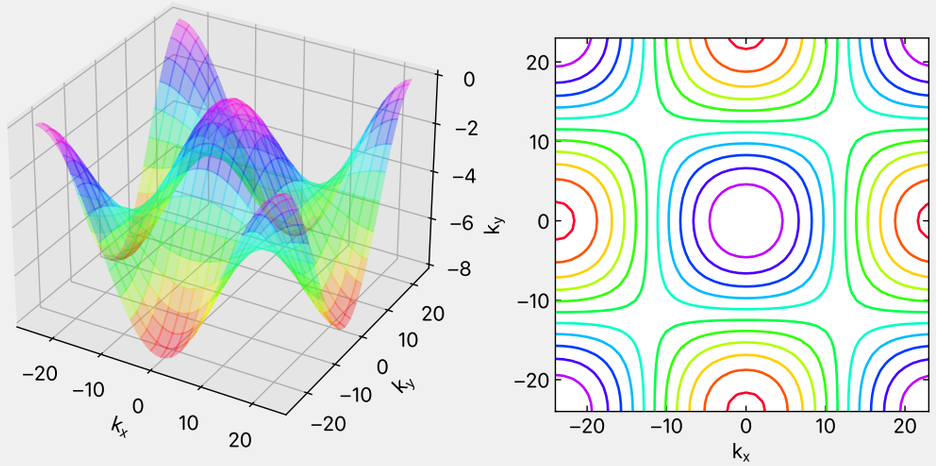
$$\Delta_s = 2\cos(k_1) + 2\cos(k_2) - 4$$



"Tilted" laplacian stencil

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix} /2$$

$$\Delta_t = 2\cos(k_1)\cos(k_2) - 2$$



Consider linear combinations of the "standard" and "tilted" laplacian

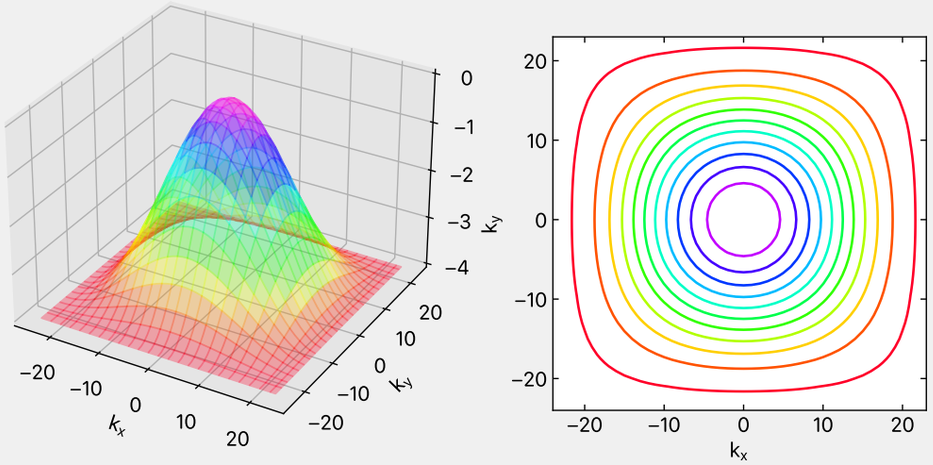
$$\alpha\Delta_s + (1-\alpha)\Delta_t$$

Construction – Laplacian

Two notable linear combinations of $\alpha\Delta_s + (1-\alpha)\Delta_t$

"Brillouin" laplacian, α=1/2

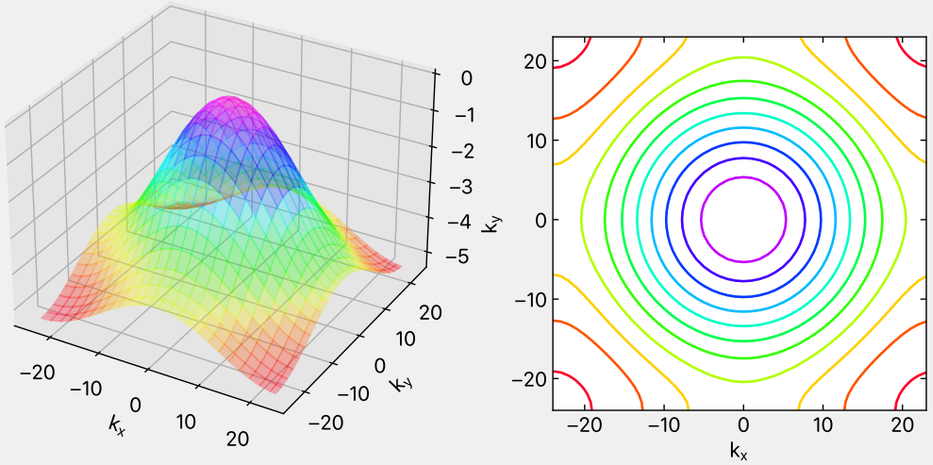
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -12 & 2 \\ 1 & 2 & 1 \end{bmatrix} /4$$



Doublers "lifted" equally

"Isotropic" laplacian, α=2/3

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} /6$$

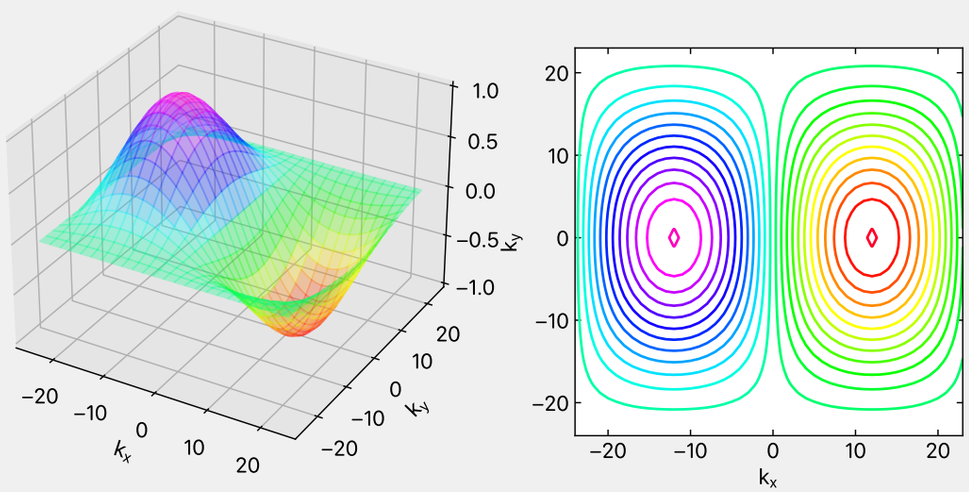


Isotropic around origin

Construction – Derivative term

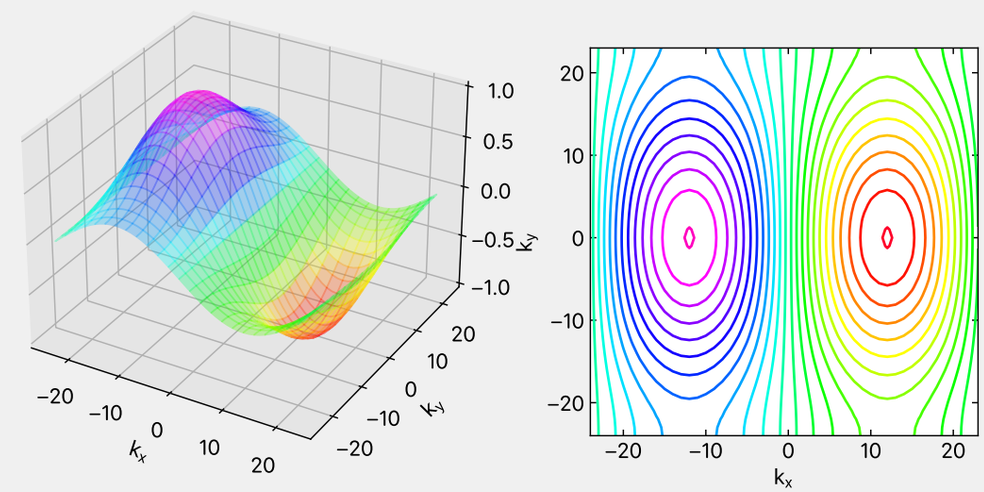
"Brillouin" (in , x–direction)

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} /8$$



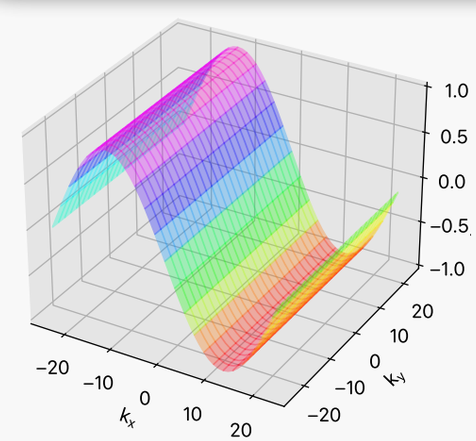
"Isotropic" (in , x–direction)

$$\begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} /12$$



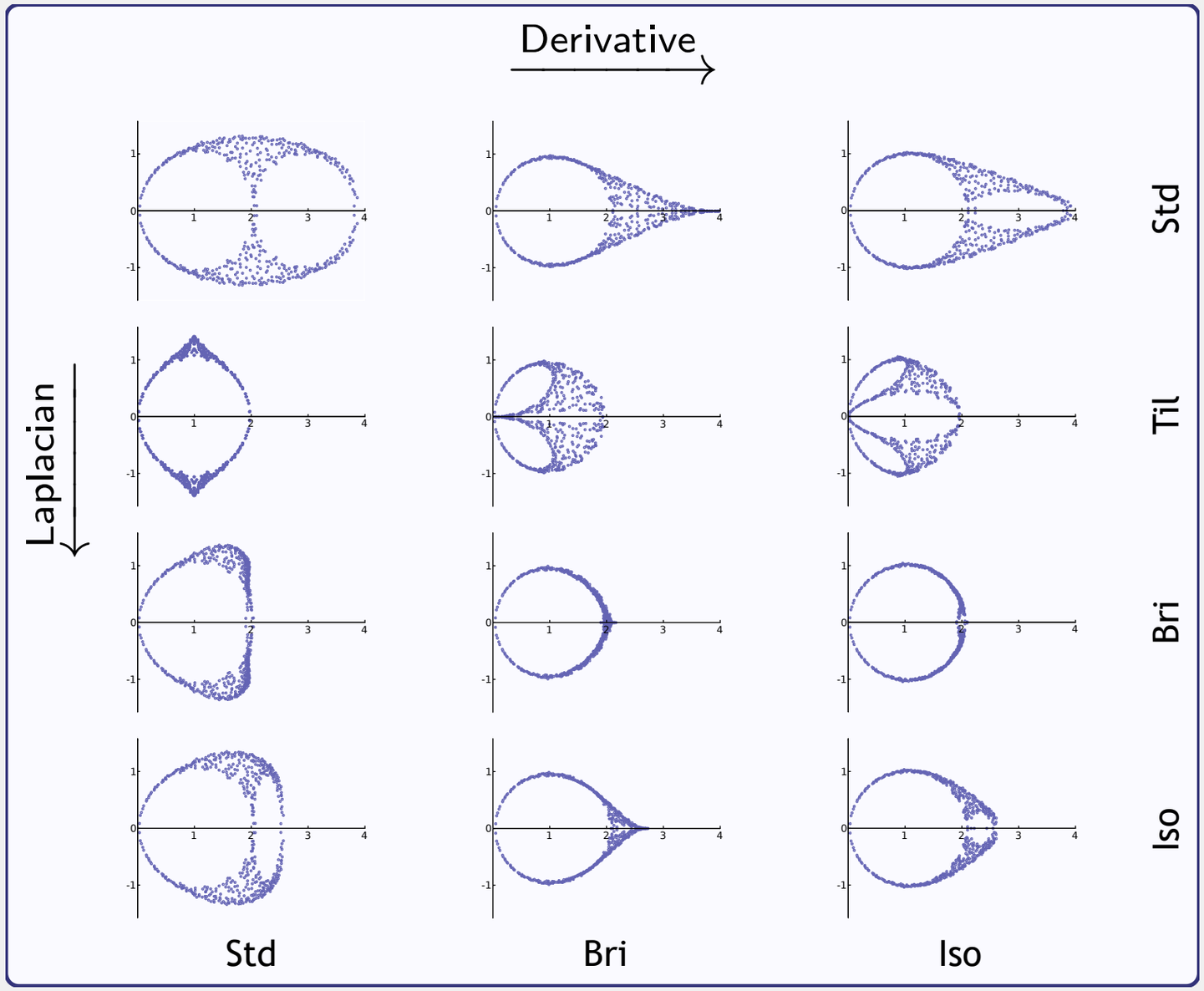
Compare to standard case:

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} /2$$



Construction – Full operator

Here for $U(1)$, 16×16, β=4.4, c_{SW}=1



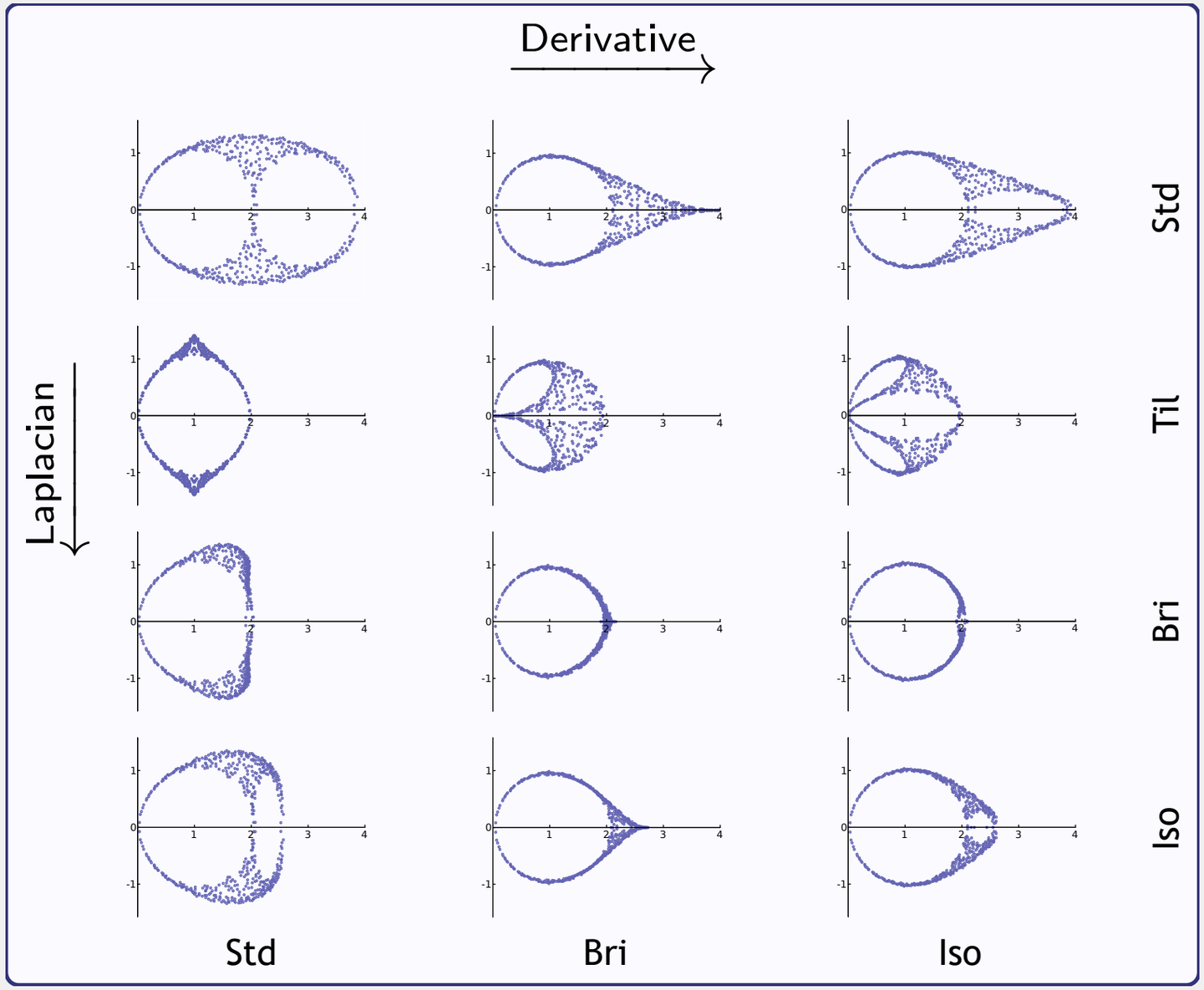
3 derivatives

×

4 laplacians

Construction – Full operator

Here for $U(1)$, 16×16, β=4.4, c_{SW}=1



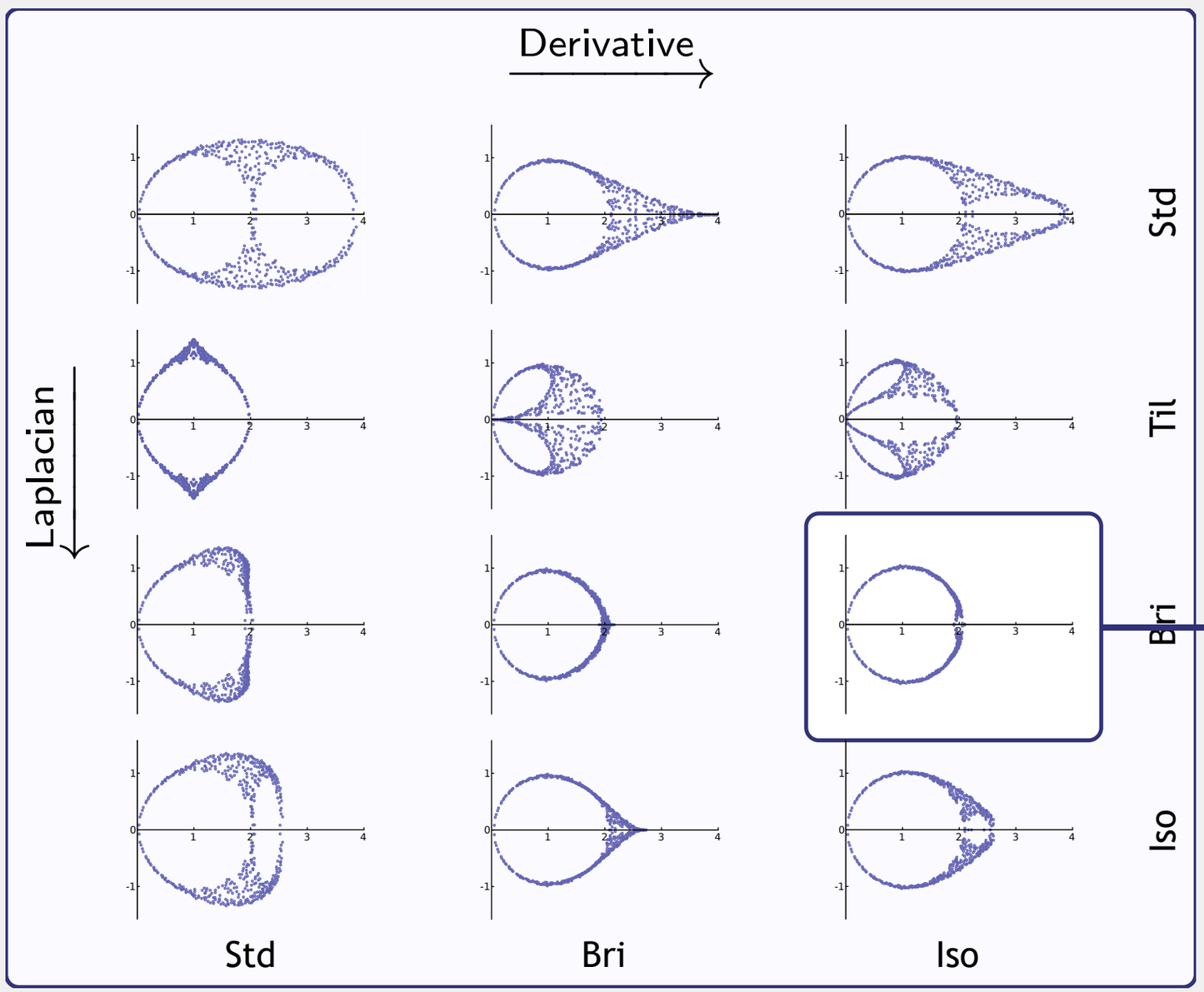
Expect larger condition number

Doublers

Better condition number, rotational symmetry

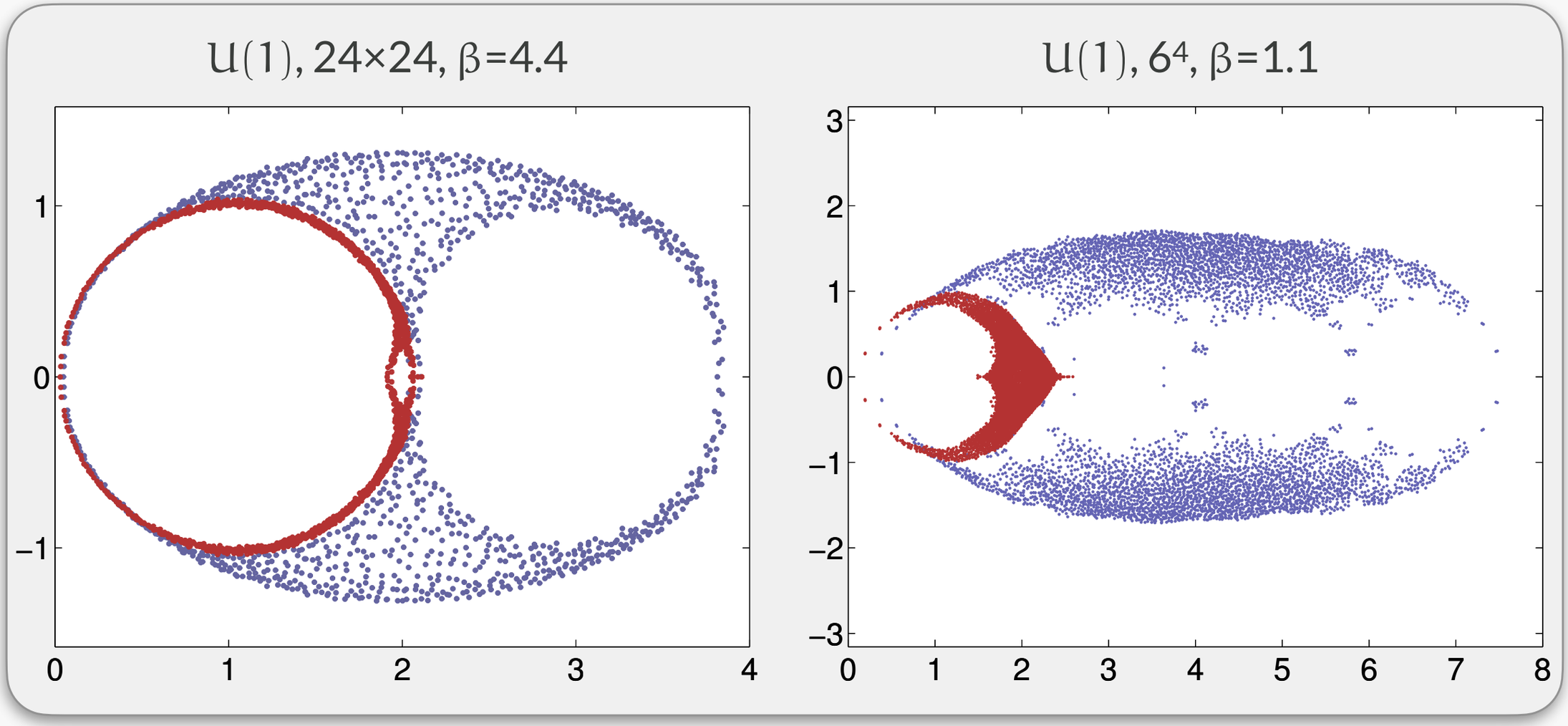
Brillouin Laplacian, isotropic derivative

Here for $U(1)$, 16×16, β=4.4, c_{SW}=1



Brillouin Laplacian,
Isotropic derivative,
⇒ our "*Brillouin*" operator

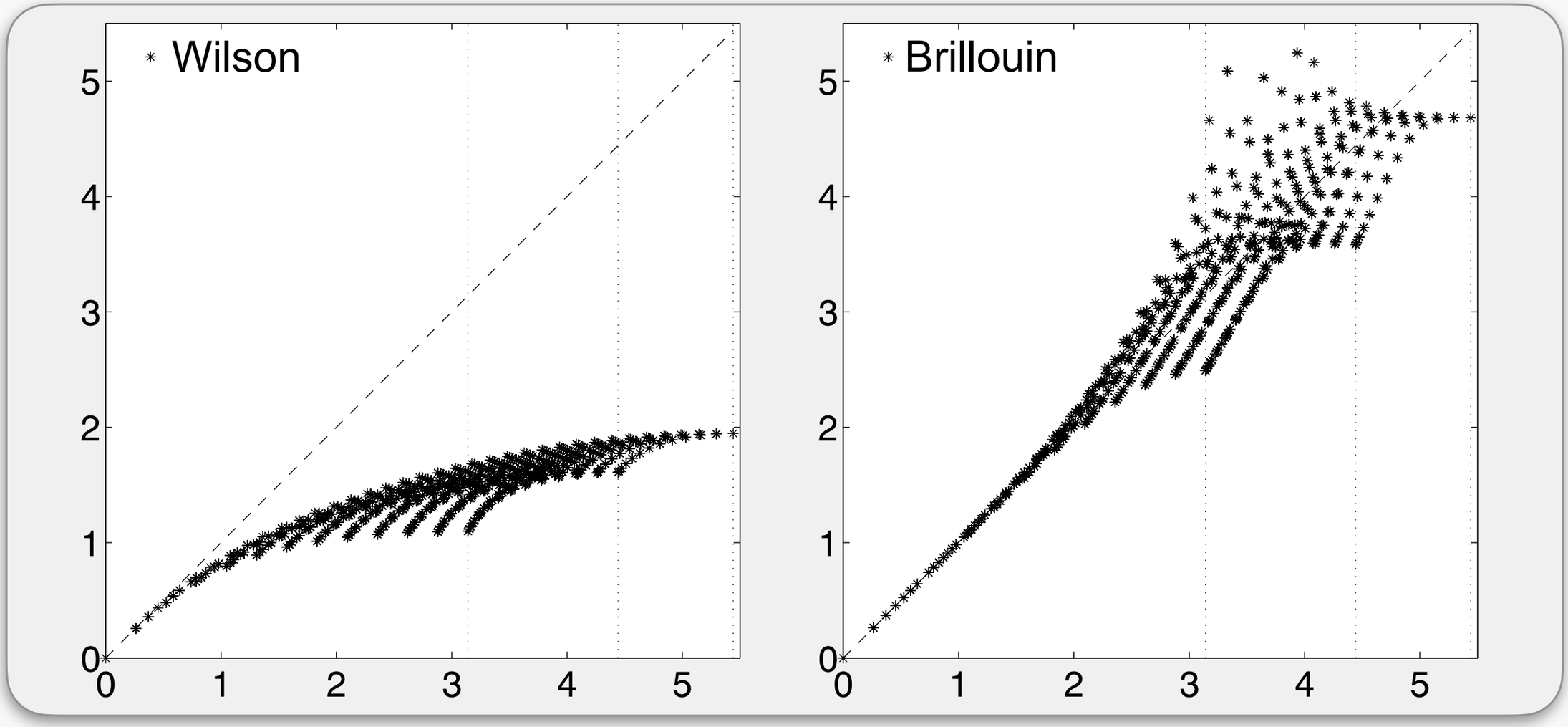
Brillouin Laplacian, isotropic derivative



