Lattice fermions based on graph theory and a new conjecture about species doubling

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Introduction

Fermion-doubling and species

The lattice field theory has a serious problem.

It is called as "**Fermion-doubling**".

What is Fermion-doubling?

Multiple **species** appear when we take naive fermion formulations on a lattice.

Why is fermion-doubling a serious problem?

- The reconcilement of a desirable number of fermions and chiral symmetry is difficult.
- We cannot distinguish between these species because they are degenerate.

  Wilson fermion : species-splitting mass fermion

$$\sum_{n,\mu} \bar{\psi}_n \left(2\psi_n - \psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}\right)$$

Fermion-doubling and species

As known results, the number of species on D-dim square lattice with periodic boundary condition (PDC) is 2^D.

e.g. Species on the 4-dim lattice with PDC.

Naive and free lattice action is

$$S = \frac{1}{2}\sum_{n,\mu} \bar{\psi}_n \gamma_\mu \left(\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}\right)$$

$$\Longrightarrow \quad D(p) = \frac{1}{a}\sum_{\mu} i\gamma_\mu \sin ap_\mu$$

16 species appear such as

$$p = (0,0,0,0),\ (\pi/a,0,0,0),\ (0,\pi/a,0,0)\ (0,0,\pi/a,0),\ (0,0,0,\pi/a),$$
$$(\pi/a,\pi/a,0,0),\ (\pi/a,0,\pi/a,0)\ (\pi/a,0,0,\pi/a),$$
$$(0,\pi/a,\pi/a,0),\ (0,\pi/a,0,\pi/a),\ (0,0,\pi/a,\pi/a),$$
$$(\pi/a,\pi/a,\pi/a,0),\ (\pi/a,\pi/a,0,\pi/a)\ (\pi/a,0,\pi/a,\pi/a),\ (0,\pi/a,\pi/a,\pi/a),$$
$$(\pi/a,\pi/a,\pi/a,\pi/a)$$

Motivation

We have found numerous evidences that the maximal # of species depend on a certain topological invariant of the lattice.

The topological invariant is **sum of the Betti number** $\sum_r \beta_r$.

Here, we define the number of species as the number of exact Dirac zero-modes of free theory

Table: Topological invariant and maximal # of the species

lattice	$\sum_r \beta_r$	maximal # of species d
4-d torus	$1+4+6+4+1$	16
Torus T^D	$(1+1)^D$	2^D
Hyperball B^D	$1+0+0+\cdots$	1
Sphere S^D	$1+0+0+\cdots+1$	2
$T^D \times B^d$	2^D+0	2^D

Motivation and our work

- Why does the maximal # of species depend on the topological invariant of the lattice?

  $\longrightarrow$ This question is a motivation of our work.

- How do you mathematically explain or prove that?

  $\longrightarrow$ **Spectral graph theory** and **topology**.

Our work

- # of species $\Longrightarrow$ Nullity of spectral matrix.
  [J.Y, T.Misumi, JHEP **02**, 104 (2022)]
- New conjecture on species doubling of lattice fermions!
  [J.Y, T.Misumi, arXiv: 2301.09805]

Lattice fermions as spectral graph theory

- Spectral graph theory
- Lattice fermions as spectral graphs

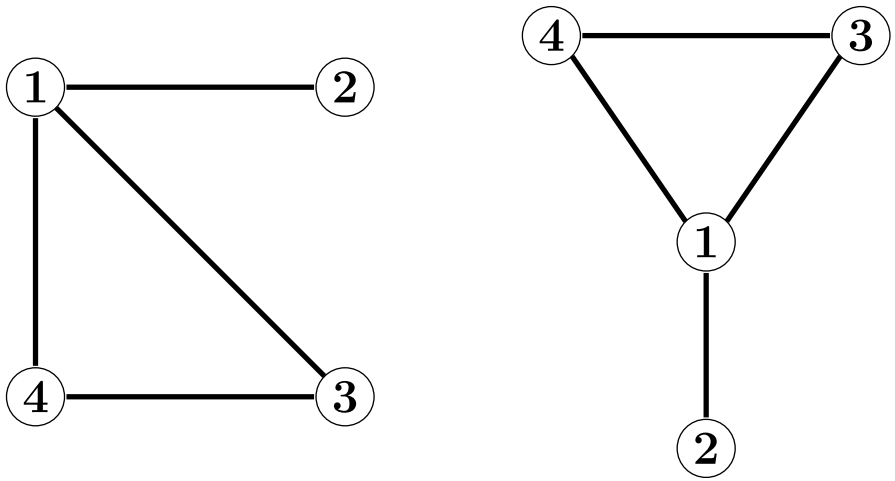
Spectral graph theory

We introduce basic concepts in spectral graph theory.

Definition (graph)

A graph G is a pair $G = (V, E)$. V is a set of vertices and E is a set of edges.

e.g. Two graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{e_{12}, e_{13}, e_{14}, e_{34}\}$.



