Exact lattice anomalies and a new path to lattice chiral gauge theories ?

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Talk

Motivation

1. Folklore: anomalies cannot be realized on lattice... Wrong ! – counterexample: **Kähler–Dirac** fermions.
2. Folklore: Hard (impossible ?) to put chiral gauge theories on lattice. **Kähler–Dirac** fermions may offer new path ...

Plan

- Kähler–Dirac and relation to Dirac. Discretization on curved space.
- New **gravitational** anomalies – survive discretization. Place constraints on IR behavior - in particular whether models can be trivially gapped.
- Embed chiral fermions in Kähler–Dirac fields. Build **mirror** models. Simplest anomaly free model $\rightarrow$ **Pati-Salam** GUT. Lattice realization ?

Kähler–Dirac equation

An alternative solution to the problem of square rooting the Laplacian:

Kähler-Dirac equation

$$(K - m)\Phi = 0 \quad K = d - d^\dagger$$
$$\text{where } K^2 = -dd^\dagger - d^\dagger d = \Box$$

Kähler-Dirac field $\Phi = (\phi, \phi_\mu, \phi_{\mu\nu}, \ldots)$.

Ex. 2d

$$\begin{aligned} \partial^\mu \phi_\mu - m\phi &= 0 \\ \partial_\mu \phi + \partial^\nu \phi_{\nu\mu} - m\phi_\mu &= 0 \\ \partial_\mu \phi_\nu - \partial_\nu \phi_\mu - m\phi_{\mu\nu} &= 0 \end{aligned}$$

Connection to Dirac

Form matrix

$$\Psi = \sum_{p=0}^{D} \sum_{n_i} \phi_{n_1 \ldots n_p}(x) \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$$

eg in 2d:

$$\Psi = \phi I + \phi_i \sigma_i + \phi_{12} \sigma_1 \sigma_2$$

In flat space can show

$$(\gamma^\mu \partial_\mu - m)\Psi = 0$$

Kähler–Dirac field describes $2^{D/2}$ degenerate Dirac fermions !

Kähler–Dirac in curved space...

Curved space

$$(d - d^\dagger - m)\Phi = 0 \quad \text{unchanged}$$

Kähler–Dirac fermions can be formulated on **any** smooth manifold.
No need for **spin structure**
No need for spin connection/vielbein formalism
quite different from Dirac

Locally Kähler–Dirac decompose into $2^{D/2}$ Dirac.
Global properties K differ from $\not\partial$ eg. K has zero modes on S^D
Expect corrections $\sim \frac{\text{wavelength}}{\text{radius of curvature}}$

A $U(1)$ symmetry for Kähler–Dirac fermions

Kähler–Dirac Action:

$$\int \overline{\Phi} K \Phi \equiv \int d^D x \sqrt{g} \sum_{p=0}^{D} \overline{\Phi}_p [(K - m)\Phi]_p$$

Operator Γ : $\phi_{\mu_1 \ldots \mu_p} \to (-1)^p \, \phi_{\mu_1 \ldots \mu_p}$

Key property $\{\Gamma, K\}_+ = 0$

Generates exact $U(1)$ symmetry of massless action

$$\Phi \to e^{i\alpha\Gamma}\Phi$$
$$\overline{\Phi} \to \overline{\Phi} e^{i\alpha\Gamma}$$

Matrix rep $\Psi \xrightarrow{\Gamma} \gamma_5 \Psi \gamma_5$ **twisted chiral symmetry**

Reduced Kähler–Dirac (RKD) fermions

Define: $\Phi_\pm = \frac{1}{2}(1 \pm \Gamma)\,\Phi$
if $m = 0$:

$$S_{\rm KD} = \int \overline{\Phi}_+ K \Phi_- + \overline{\Phi}_- K \Phi_+ \to S_{\rm RKD} = \int \overline{\Phi}_+ K \Phi_-$$