Expectations

- Smaller condition number
- Smaller additive mass renormalisation
- Smaller rotational symmetry violation

Free-field quark dispersion relation



Noticeable improvement in dispersion relation for Brillouin improved fermions compared to standard Wilson

More during Stephan's later today

Test case in quenched QCD

S. Durr & G. K., arXiv:1012.3615

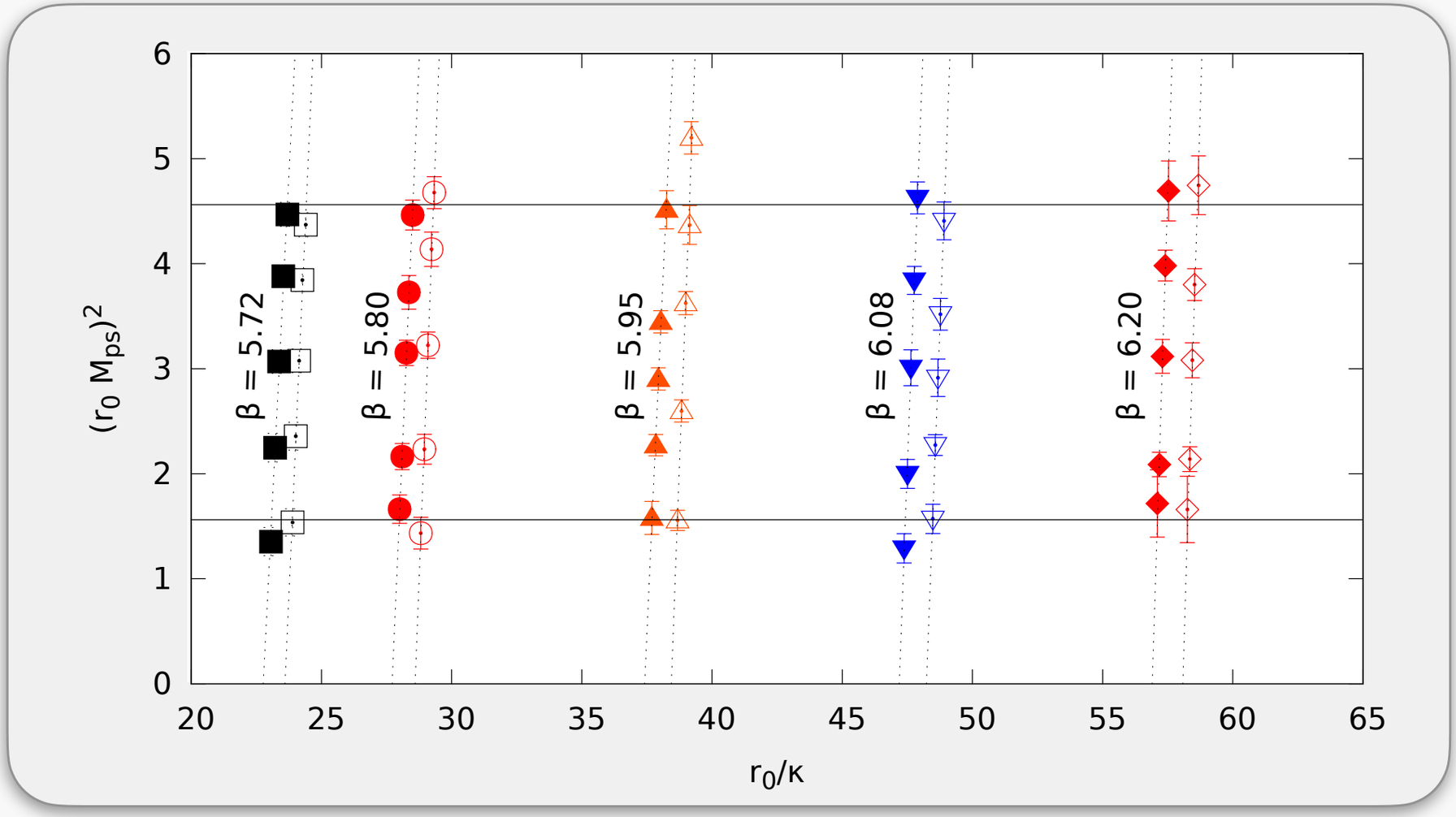
size	β	a (fm)	a^{-1} (GeV)
$10^3 \times 20$	5.72	0.160	1.236
$12^3 \times 24$	5.80	0.133	1.479
$16^3 \times 32$	5.95	0.100	1.978
$20^3 \times 40$	6.08	0.080	2.463
$24^3 \times 48$	6.20	0.067	2.964

- Constant $L \simeq 1.6$ fm
- 1 iteration APE, α=0.72
- c_{SW}=1

Compare to standard Wilson:

- Cut-off effects for pseudo-scalar decays
- Inversion iteration count (or condition number)
- Eigenvalue spectrum

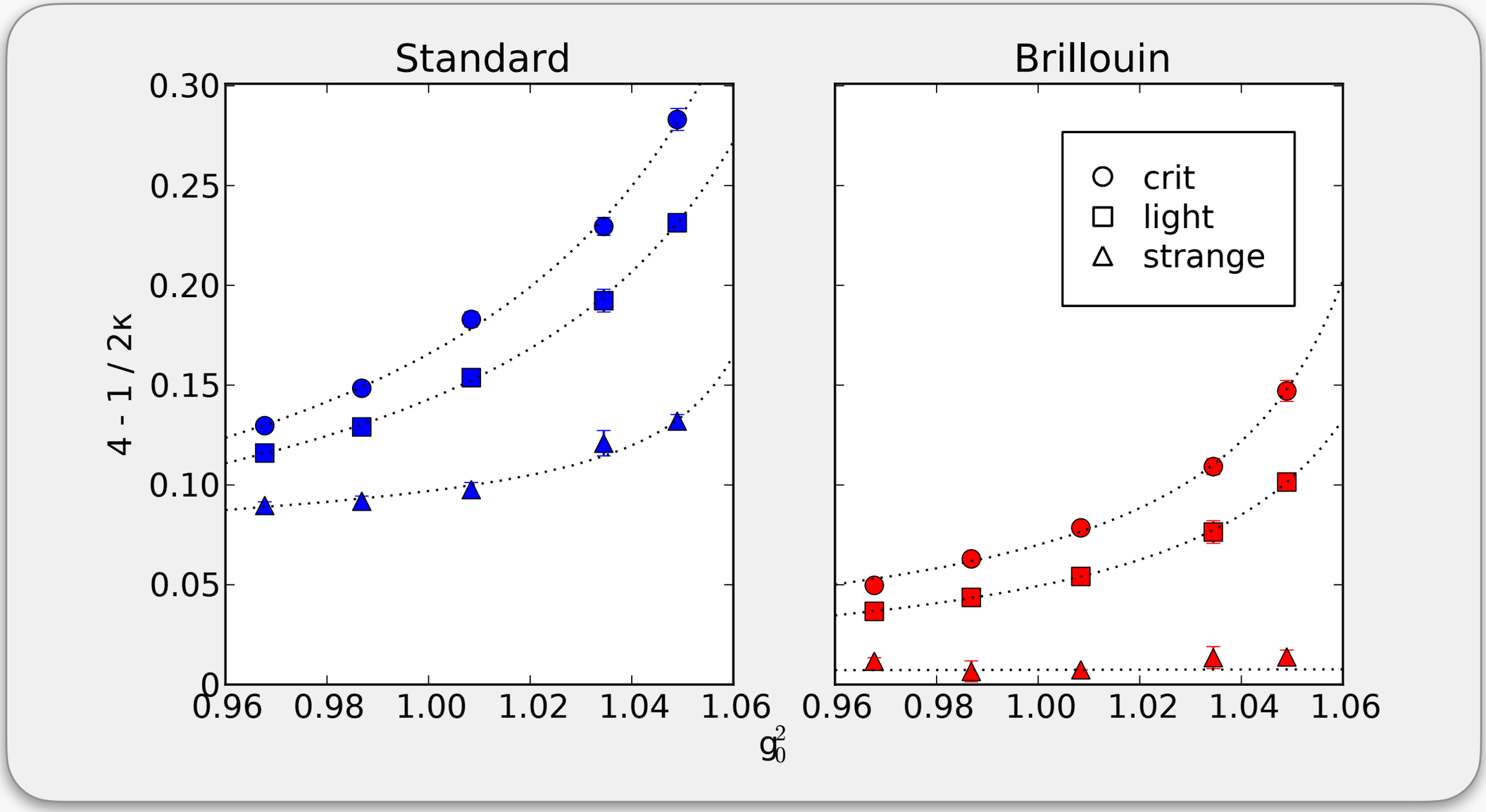
Test case in quenched QCD



- Horizontal line: target κ for both Wilson and Brillouin:

  $(r_0 M_{PS})^2 = 1.25^2, 2.125^2 = 500^2, 860^2\,\mathrm{MeV}^2$

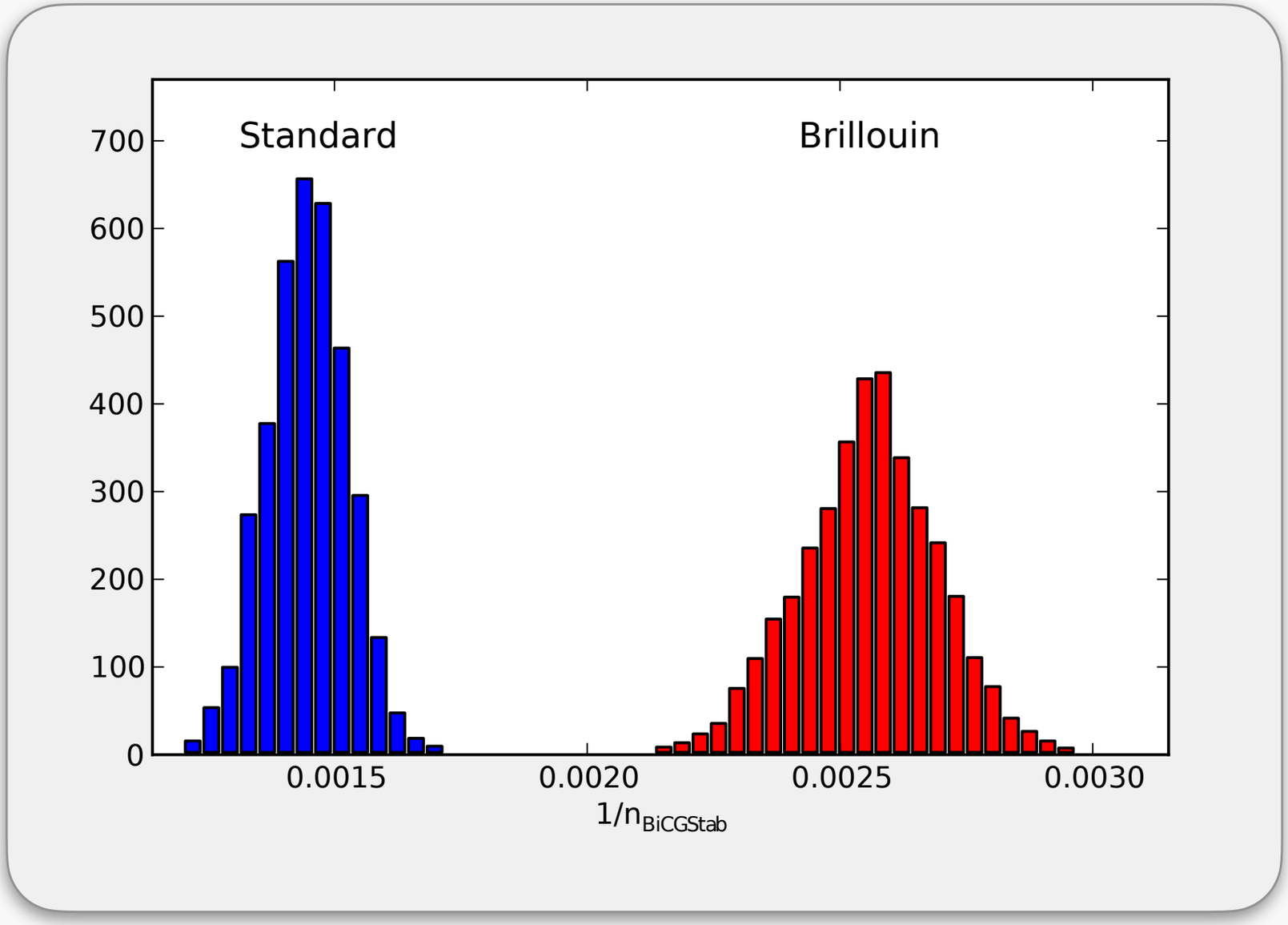
Critical κ



- Fits to rational ansatz

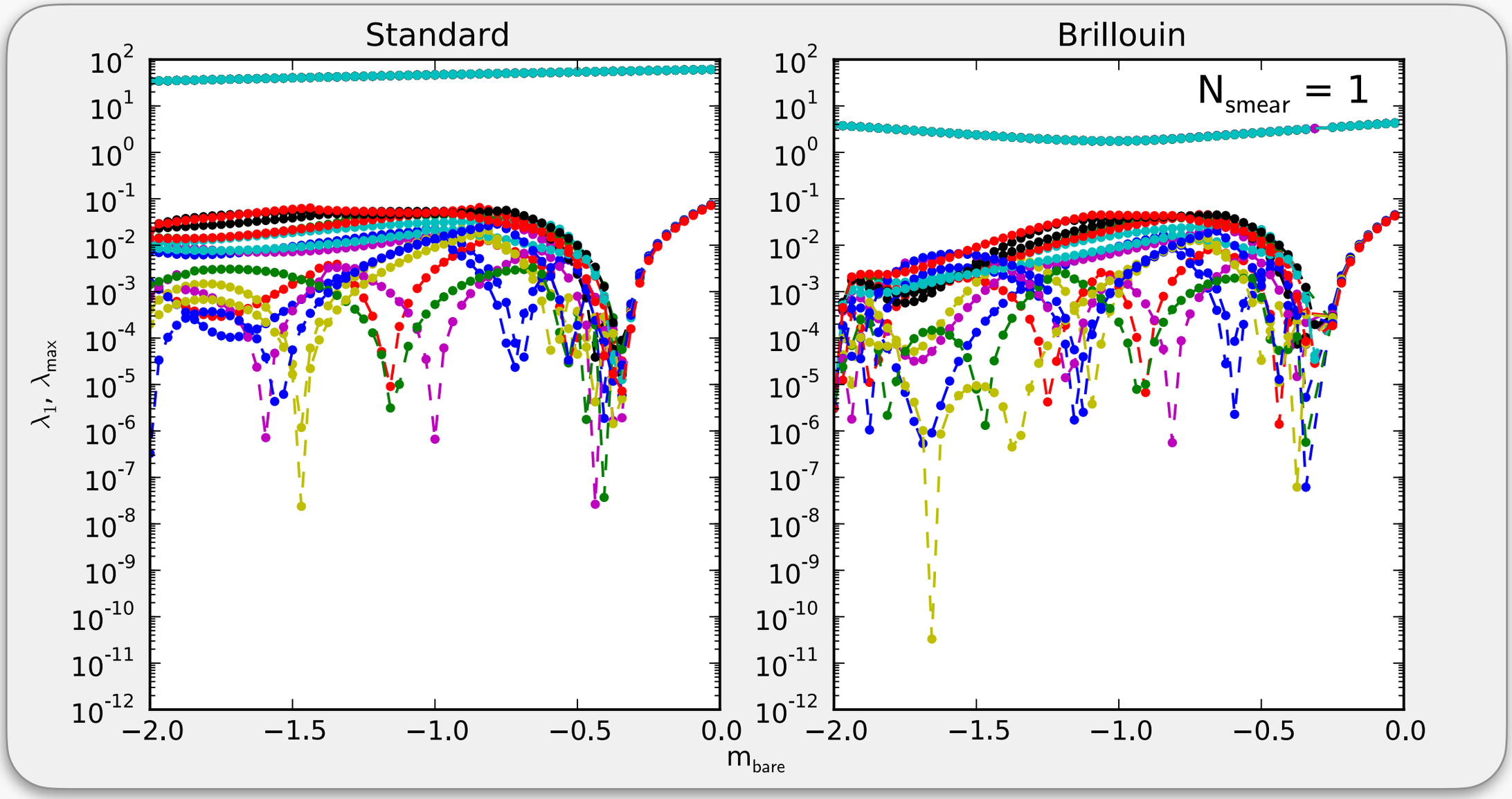
$$-am_{\text{crit}} = \frac{c_1 g_0^2 + c_2 g_0^4}{1 + c_3 g_0^2}$$

BiCGstab iteration count



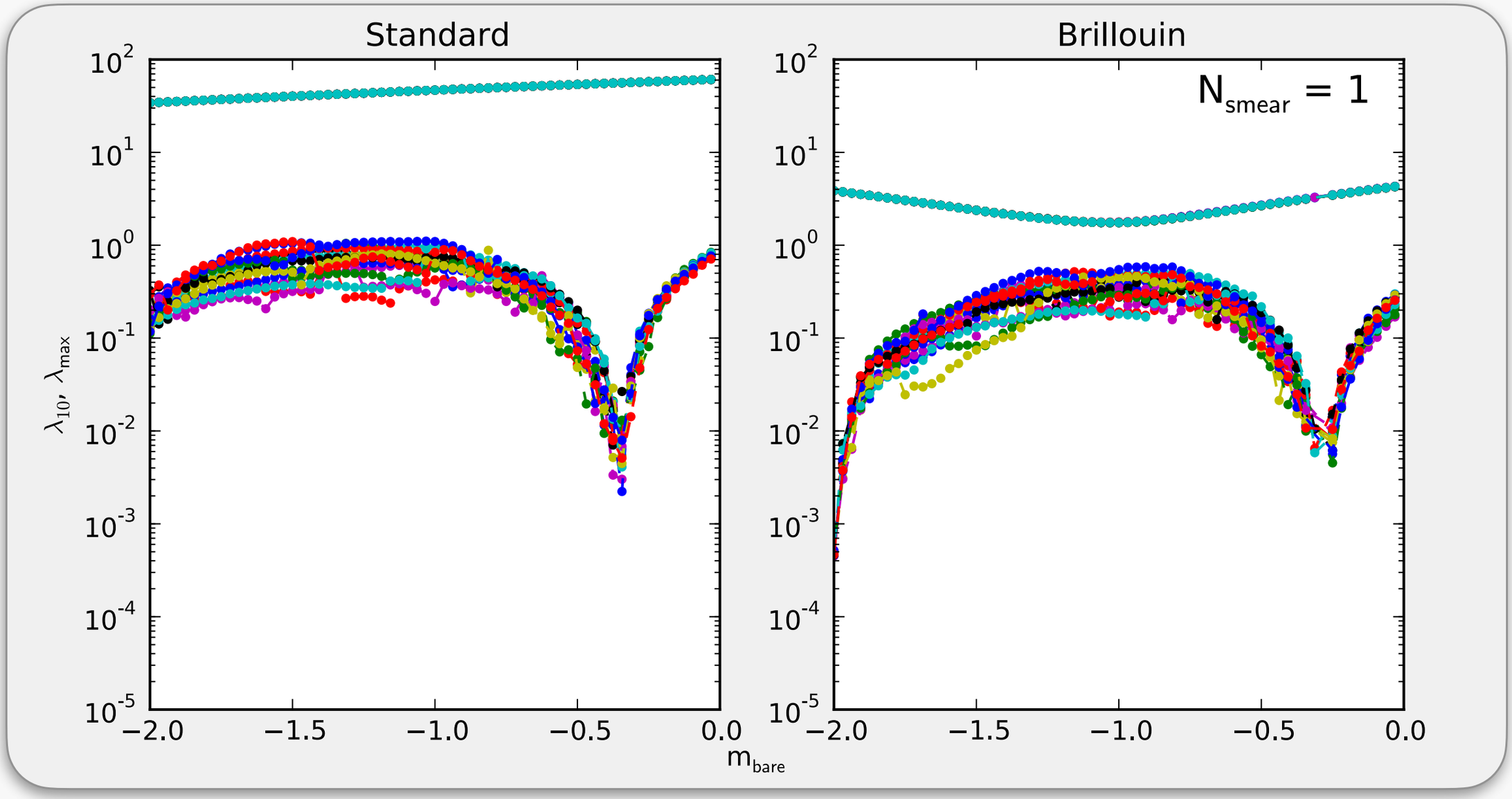
- 1.7× less iterations for Brillouin compared to Wilson

Eigenvalues



- Comparing smallest to largest eigenvalue

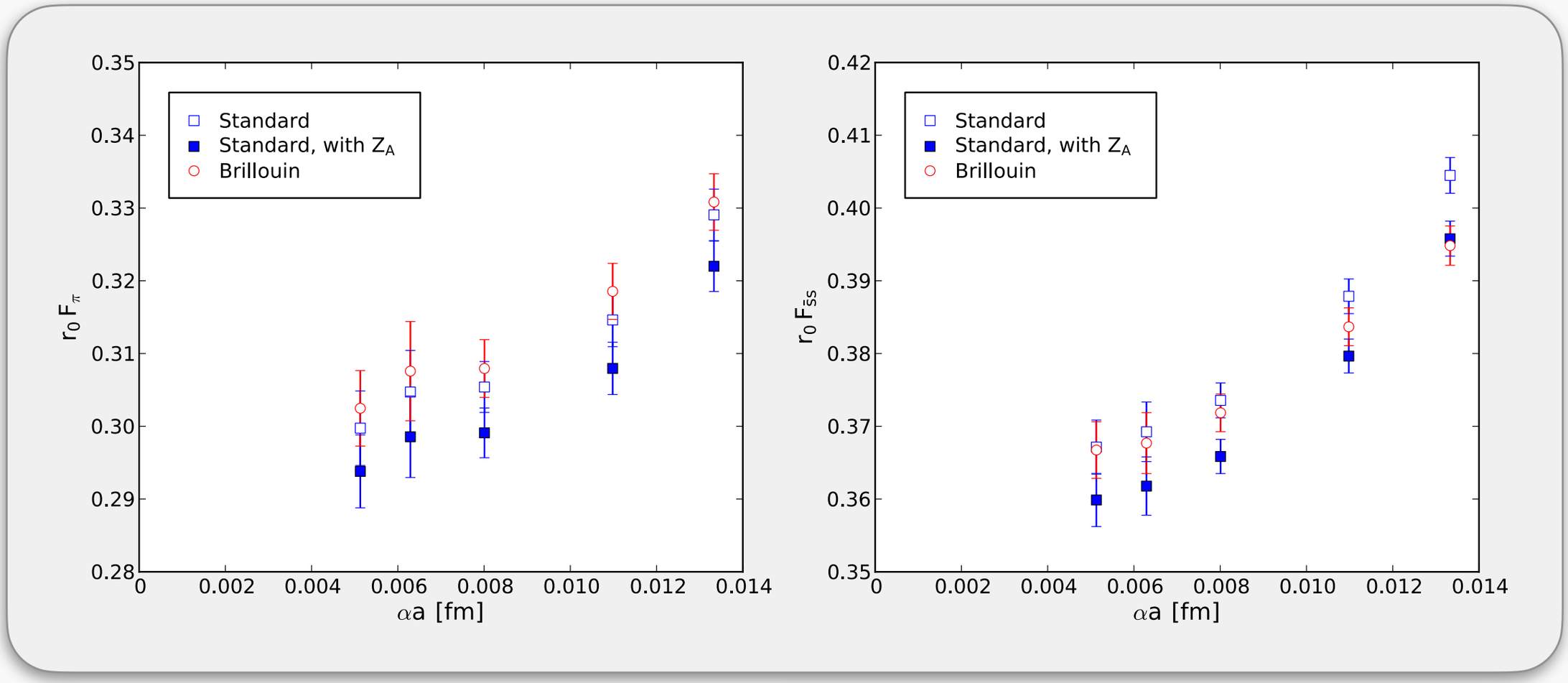
Eigenvalues



- Comparing 10th smallest to largest eigenvalue
- May be more amenable to deflation