Note that we can commutate two vertices in a edge, so $e_{12} = e_{21}$.

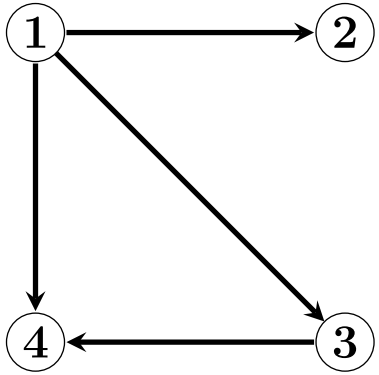
Spectral graph theory

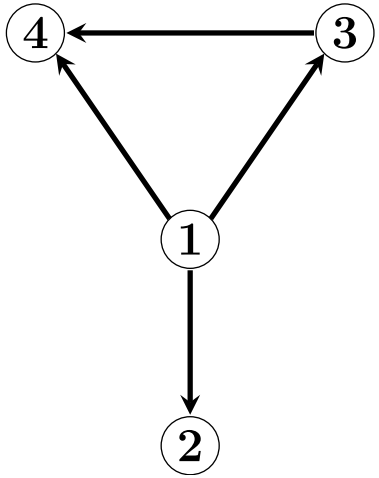
Definition (directed graph)

A directed graph is a pair (V, E) *of sets of vertices and edges together with two maps* $\mathrm{init} : E \to V$ *and* $\mathrm{ter} : E \to V$*. The two maps are assigned to every edge* e_{ij} *with an initial vertex* $\mathrm{init}(e_{ij}) = v_i \in V$ *and a terminal vertex* $\mathrm{ter}(e_{ij}) = v_j \in V$*. If* $\mathrm{init}(e_{ij}) = \mathrm{ter}(e_{ij})$*, the edge* e_{ij} *is called a loop.*

Spectral graph theory

e.g. Two directed graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{e_{12}, e_{13}, e_{14}, e_{34}\}$.





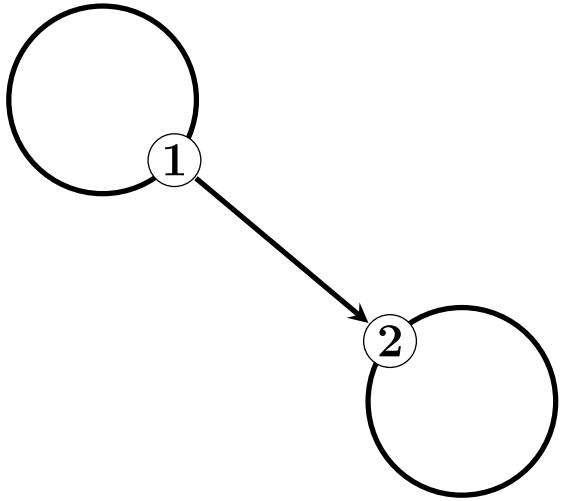
Unlike previous graphs, we cannot commutate two vertices in a directed edge, so $e_{12} \neq e_{21}$.

Spectral graph theory

Definition (directed graph)

A directed graph is a pair (V, E) of sets of vertices and edges together with two maps $\mathrm{init} : E \to V$ and $\mathrm{ter} : E \to V$. The two maps are assigned to every edge e_{ij} with an initial vertex $\mathrm{init}(e_{ij}) = v_i \in V$ and a terminal vertex $\mathrm{ter}(e_{ij}) = v_j \in V$. If $\mathrm{init}(e_{ij}) = \mathrm{ter}(e_{ij})$, the edge e_{ij} is called a loop.

e.g. A loop graph $G = (V, E)$ with $V = \{1, 2\}$ and $E = \{e_{11}, e_{12}, e_{22}\}$.

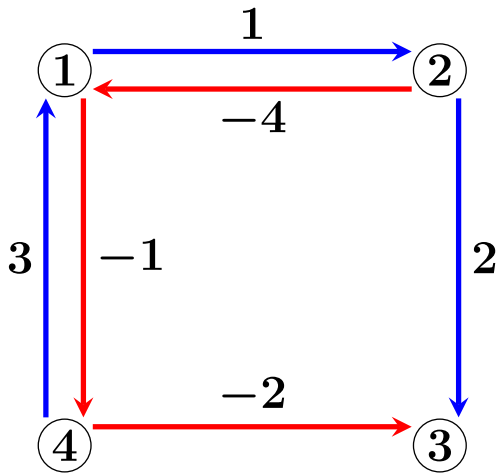


Spectral graph theory

Definition (weighted graph)

A weighted graph has a value (weight) for each edge in a graph.

e.g. A weighted graph with $V = \{1, 2, 3, 4\}$ and $E = \{e_{12}, e_{23}, e_{41}, e_{14}, e_{21}, e_{43}\}$.