Analogous to decomposition of massless Dirac field into 2 Weyl fields

Introducing $\Psi = \begin{pmatrix} \overline{\Phi}_+^T \\ \Phi_- \end{pmatrix}$

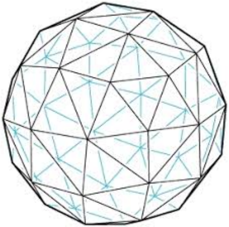
$$S_{\rm RKD} = \int \Psi^T \mathcal{K} \Psi \quad \mathcal{K} = \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix}$$

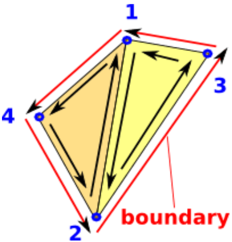
Reduced fields naturally massless

$$\overline{\Phi}_- \Phi_+ = \Phi_- \Phi_- = \overline{\Phi}_+ \overline{\Phi}_- = 0$$

Flat space continuum limit: $2^{D/2-1}$ Dirac or $2^{D/2}$ Majorana

Discrete curved space → triangulation





p-simplex $C^p = [a_0, \ldots, a_p]$

Boundary operator δ: $\delta[a_0 \ldots a_p] = \sum_{i=0}^{p} (-1)^i \, [a_0 \ldots \hat{a}_i \ldots a_p]$

where $\hat{a}_i$ indicates that vertex is omitted.

eg

$$\begin{aligned}\delta([142] + [123]) &= [42] - [12] + [14] + [23] - [13] + [12] \\ &= [42] + [23] + [31] + [14]\end{aligned}$$

Note:

$$\delta^2([142] + [123]) = [2] - [4] + [4] - [1] + [3] - [2] - [3] + [1] = 0!$$

Lattice p-forms

Continuum p-forms $\rightarrow$ each p-simplex $C_p \equiv [a_0, \ldots a_p]$ carries a lattice field $\phi(C_P)$:

$$\delta\phi(C_p) = \sum_{C_{p-1}} I(C_p, C_{p-1})\phi(C_{p-1})$$

where $I(C_p, C_{p-1})$ is zero unless C_{p-1} lies in boundary of C_p when it is ± 1 according to orientation

Similarly co-boundary operator $\overline{\delta}$:

$$\overline{\delta}\phi(C_p) = \sum_{C_{p+1}} I(C_{p+1}, C_p)^T \phi(C_{p+1})$$

Note $\delta^2 = \overline{\delta}^2 = 0$

Lattice Kähler–Dirac equation

$$\phi_p(x) \to \phi(C_p)$$
$$\delta \to d^\dagger$$
$$\overline{\delta} \to d$$

$$\boxed{(\delta - \overline{\delta} - m)\Phi = 0 \quad \text{with } \Phi = (\phi(C_0), \phi(C_1), \ldots \Phi(C_D))}$$

- Discrete Laplacian $\delta\overline{\delta} + \overline{\delta}\delta$.
- Exact zero modes of $\delta - \overline{\delta}$ match those of $d - d^\dagger$. Given by ranks of homology groups.
- No fermion doubling ! Continuum limit describes $2^{D/2}$ Dirac fermions **just like** continuum theory.
- $U_\Gamma(1)$ remains exact symmetry of lattice theory
- Can include arbitrary random triangulations with any topology and even non-orientable triangulations

Special case - staggered fermions

Decompose on p-cells of regular hypercubic lattice
Introduce second lattice with 1/2 lattice spacing

$$\chi(x + \hat{\mu}_1 + \hat{\mu}_2 + \ldots + \hat{\mu}_p) = \phi_{[\mu_1 \ldots \mu_p]}(x)$$