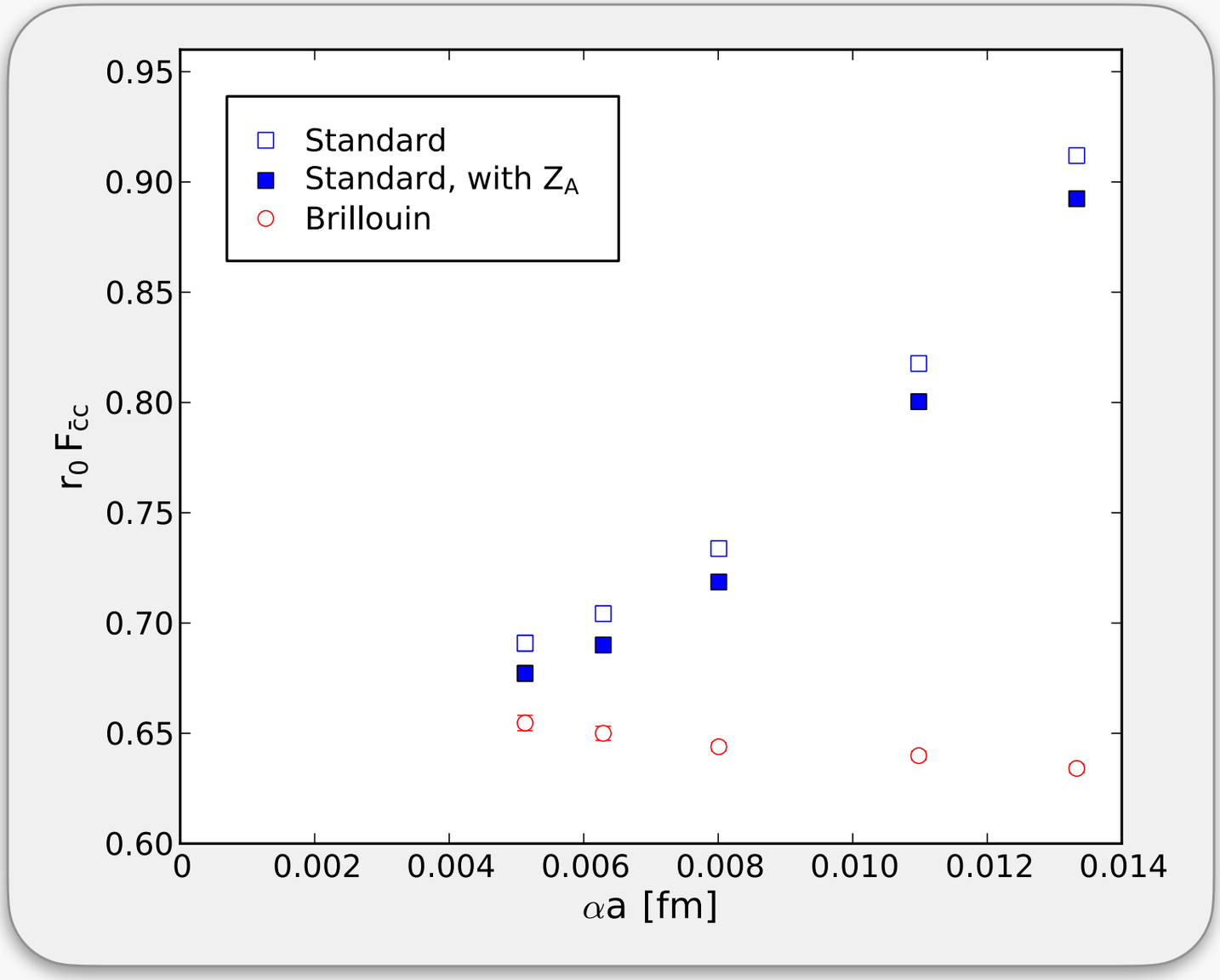
Meson decay constants

- Investigated scaling of pseudo scalar decay constants, f_π, $f_{s\bar{s}}$, $f_{c\bar{c}}$
- Five values of a at matched pseudo scalar masses



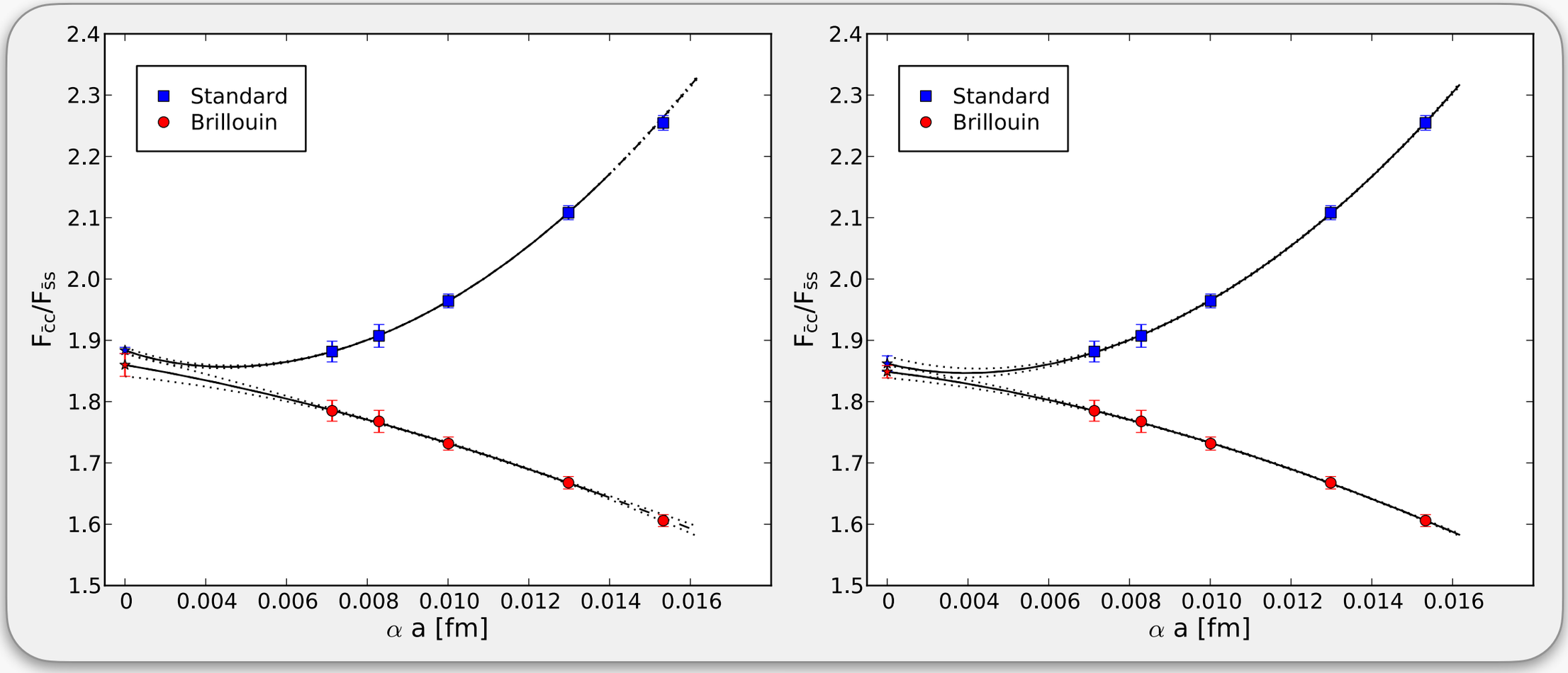
- Z_A available for Wilson only
- Similar approach to continuum (note vs αa)

Meson decay constants



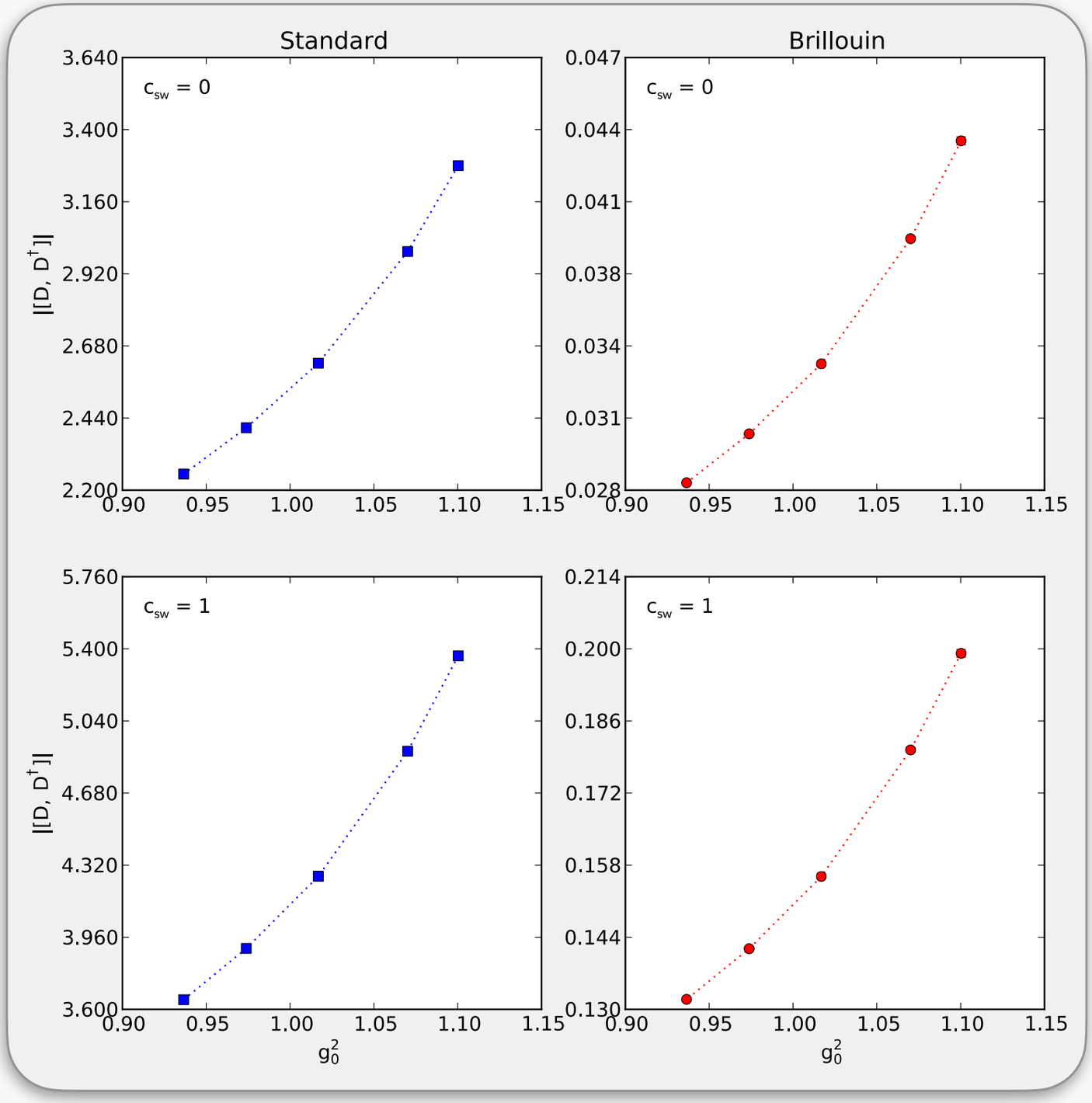
- Charmed pseudo scalar meson decay
- Milder approach to continuum when using Brillouin fermions compared to standard Wilson

Meson decay constants



- Ratio – eliminate Z_A
- Fit including αa term: $f_{c\bar{c}}/f_{s\bar{s}} = d_0 + d_1\alpha(a)a + d_2 a^2$
- Left: excluding coarsest a, right: all points included

Normality check



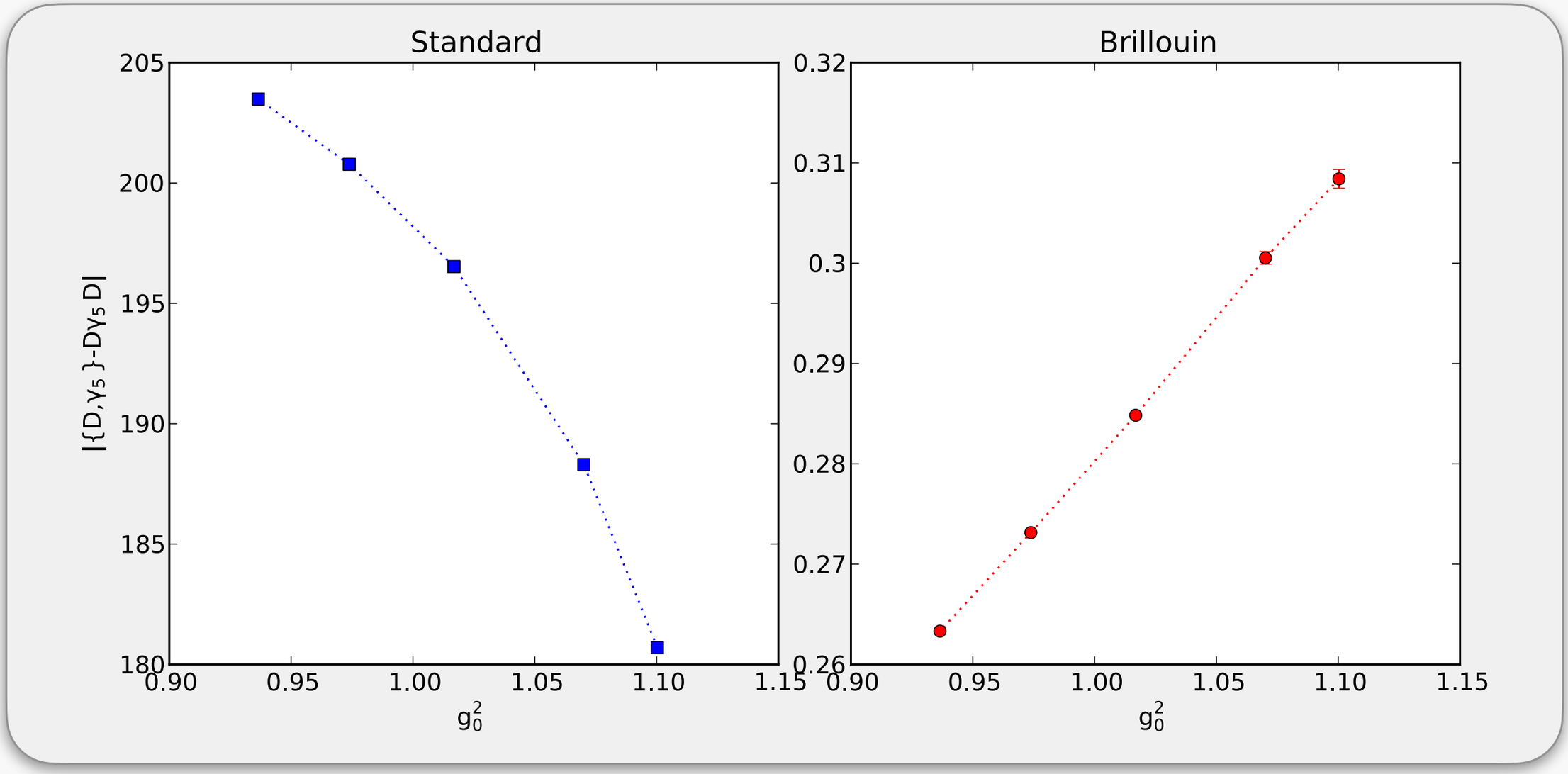
Normality "violation":

$$\|(DD^\dagger - D^\dagger D)\eta\|$$

where η are Gaussian random vectors

- Clover term increases violation for both cases
- Considerably smaller violation for Brillouin