These weights are as follows:

$$w_{12} = 1, \quad w_{23} = 2, \quad w_{41} = 3, \quad w_{14} = -1, \quad w_{21} = -4, \quad w_{43} = -2$$

Spectral graph theory

Definition (adjacency matrix)

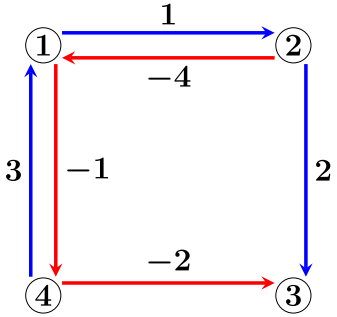
An adjacency matrix A of graph is the $|V| \times |V|$ matrix is given by

$$A_{ij} \equiv \begin{cases} w_{ij} & \text{if there is a edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

where w_{ij} is the weight of an edge from i to j.

Spectral graph theory

e.g. An adjacency matrix of a previous weighted and directed graph.



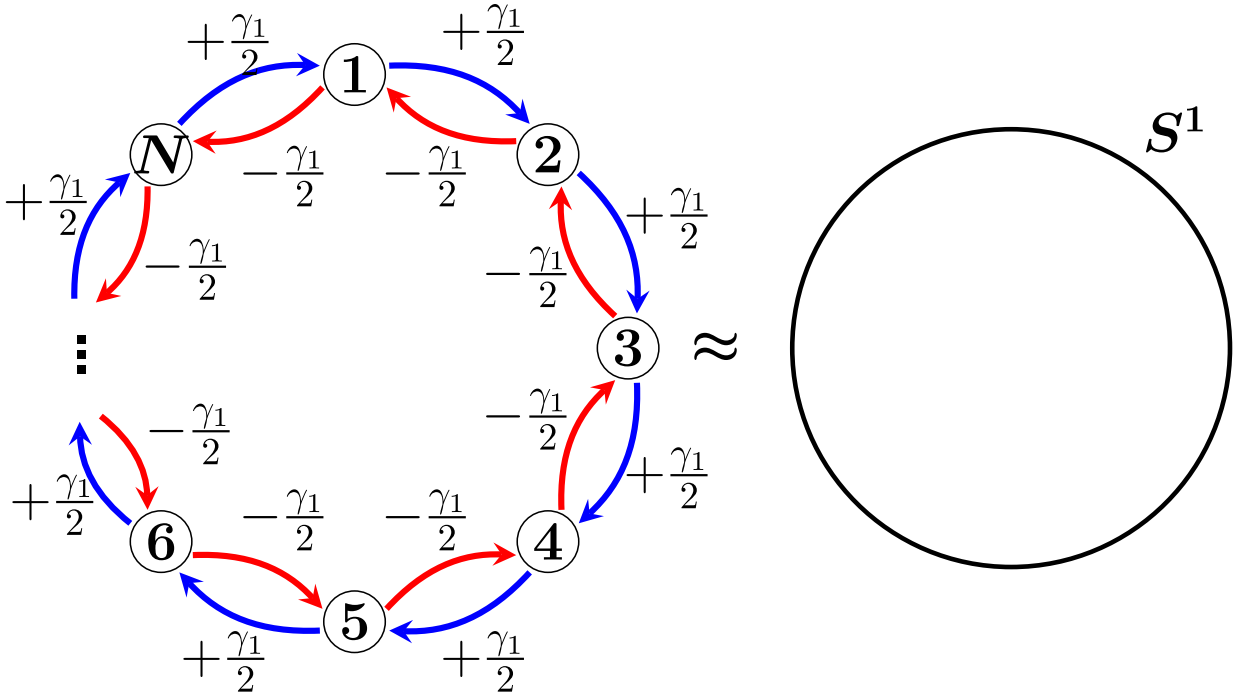
An adjacency matrix of this graph is

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \end{pmatrix}$$

Lattice fermions as spectral graphs

- Naive fermion on 1-dim N lattice with PDC (T^1 or S^1)

A weighted and directed graph like naive fermion on T^1 is depicted as



This graph schematically shows a circle S^1.

Lattice fermions as spectral graphs

- Naive fermion on 1-dim N lattice with PDC (T^1 or S^1)

An adjacency matrix $A^{1\mathrm{d}}$ of the previous graph is

$$A^{1\mathrm{d}} = P_N \otimes \gamma_1$$

where P_N is N square matrix below,

$$P_N = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}.$$

Lattice fermions as spectral graphs

- Naive fermion on 1-dim N lattice with PDC (T^1 or S^1)

We show that the bilinear form of the adjacency matrix $A^{1\mathrm{d}}$ for the field vector $\boldsymbol{\psi}$ is the Lagrangian about naive fermion with PDC.