Form discrete Kähler–Dirac matrix field using

$$\Psi(x) = \sum \chi(\mathbf{x} + \hat{\mu}_1 + \ldots + \hat{\mu}_p)\gamma^{\mu_1} \cdots \gamma^{\mu_D}$$
$$= \sum_{\mathrm{b_i=0,1\ in\ hyp\ cube}} \chi(x+b)\gamma^{x+b} \quad \gamma^x = \gamma_1^{x_1}\gamma_2^{x_2} \ldots \gamma_D^{x_D}$$

Plug into $\sum \mathrm{Tr}(\overline{\Psi}\not{\Delta}\Psi)$ and do trace $\rightarrow$

$$S = \sum_{x,\mu} \eta_\mu(x)\overline{\chi}(x)\Delta_\mu\chi(x) \quad \text{with} \quad \eta_\mu(x) = (-1)^{\sum_i^{\mu-1} x_i}$$

Discrete Kähler–Dirac on regular lattice = staggered action !
$\Gamma \rightarrow \epsilon(x_1 + \ldots x_D) -$ site parity

Gravitational anomaly for Kähler–Dirac fermions

Work on lattice in d dims

Under $(\Phi, \overline{\Phi}) \to e^{i\alpha\Gamma}(\Phi, \overline{\Phi})$

$$\delta S_{\rm KD}(\overline{\Phi}, \Phi) = 0$$

But measure not invariant

$$D\Phi D\overline{\Phi} = \prod_p d\phi_p d\overline{\phi}_p \to e^{2iN_0\alpha} e^{-2iN_1\alpha} .. e^{2i(-1)^d N_d\alpha} \prod_p d\phi_p d\overline{\phi}_p$$
$$= \; e^{2i\chi\alpha} D\overline{\Phi} D\Phi \quad \chi \equiv \text{Euler}$$

Anomaly in even dimensions

Compactify $R^{2n} \to S^{2n}$. Breaks $U(1) \to Z_4$.

Note

Example of QM anomaly for finite number dof ...

$$\text{Index}(K) = n_+ - n_- = \chi = \int \text{Pf}\,(R \wedge \ldots \wedge R)$$

Consequences

Global $U(1)$ symmetry of Kähler–Dirac field broken to Z_4. Prohibits mass terms but allows for eg. four fermion ops. in $S_{\rm eff}$.

Theories of **reduced** Kähler–Dirac fermions with $U(1)$ symmetries cannot be consistently coupled to gravity – breakdown in gauge invariance
Analog: ABJ anomaly for Dirac implies cannot couple single Weyl fields to $U(1)$ gauge field

Can think of anomaly as **'t Hooft** anomaly for lattice fermions in flat space that arises when I try to couple them to gravity

’t Hooft anomalies

Represent an obstruction to gauging a global symmetry.
Can be seen by coupling to classical background field
Non-zero anomaly coeff in U.V $\stackrel{\text{RG invariant}}{\rightarrow}$ **physics of I.R non-trivial**:

- Massless (composite) fermions (CFT)
- Goldstone bosons from SSB
- TQFT

In particular:

Cannot gap all states in I.R (symmetric mass generation) **unless all ’t Hooft anomalies cancel**

Are there any (more) ’t Hooft anomalies for Kähler–Dirac ?

Try to gauge Z_4 ...

Typical term in action:

$$\overline{\phi}(C_p)I(C_p, C_{p-1})\phi(C_{p-1})$$

Under local Z_4:

$$\phi(C_p) \to e^{i\frac{\pi}{2}\Gamma n(C_p)}\phi(C_p) \quad n(C_p) = 0, 1, 2, 3$$

To keep invariant need to promote $I(C_p, C_{p-1})$ to Z_4 gauge field $U(C_p, C_{p-1})$ transforming as

$$e^{-i\frac{\pi}{2}\Gamma n(C_p)}U(C_p, C_{p-1})e^{-i\frac{\pi}{2}\Gamma n(C_{p-1})}$$

Measure ? $\int d\phi(C_p)d\overline{\phi}(C_p)$ **NOT** invariant $\to$ 't Hooft anomaly !