Ginsparg – Wilson “violation”

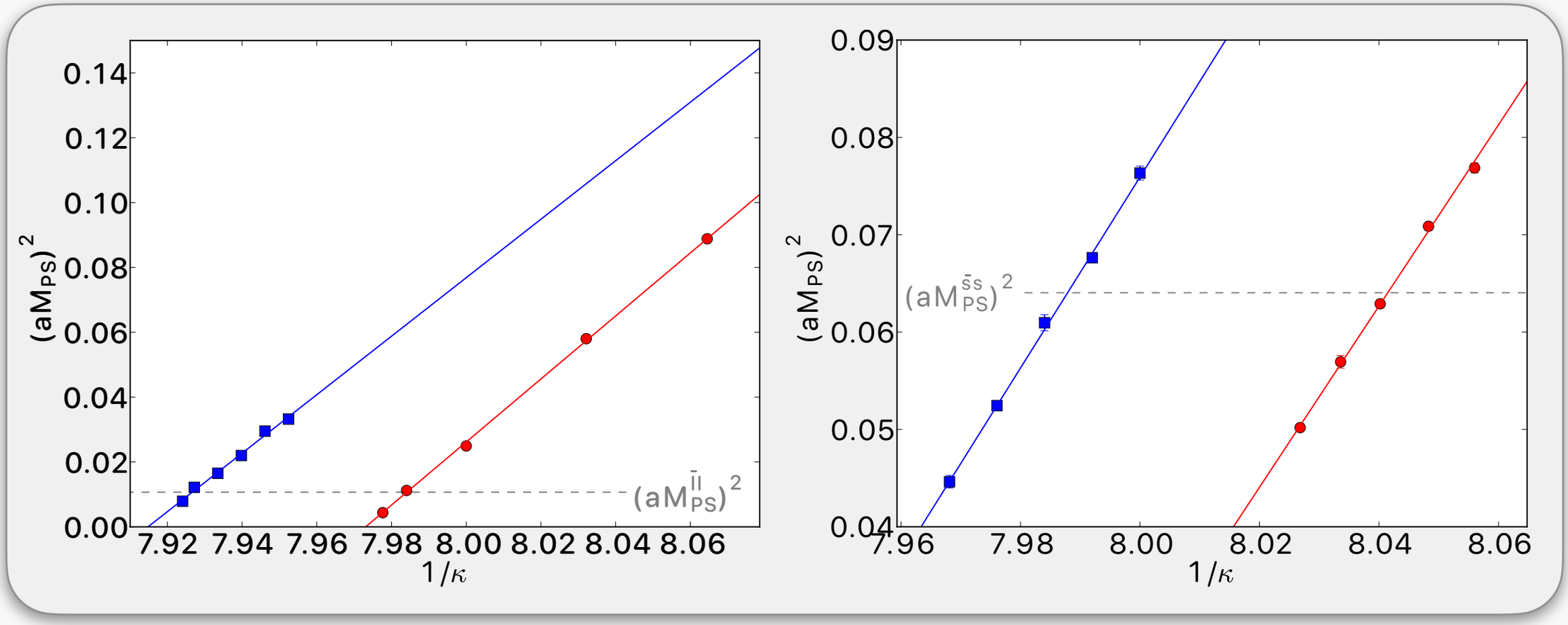


- Here for $\kappa = \kappa_{crit}$ (only known here for c_{SW}=1)
- Note the magnitude of the y-axes

Meson and baryon dispersion relations

S. Durr, G. K., & T. Lippert, arXiv:1208.6270

Using a QCDSF N_f=2, $(L/a)^3$=40^3 ensemble, m_π≈280 MeV, a≈0.073 fm

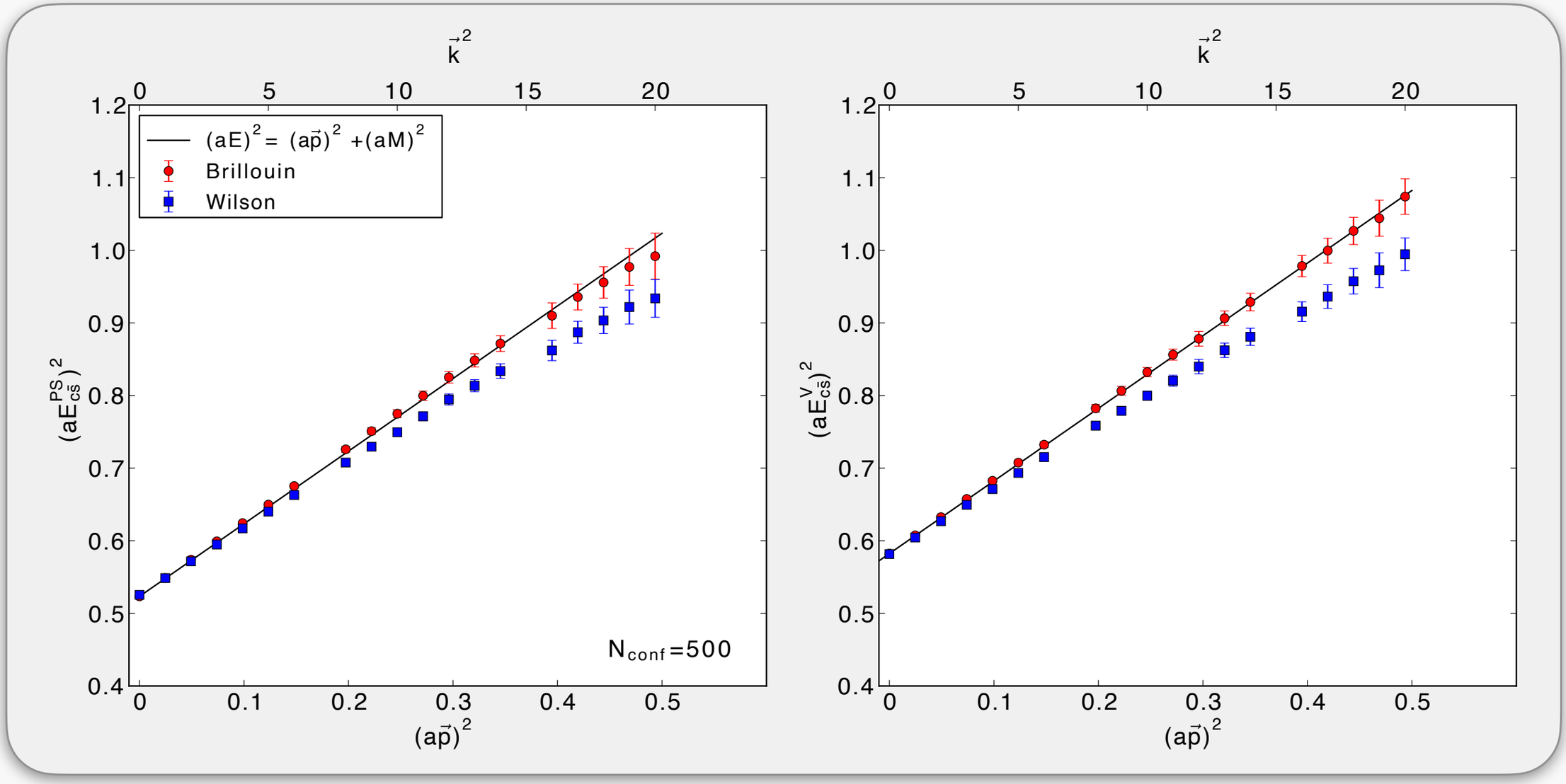


- Tuning of Wilson and Brillouin κ to same light and strange pseudo-scalar meson masses
- Target values (horizontal dashed lines) correspond to:

$$M_{PS}^{\bar{l}l} \simeq 280\,\mathrm{MeV} \qquad M_{PS}^{\bar{s}s} \simeq 680\,\mathrm{MeV}$$

Meson and baryon dispersion relations

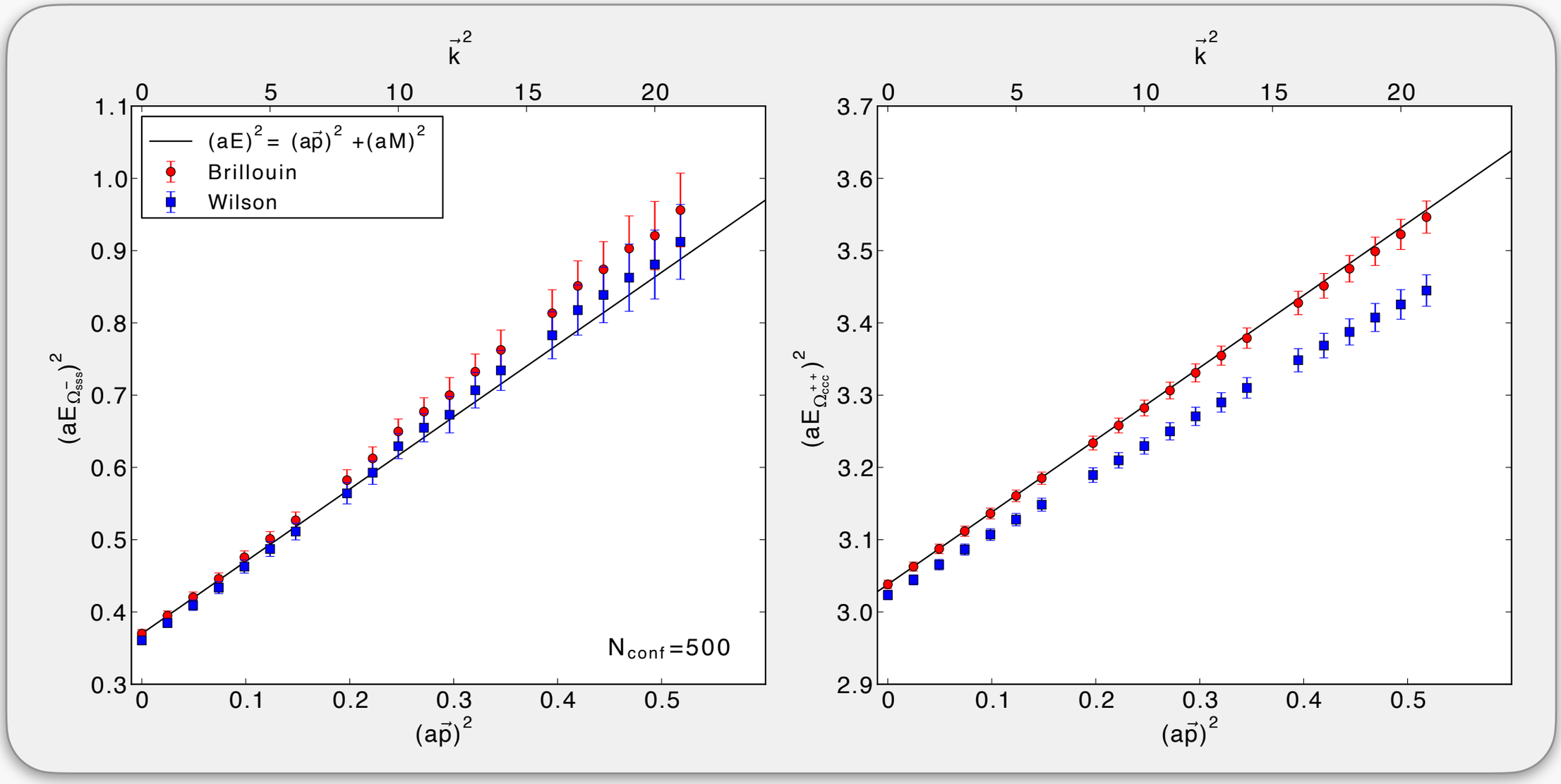
Sea: Wilson N_f=2, $(L/a)^3=40^3$, $m_\pi \approx 280$ MeV, $a \approx 0.073$ fm (by QCDSF)



- Left: pseudo scalar meson, Right: vector meson
- Solid line: continuum dispersion relation

Meson and baryon dispersion relations

Sea: Wilson N_f=2, $(L/a)^3=40^3$, $m_\pi \approx 280$ MeV, $a \approx 0.073$ fm (by QCDSF)



- Left: strange Omega, Right: charm Omega
- Solid line: continuum dispersion relation

Overlap with a Brillouin kernel

S. Durr & G. K., arXiv:1701.00726

- The massless Overlap operator:

$$aD_0^{ov} = \left[aD_{-\rho/a}(a^2 D^\dagger_{-\rho/a} D_{-\rho/a})^{-1/2} + 1\right] = \left[\gamma_5 \,\mathrm{sign}(\gamma_5 aD_{-\rho/a}) + 1\right]$$

where there is a choice of the kernel operator $D_{-\rho/a}$

- Approximate the sign function via Kenney-Laub iterates*:

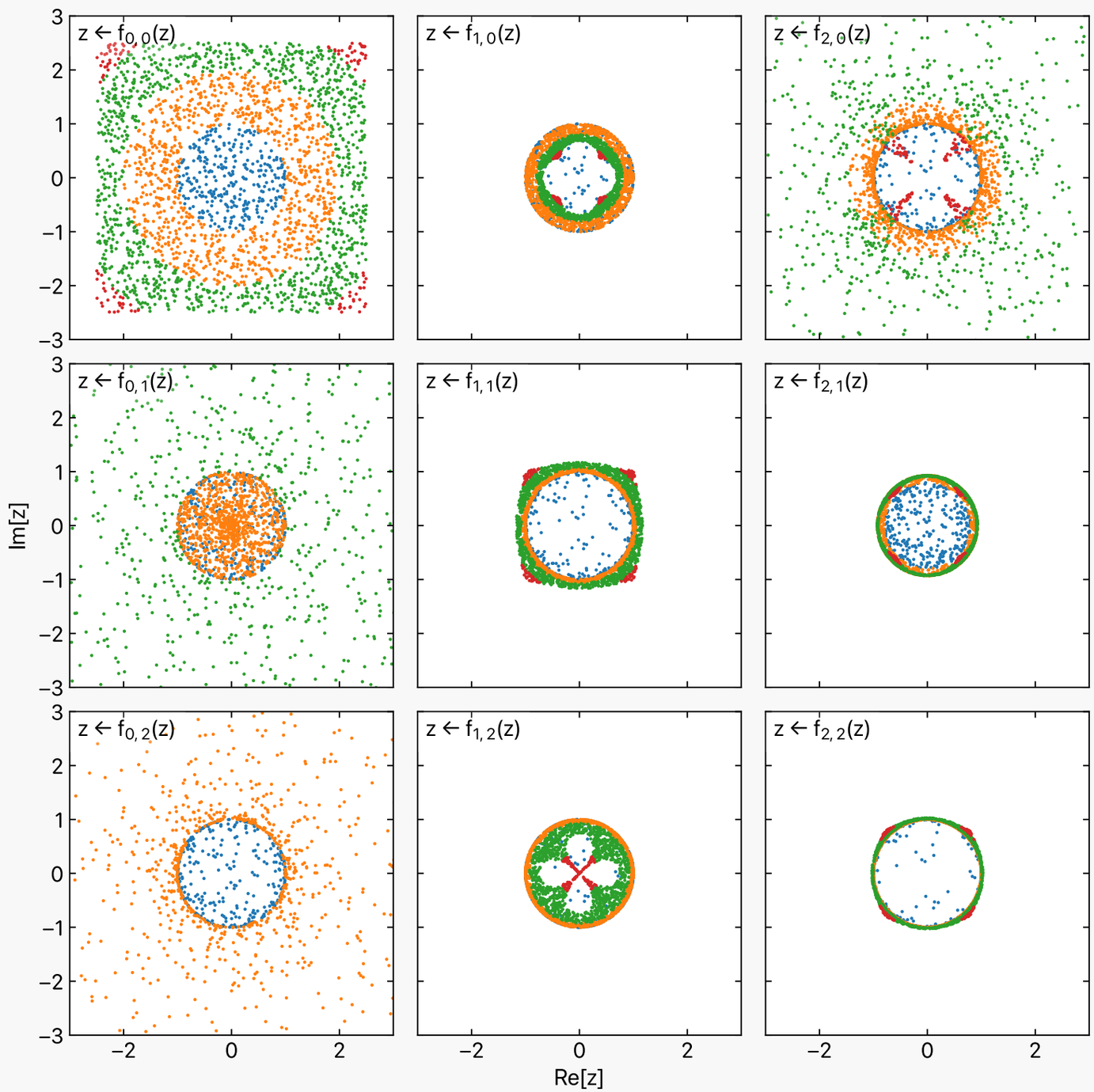
$$X_{k+1} = X_k \frac{p_{mn}(I - X_k^2)}{q_{mn}(I - X_k^2)} \equiv f_{mn}(X_k)$$

$r_{mn}(t) = \dfrac{p_{mn}(t)}{q_{mn}(t)}$ the (m, n) Padé approximant to: $h(t) = (1-t)^{1/2}$

	$n = 0$	$n = 1$	$n = 2$
$m = 0$	—	$\frac{2x}{1+x^2}$	$\frac{8x}{3+6x^2-x^4}$
$m = 1$	$\frac{x(3-x^2)}{2}$	$\frac{x(3+x^2)}{1+3x^2}$	$\frac{4x(1+x^2)}{1+6x^2+x^4}$
$m = 2$	$\frac{x(15-10x^2+3x^4)}{8}$	$\frac{x(15+10x^2-x^4)}{4(1+5x^2)}$	$\frac{x(5+10x^2+x^4)}{1+10x^2+5x^4}$

*C.S. Kenney and A.J. Laub, SIAM J. Matrix Anal. Appl. 12, 273 (1991)

Kenney-Laub iterates

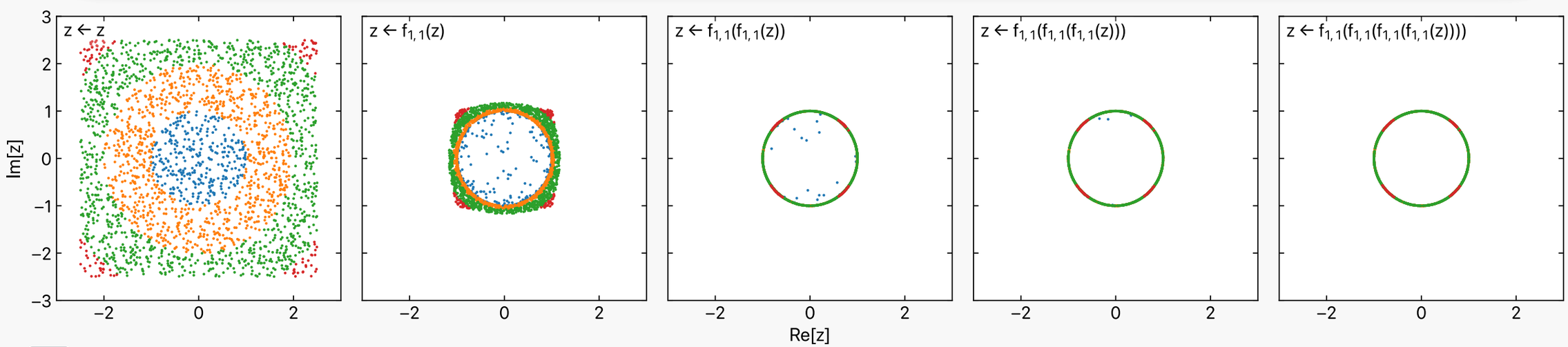


- Fast convergence to the unit circle
- For what follows: restrict to diagonal, $n = m$

Kenney-Laub iterates

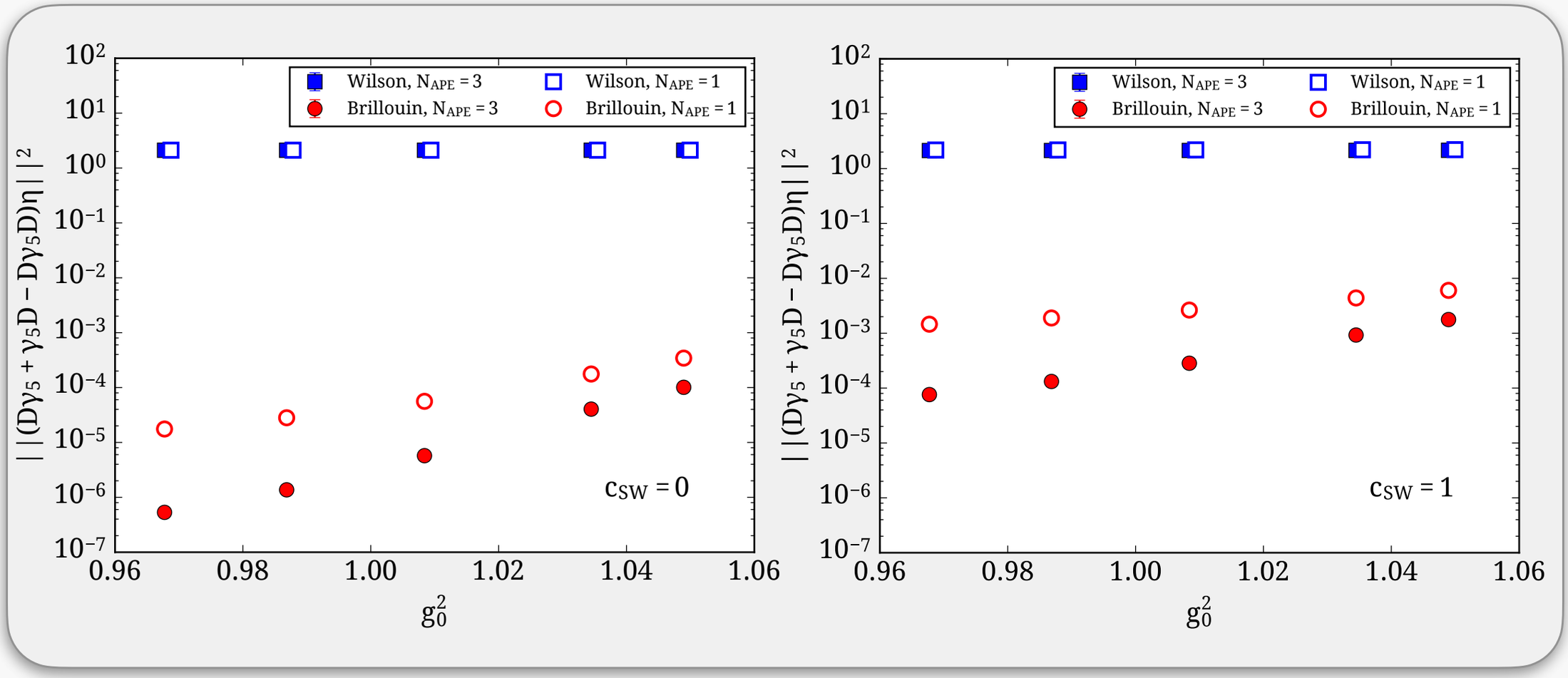
- Nesting, e.g. $f_{11}^{(3)}(x) \equiv f_{11}(f_{11}(f_{11}(x))) = f_{13,13}(x)$
- In a more practical setting, consider a "zero-mode" λ = 0.2
  - Via overlap prescription, shift by -1
  - Take the sign (here, apply KL)
  - Shift back by +1

	$f_{1,1}^{(k)}$	$f_{2,2}^{(k)}$	$f_{3,3}^{(k)}$	$f_{4,4}^{(k)}$	$f_{5,5}^{(k)}$
$k=1$	$2.7\,10^{-3}$	$3.4\,10^{-5}$	$4.2\,10^{-7}$	$5.2\,10^{-9}$	$6.3\,10^{-11}$
$k=2$	$5.2\,10^{-9}$	$2.8\,10^{-24}$	$3.5\,10^{-47}$	$1.0\,10^{-77}$	$6.9\,10^{-116}$
$k=3$	$3.4\,10^{-26}$	$1.0\,10^{-119}$	$1.0\,10^{-327}$	$4.6\,10^{-696}$	$1.6\,10^{-1270}$



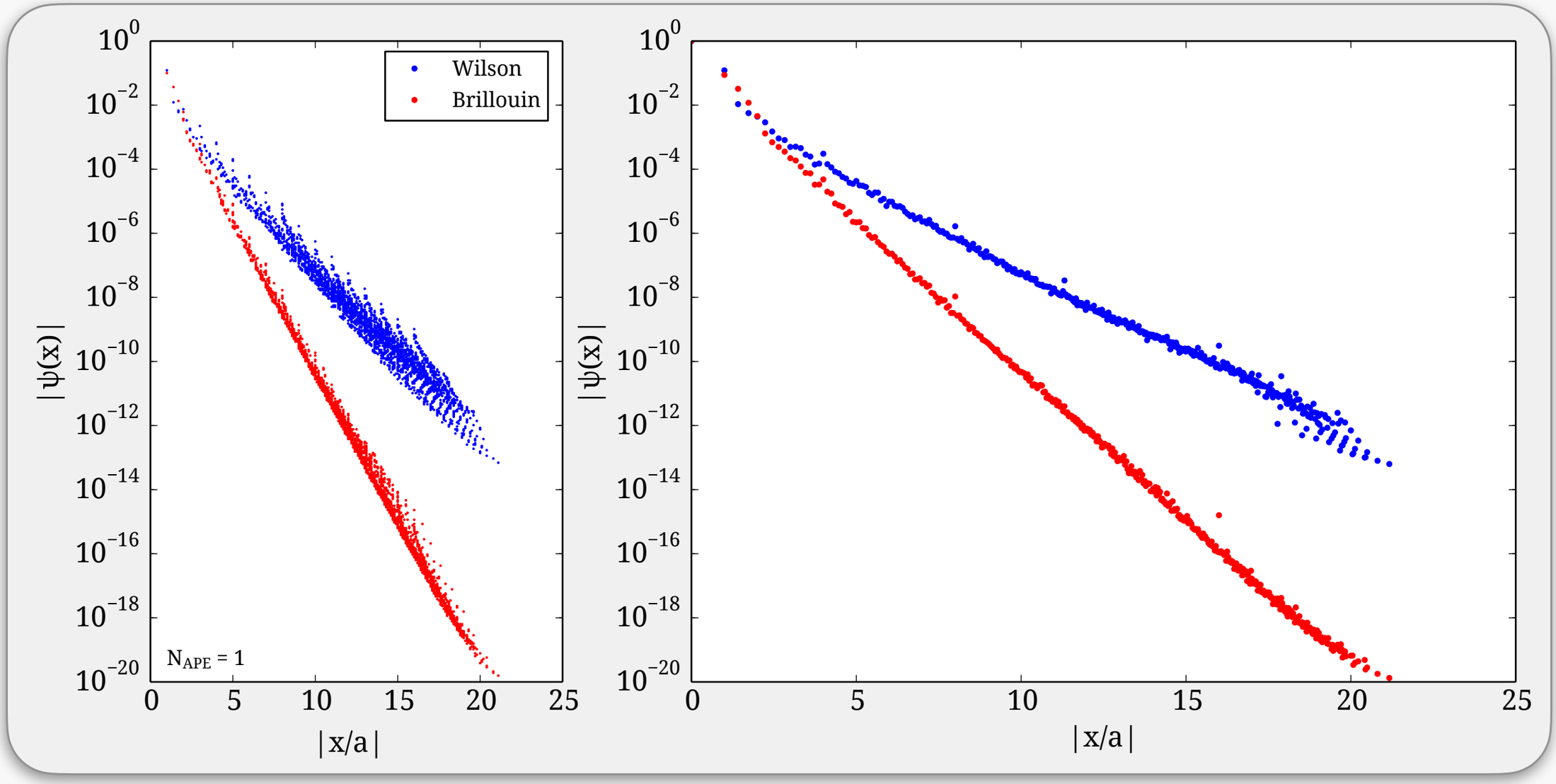
Overlap with a Brillouin kernel

- Same five quenched ensembles as prior analysis
- Ginsparg-Wilson "violation"
- Single iteration of $f_{1,1}$