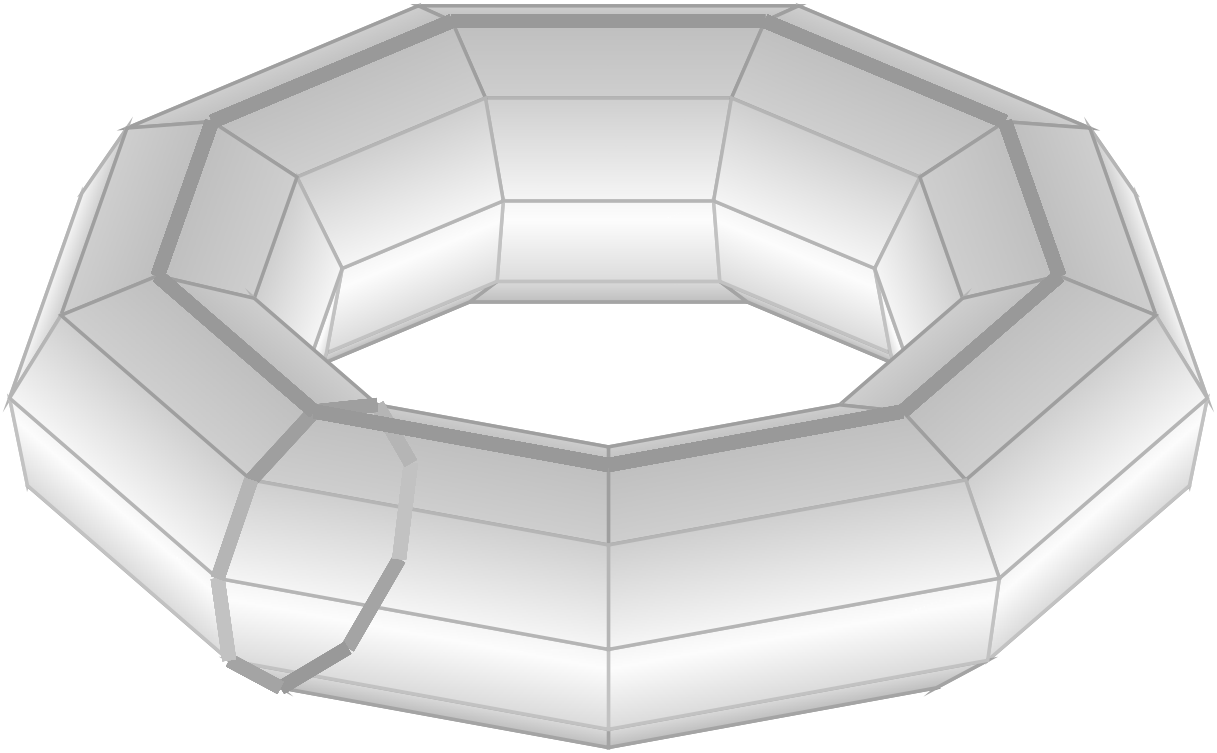
$$\bar{\boldsymbol{\psi}} A^{1\mathrm{d}} \boldsymbol{\psi} = \sum_{n=1}^{N} \bar{\psi}_n \gamma_1 \left(\psi_{n+1} - \psi_{n-1}\right) = \sum_{n=1}^{N} \bar{\psi}_n \gamma_1 D \psi_n$$

Adjacency matrix of a graph showing lattice $\Longrightarrow$ Dirac operator on lattice

Lattice fermions as spectral graphs

- Naive fermion on 2-dim N^2 lattice with PDC (T^2)

A graph corresponds the 2-dim lattice with PDC is a figure below.



This graph schematically shows T^2.

Lattice fermions as spectral graphs

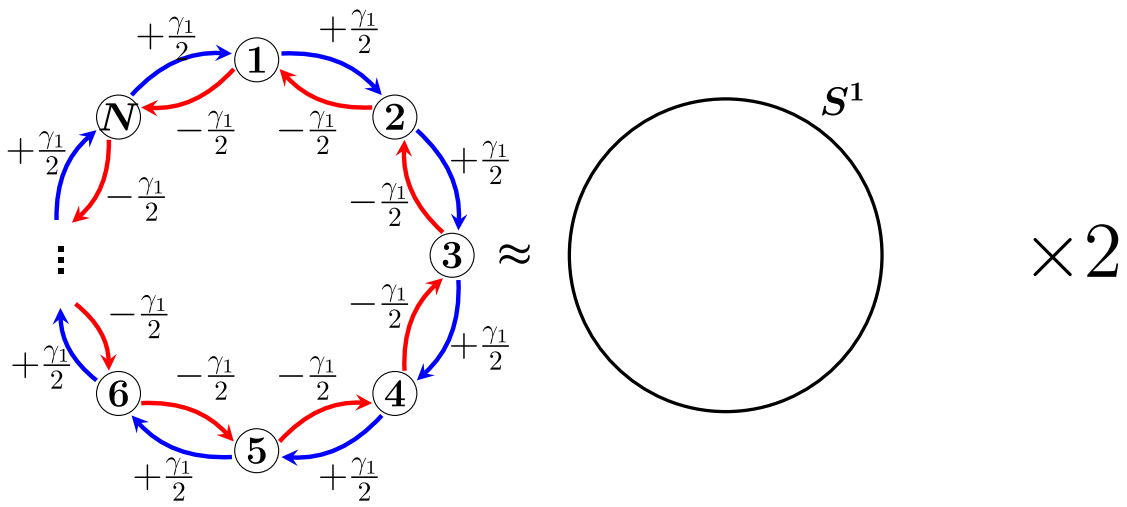
- Naive fermion on 2-dim N^2 lattice with PDC (T^2)