Cancels for multiples of 2 flavors

Consequences

't Hooft anomalies for Kähler–Dirac fields cancelled for $N_f = 2k$

2 Kähler–Dirac $\equiv$ 4 reduced fields
Yield $2^{D/2+1}$ Dirac or $2^{D/2+2}$ Majorana fermions in continuum limit

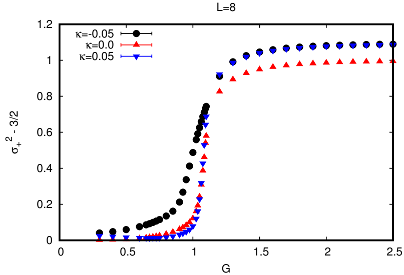
Agrees with results for gapping boundary fermions in topological superconductors and cancellation of discrete anomalies of Weyl/Majorana fermions in variety dims

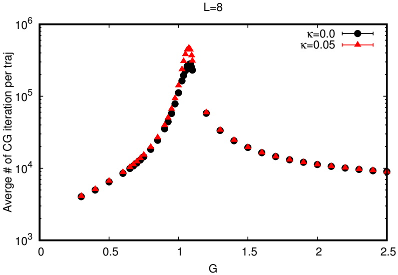
D=1	Time reversal $T^2 = 1$	8 Majorana	4 RKD
D=2	Chiral fermion parity	8 Majorana/Weyl	4 RKD
D=3	Time reversal $T^2 = -1$	16 Majorana	4 RKD
D=4	Spin-Z_4 symmetry	16 Majorana/Weyl	4 RKD

Explains observations of SMG for certain interacting staggered fermions in 4d

Massive symmetric phase for 2 staggered fermions

Higgs-Yukawa model: $S = \sum \chi(\eta.\Delta)\chi + \frac{1}{2}\sigma^2 - \kappa\sigma\Box\sigma + G\sigma\chi\chi$





Evidence for direct, continuous phase transition between massless and massive phases with no symmetry breaking (S.C et al. PRD98 (2018) 114514)

Summary so far

- Kähler–Dirac fermions admit gravitational anomalies which **survive discretization.** Break $U(1) \to Z_4$ in even dims $\to$ Cannot couple reduced Kähler–Dirac field to gravity.

Notice - Kähler–Dirac have no γ_5 anomalies.

- 't Hooft anomalies for Z_4 and mixed $Z_4 \times R$ cancel for multiples of 2 Kähler–Dirac /staggered fields. $\equiv$ 16 Majorana in 4d.
- Cancellation of all 't Hooft anomalies **necessary condition for symmetric mass generation (SMG)**
- Explains phase diagram of certain staggered fermion models.

What is SMG good for ?

Use SMG to gap mirrors in lattice models targeting chiral gauge theories ..?

Chiral lattice fermions

Lack a **non-perturbative** definition of a chiral gauge theory (Weyl fields in complex representation of gauge group)

Naive lattice approach fails because of fermion doubling: Nielsen-Ninomiya theorem always leads to equal numbers of left ψ_L and right ψ_R fields.

Mirror models Try to give mass to say ψ_R using multifermion interactions without touching ψ_L.

New idea:

Embed Weyl fermions in Kähler–Dirac fields.
Mirrors defined by Γ **not** γ_5. Use SMG to gap.
Arrange low energy theory flow to (anomaly free) chiral theory

Minimal model - continuum

Start: theory of **full** Kähler–Dirac fields with exact Z_4 symmetry. Decompose into **reduced** fields (Ψ_-, Ψ_+). Treat Ψ_+ as mirror. Need at least 4 copies for SMG

Consider “light" fields Ψ_- in (Euclidean) chiral basis $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix}$ where $\sigma_\mu = (I, \sigma_i)$. **Continuum matrix form in flat space**

$$\Psi_- = \begin{pmatrix} 0 & \psi_R \\ \psi_L & 0 \end{pmatrix}$$

L and R handed doublet of Weyl fields transforming as $(1, 2)$ and $(2, 1)$ under an $SU(2) \times SU(2)$ flavor symmetry.