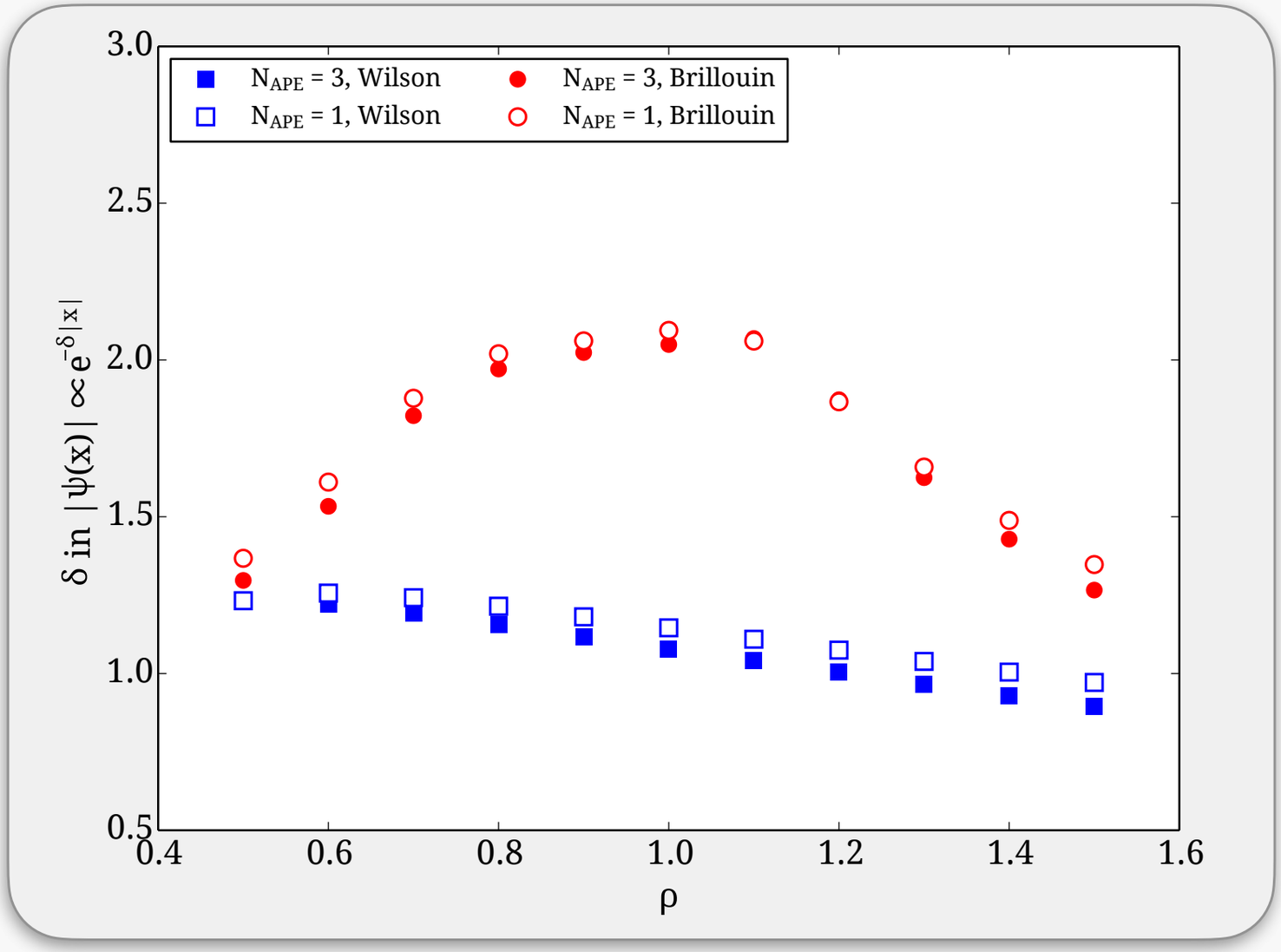
Overlap with a Brillouin kernel

- Fall-off of overlap operator $\psi = D^{ov}\eta$ with η Gaussian noise
- Single iteration of $f_{1,1}$
- Right: averaged over equal directions; $(L/a)^3 = 24^3$



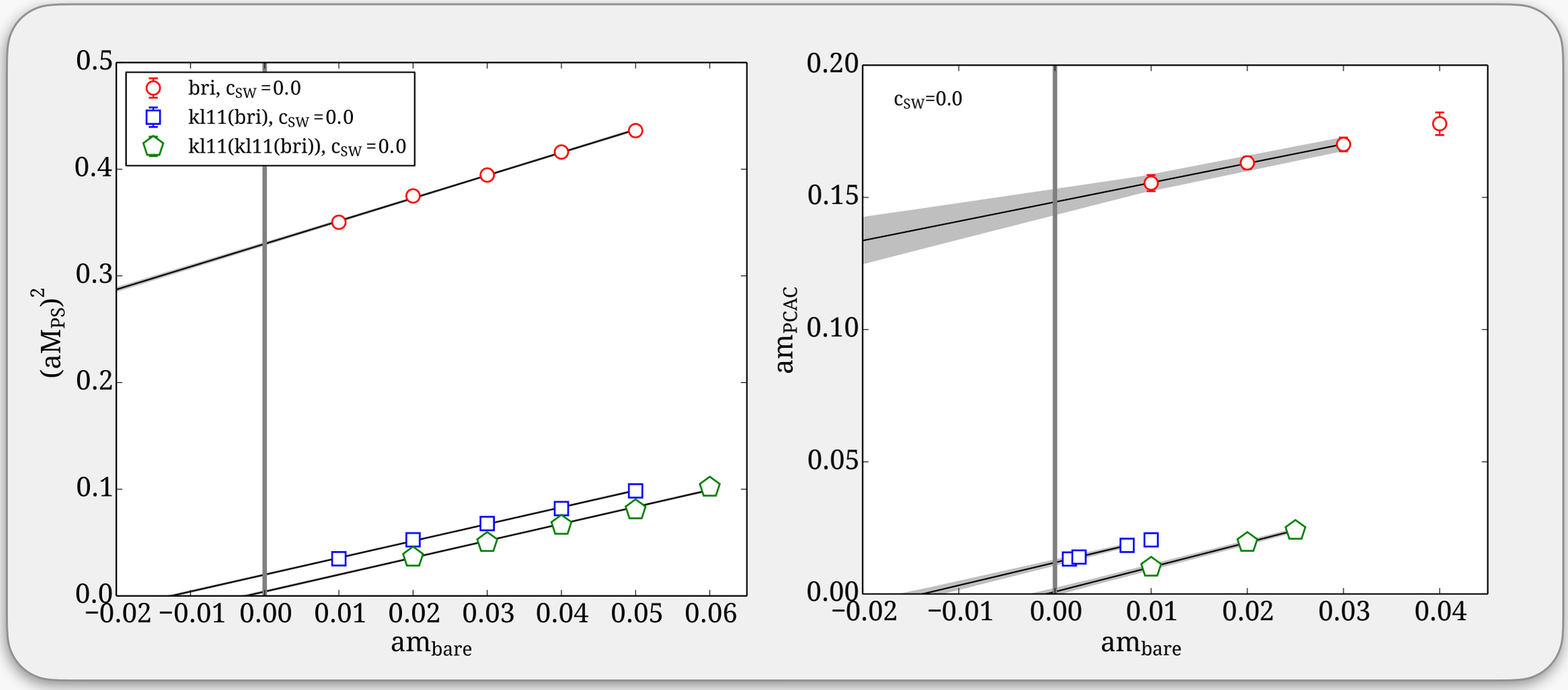
Overlap with a Brillouin kernel

- Fall-off of overlap operator $\psi = D^{ov}\eta$ with η Gaussian noise
- Single iteration of $f_{1,1}$
- Fitted to $e^{-\delta|x|}$



Overlap with a Brillouin kernel

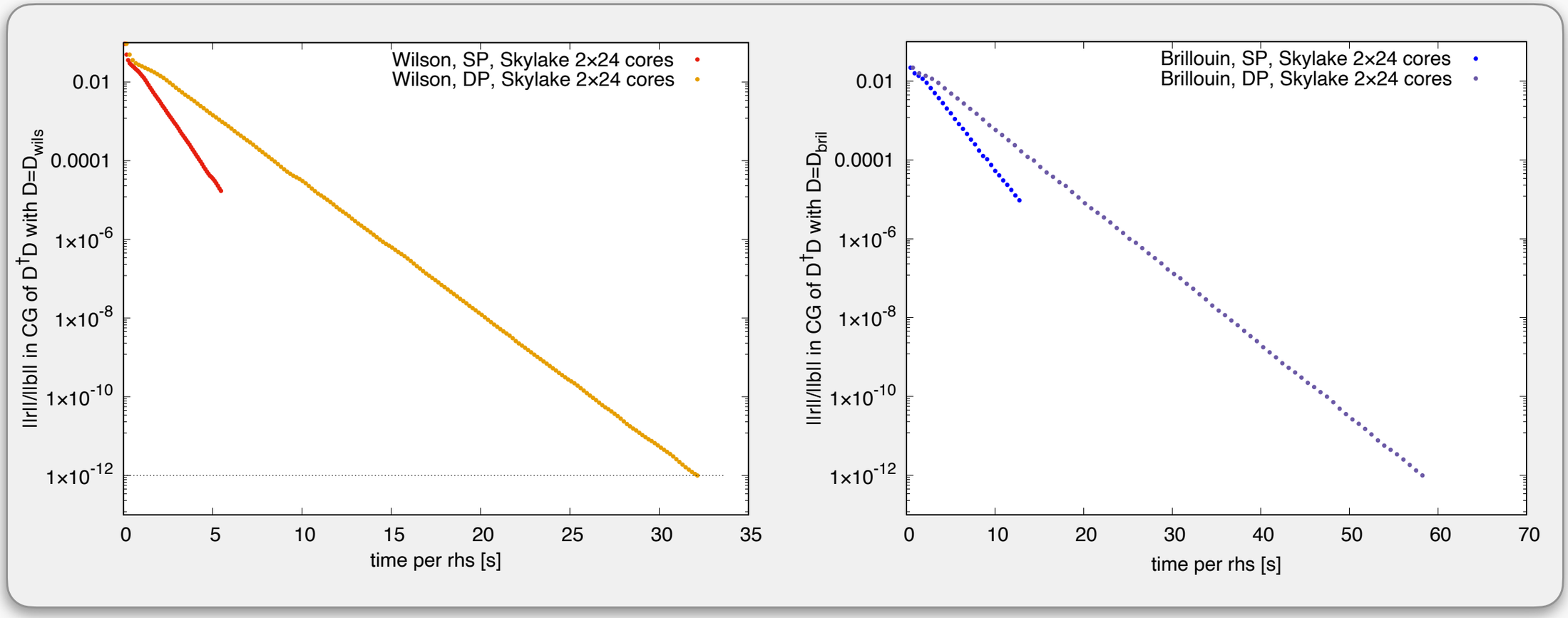
- Practical tests for massive overlap operator
- Used a QCDSF N_f=2 ensemble, m_π≈280 MeV, a≈0.073 fm



- Near-zero additive mass with just two recursions of f_{11}

Codes and performance

- Single-node, OpenMP:
  - → S. Durr, Comput. Phys. Commun. 282 (2023) 108555 arXiv:2112.14640
- Inversion of a $24^3 \times 48$ quenched lattice



- MPI/OpenMP code [*undocumented!*]
  - → https://github.com/g-koutsou/qpb

Summary & Outlook

Brillouin improvement

- Modified laplacian and derivative operators in Wilson action
- No parameters tuning needed
- Motivated by symmetry and value at Brillouin zone

Practical tests reveal:

- Reduced condition number
- Better dispersion relation – verified for hadrons which include heavy quarks
- Smaller additive mass renormalisation compared to standard Wilson

Suitability of Brillouin operator as a kernel to Overlap

- More circular-like spectrum motivates its investigation as an Overlap kernel
- Here, tests carried out using Kenney-Laub iterates as approximations to the sign function – fixed coefficients in ratio
- Nice features of Brillouin include: smaller violation of Ginsparg-Wilson relation, increased locality of operator, few KL-iterates for reaching near-zero additive mass

Acknowledgements

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