An adjacency matrix $A^{2\mathrm{d}}$ of the graph like T^2 is below

$$A^{2\mathrm{d}} = \mathbf{1}_N \otimes P_N \otimes \gamma_1 + P_N \otimes \mathbf{1}_N \otimes \gamma_2.$$

$\mathbf{1}_N$ is the identity matrix of order N.

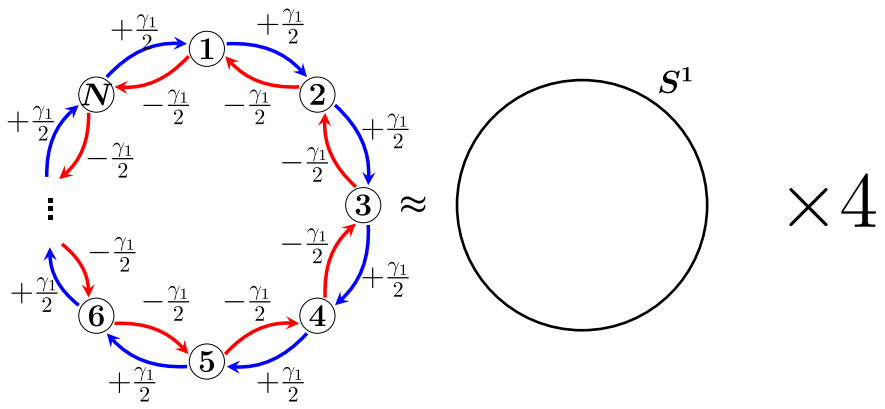
It is topologically consistent since $T^2 = S^1 \times S^1$



Lattice fermions as spectral graphs

- Naive fermion on 4-dim N^4 lattice with PDC (T^4)

Topologically, we construct a graph showing 4-dim naive fermion below,



We obtain an adjacency matrix of this graph

$$
\begin{aligned}
A^{4\mathrm{d}} \quad &= \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \gamma_1 \\
&+ \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \gamma_2 \\
&+ \mathbf{1}_N \otimes p_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\
&+ P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4
\end{aligned}
$$

Lattice fermions as spectral graphs

- Naive fermion on 4-dim N^4 lattice with PDC (T^4) :

Diagonalization of the matrix P_N

$$P_N X = \sum_k i \sin\left(\frac{2\pi(k-1)}{N}\right) |k\rangle \langle k| \equiv \Lambda_N X$$

From above diagonalization, we can diagonalize the adjacency matrix $A^{4\mathrm{d}}$

$$\begin{aligned}\mathcal{U}^\dagger A^{4\mathrm{d}} \mathcal{U} = &\mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_N \otimes \gamma_1 \\ &+ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ &+ \mathbf{1}_N \otimes \Lambda_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ &+ \Lambda_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4\end{aligned}, \qquad \mathcal{U} = \bigotimes_{\mu=1}^{4} X \otimes \mathbf{1}_4$$

Lattice fermions as spectral graphs

- Naive fermion on 4-dim N^4 lattice with PDC (T^4) :

Due to obtaining species, we take an equation below

$$\mathcal{U}^\dagger A^{4\mathrm{d}} \mathcal{U} = \mathbf{0} \quad \Longrightarrow \quad \sum_{\mu=1}^{4} i\gamma_\mu \sin\left(\frac{2\pi(k_\mu - 1)}{N}\right) = \mathbf{0}$$

Linear independence of γ matrices

$$\sin\left(\frac{2\pi(k_\mu - 1)}{N}\right) = 0 \quad \Longrightarrow \quad k_\mu = 1 \text{ or } \frac{N}{2} + 1$$

We get 2^4 solutions when we take N the even number.

So, there are **16 species!**

Note that 16 species is the maximal number of species, here.

Appearing 16 species is consistence with fermion-doubling.

Lattice fermions as spectral graphs

Lattice field theory and Spectral graph theory

Lattice field theory	$\Longrightarrow$	Spectral graph theory
Lattice fermion	$\Longrightarrow$	Directed and Weighted spectral graph
# of species	$\Longrightarrow$	Nullity of spectral matrix

We can use known theorems to study lattice field theory.

New conjecture on species doubling

New conjecture on species doubling

We have found numerous evidences that the maximal # of species is equivalent to sum of the Betti number $\sum_r \beta_r$.