4 copies – additional $SU(4)$ symmetry.

Replace $\psi_R = i\sigma_2\psi_L^*$.

Get reps $(\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ - Pati-Salam reps !

Pati-Salam - quick summary

leptons (e,ν) as fourth color
left-right symmetric weak interaction
Symmetry: $SU(4) \otimes SU_L(2) \otimes SU_R(2)$

One generation:

$$\begin{pmatrix} u_r & u_b & u_s & \nu \\ d_r & d_b & d_g & e \end{pmatrix}_L \quad \oplus \quad \begin{pmatrix} u_r^c & u_b^c & u_s^c & \nu^c \\ d_r^c & d_b^c & d_g^c & e^c \end{pmatrix}_L$$

Subsequently $SU(4) \to SU(3)$ and $SU_L(2) \otimes SU_R(2) \to SU_L(2)$

$(4,2,1) \to (3,2)_{\frac{1}{6}} \oplus (1,2)_{-\frac{1}{2}}$ q_L and l_L
$(\overline{4},1,2) \to (\overline{3},1)_{\frac{1}{3}} \oplus (\overline{3},1)_{-\frac{2}{3}} \oplus (1,1)_1 \oplus (1,1)_0$ d^c,u^c, e^c and ν^c
1 family of SM !

need eg GUT scale Higgs in $(4,1,2)$ rep. to do this

Gapping mirrors

Add Z_4 symmetric four fermion interactions in mirror sector. No effect on Pati-Salam fields

$$\frac{G^2}{2}\int d^4x\, \epsilon_{abcd}\left[\mathrm{tr}\,(\overline{\Psi}^a_-\overline{\Psi}^b_-)\mathrm{tr}\,(\overline{\Psi}^c_-\overline{\Psi}^d_-)+\mathrm{tr}\,(\Psi^a_+\Psi^b_+)\mathrm{tr}\,(\Psi^c_+\Psi^d_+)\right]$$

Better: gauge $SU(4)$ of mirror sector and use **confinement** to generate four fermion condensate + massive hadrons

Notice: mirror sector fields do not couple to Pati-Salam except gravitationally. Composite dark matter ?

Lattice chiral gauge theory

Replace continuum Kähler–Dirac field by staggered field χ.

$$S = \sum_{x,\mu} \eta_\mu(x) \left[\overline{\chi}_+ \Delta_\mu \chi_- + \overline{\chi}_- \Delta^c_\mu \chi_+\right] +$$

$$G \sum_x \hat{\phi}_{ab} \left[\overline{\chi}^a_- \overline{\chi}^b_- + \chi^a_+ \chi^b_+\right] + \frac{1}{2} \sum_x \hat{\phi}^2_{ab}$$

with

$$\hat{\phi}_{ab} = \frac{1}{2}\left(\phi_{ab} + \frac{1}{2}\epsilon_{abcd}\phi_{cd}\right)$$

and

$$\Delta^c_\mu \chi_+(x) = U_\mu(x)\chi_+(x+\mu) - U^\dagger(x-\mu)\chi_+(x-\mu)$$

Continuum limit

Sixteen free Weyl fermions in PS rep. Gapped mirror sector with $SU(4)$ invariant four fermion condensate + heavy $SU(4)$ hadrons

Conclusions

- Possible to build mirror models using (lattice) Kähler–Dirac fields and Γ. Anomaly cancellation conditions allow for SMG in mirror sector.
- Simplest model: remaining light fields $\rightarrow$ Pati-Salam. Mirror sector as composite dark matter ?
- For lattice chiral gauge theory: need to understand how to gauge $SU(2) \times SU(2)$ sector in lattice ?
- Sign problems for (gauged) reduced Kähler–Dirac fermions

Thanks !