Table: Betti numbers and Maximal numbers of species

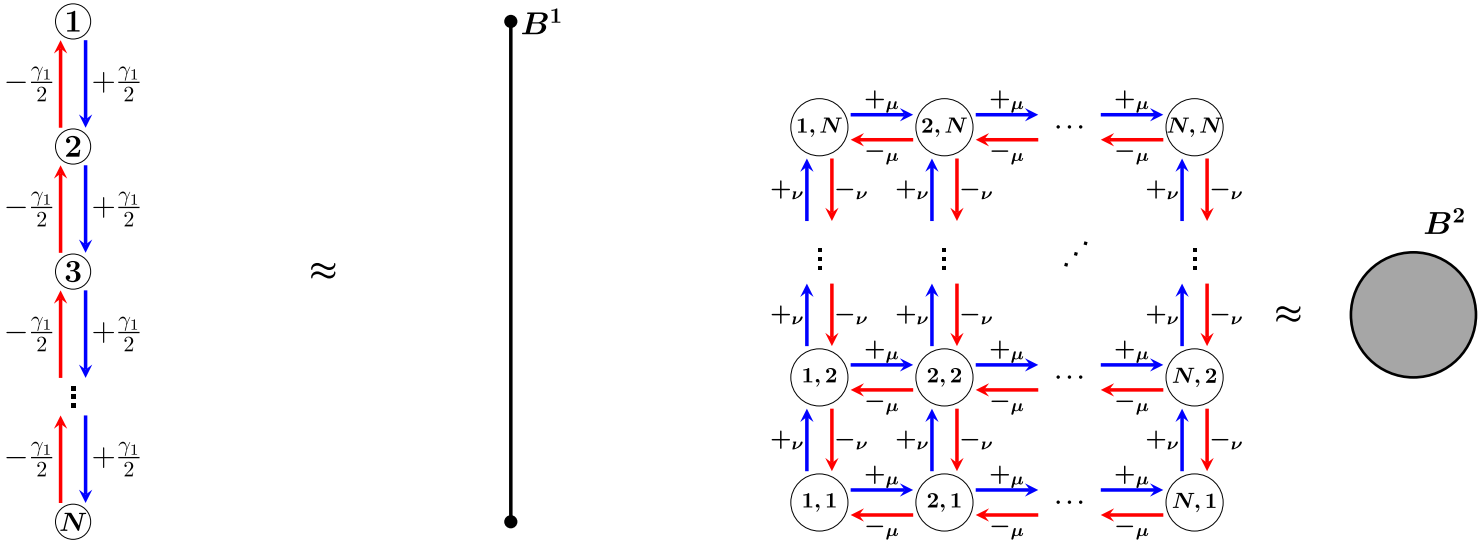
manifold M	sum of $\beta_r(M)$	maximal # of species
1-d torus	$1+1$	2
2-d torus	$1+2+1$	4
3-d torus	$1+3+3+1$	8
4-d torus	$1+4+6+4+1$	16
Torus T^D	$(1+1)^D$	2^D
Hyperball B^D	$1+0+0+\cdots$	1
Sphere S^D	$1+0+0+\cdots+1$	2
$T^D \times B^d$	$2^D \times 1$	2^D

Next some slides, briefly explain B^D, S^D, and $T^D \times B^d$.

New conjecture on species doubling

- B^D

Graphs of 1-dim ball B^1 and 2-dim ball B^2



New conjecture on species doubling

- The number of species on B^D

Diagonalized adjacency matrix of D-dim hyperball B^D with N^D vertices

$$\mathcal{U}^\dagger A^{B^D} \mathcal{U} = \sum_k \sum_{\mu=1}^{D} i\gamma_\mu \cos\left(\frac{k_\mu \pi}{N+1}\right) |k\rangle \langle k|$$

$$\Longrightarrow \quad \cos\left(\frac{k_\mu \pi}{N+1}\right) = 0$$

$$\Longrightarrow \quad k_\mu = \frac{N+1}{2}$$

We get a solution when we take N the odd number.

When we take N the even number, the number of species is lower than 1.

So, there is **a single species at maximum** on B^D.

New conjecture on species doubling

- S^D

e.g. Two graphs of 2-dim sphere S^2

Left graph has $4 + 2$ vertices.

Right graph has $6 + 2$ vertices.

New conjecture on species doubling

- The number of species on S^D

Diagonalized adjacency matrix of 2-dim sphere S^2 with $M+2$ vertices

$$\mathcal{U}^\dagger A^{S^2} \mathcal{U} = \sum_{k=1}^{M} \gamma_1 \sin\left(\frac{2\pi(k-1)}{M}\right) |k\rangle \langle k| - i\sqrt{\frac{M}{2}}\gamma_2 |M+1\rangle \langle M+1| + i\sqrt{\frac{M}{2}}\gamma_2 |M+2\rangle \langle M+2|$$

$$\Longrightarrow \quad \sin\left(\frac{2\pi(k-1)}{M}\right) = 0$$

$$\Longrightarrow \quad k = 1 \text{ or } \frac{N}{2} + 1$$

We get two solutions when we take M the even number.

When we take M the odd number, the number of species is lower than 2.

So, there are **2 species at maximum** on S^2.

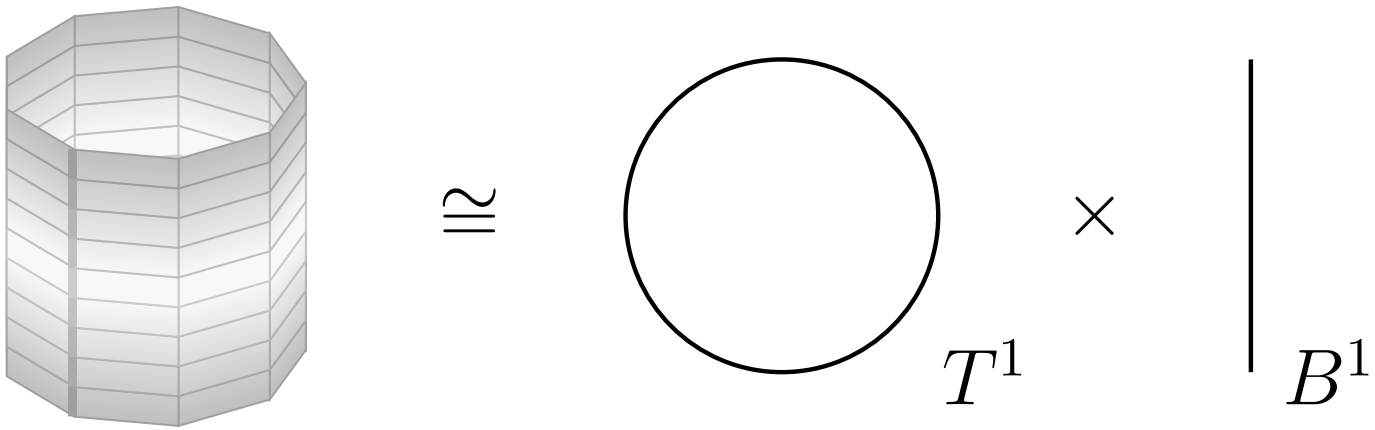
S.Kamata, S.Matsuura, T.Misumi, and K.Ohta (2016)
R.C.Brower, E.S.Weinberg, G.T.Fleming, A.D.Gasbarro, T.G.Raben, and C.-I.Tan (2017)
cf. S.Catterall, J.Laiho, and J.Unmuth-Yockey (2018)

N.Butt, S.Catterall, A.Pradhan, and G.C.Toga (2021)

New conjecture on species doubling

- $T^D \times B^d$

A graph of $T^1 \times B^1$



New conjecture on species doubling

- The number of species on $T^D \times B^d$

Diagonalized adjacency matrix of $T^D \times B^d$

$$\mathcal{U}^\dagger A^{T^D \times B^d} \mathcal{U} = i \left\{ \sum_{\mu=1}^{D} \gamma_\mu \sin\left(\frac{2\pi(k_\mu - 1)}{N}\right) + \sum_{\nu=D+1}^{D+d} \gamma_\nu \cos\left(\frac{k_\nu \pi}{N+1}\right) \right\} |k\rangle \langle k|$$

$$\Longrightarrow \quad \begin{cases} \sin\left(\dfrac{2\pi(k_\mu - 1)}{N}\right) = 0 \\ \cos\left(\dfrac{k_\nu \pi}{N+1}\right) = 0 \end{cases} \quad \Longrightarrow \quad k_\mu = 0 \text{ or } \frac{N}{2} + 1$$

We get 2^D solutions when we take N the even number.

When we take N the odd number, the number of species is lower than 2^D.

So, there are 2^D **species at maximum** on $T^D \times B^d$.

Note that Wilson fermion on $T^4 \times B^1$ is equivalent to **Domain-wall fermion** because # of species depends on mass parameter, with 16 being maximal.

New conjecture on species doubling

We have found numerous evidences that the maximal # of species is equivalent to sum of the Betti number $\sum_r \beta_r$.

Table: Betti numbers and Maximal numbers of species

manifold M	sum of $\beta_r(M)$	maximal # of species
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Is there a known theorem which informs us of maximal # of species?

New conjecture on species doubling

Is there a known theorem which informs us of maximal # of species?

No!

As a well-known theorem, there is Nielsen-Ninomiya's no-go theorem.

But, this theorem is just no-go theorem.

It never tells us how many fermion species emerge given a lattice fermion formulation.

Our work

We propose a new conjecture on species doubling of lattice fermions!

New conjecture on species doubling

- Assumptions of our new conjecture

We firstly impose the following five conditions on the fermion action of the lattice-discretized D-dimensional manifold $\mathcal{M}$:

i. **Central difference**; anti-hermiticity of the Dirac matrix in the action holds due to this condition.

ii. γ_5 **hermiticity**; even the action with the mass term or the Wilson term satisfies this condition.

iii. **Four spinors**; this condition assures the linear independence of the lattice action for each direction.

iv. **Locality**; this condition leads to finite-hopping actions although it may be unnecessary for our conjecture because non-locality usually decreases the number of species.

v. **Finite volume lattice**; our conjecture claims that the fermion action on the finite-volume lattice picks up the topology of the continuum manifold.

New conjecture on species doubling

- Our new conjecture

Our conjecture claims that, as long as these conditions hold, the maximal number of fermion species on the lattice-discretized D-dimensional manifold is equal to the summation of Betti numbers $\beta_r(\mathcal{M})$ over $0 \leq r \leq D$ for the continuum manifold $\mathcal{M}$. It is expressed as

$$\max[\mathcal{N}({}^*\mathcal{M})] = \sum_{r=0}^{D} \beta_r(\mathcal{M}) \,, \tag{1}$$

where $\mathcal{N}({}^*\mathcal{M})$ is the number of fermion species on the lattice-discretized manifold ${}^*\mathcal{M}$.

How can we prove it?

Program of proof

Program of proof

- Program of the proof of new conjecture : Outline

Introduce r-th Laplacian operator from topology and Hodge theory.
Prove each of Betti numbers ($\beta_0 = 1$ and $\beta_1 = 1$) is equivalent to the nullity of Laplacian operators L, L' on 1-dim torus or 1-dim ball by regarding lattice fermion.

$$\Delta_r \equiv \partial_{r+1}\partial^*_{r+1} + \partial^*_r \partial_r$$

$$\downarrow$$

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4-dim Torus and Hyperball.

$$H_r(C \times C') \cong \bigoplus_{p+q=r} H_p(C) \otimes H_q(C')$$

$$\downarrow$$

Classify necessary conditions and complete proof

Program of proof

- Program of the proof of new conjecture : Laplacian

To prove its conjecture, we introduce r-th Laplacian operator from topology

$$\Delta_r \equiv \partial_{r+1}\partial^*_{r+1} + \partial^*_r\partial_r$$

where ∂_r is a r-th boundary operator.

We now propose a program for proof of the conjecture in term spectral graph theory and Hodge theory.

In Hodge theory, the number of zero-eigenvalues of a r-th Laplacian defined on a complex chain coincides with the r-th Betti number.

W. Hodge, " The Theory and Applications of Harmonic Integrals, "
B. Eckmann, (1945).
J. Dodziuk, (1976).
J. Dodziuk and V. Patodi, (1976)

Program of proof

- Program of the proof of new conjecture : Laplacian

In our program, we re-interpret Laplacian operator as spectral graphs.

We define a Laplacian operator L of a graph as

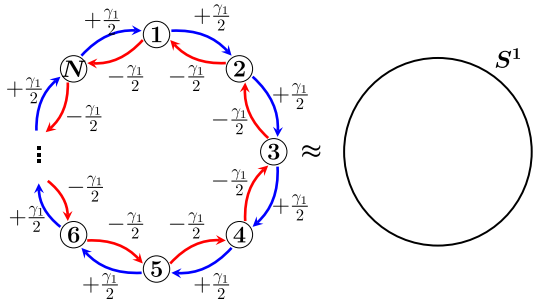
$$L_{ij} \equiv \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (i,j) \text{ are linked} \\ 0 & \text{if } i \neq j \text{ and } (i,j) \text{ are not linked} \end{cases},$$

where d_i is the number of edges sharing the site i.

Program of proof

- Program of the proof of new conjecture : 1-dim lattice

e.g. A graph of 1-dim lattice fermion like T^1 (S^1)



The Laplacian matrix $L^{1\mathrm{d}}$ of this graph

$$L^{1\mathrm{d}} = \begin{pmatrix} 4 & 2 & 0 & & 0 & 0 & -2 \\ -2 & 4 & 2 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 4 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -2 & 4 & 2 \\ 2 & 0 & 0 & & 0 & -2 & 4 \end{pmatrix}$$

Program of proof

- Program of the proof of new conjecture : 1-dim lattice

The bilinear form of the Laplacian matrix $L^{1\mathrm{d}}$ for the field vector $\boldsymbol{\psi}$

$$\bar{\boldsymbol{\psi}} L^{1\mathrm{d}} \boldsymbol{\psi} = 2 \sum_n \bar{\psi}_n \left(2\psi_n - \psi_{n+1} - \psi_{n-1}\right)$$

The Wilson term, species-splitting mass term, on 1-dim lattice

$$S_W^{1\mathrm{d}} = \frac{1}{2} \sum_n \bar{\psi}_n \left(2\psi_n - \psi_{n+1} - \psi_{n-1}\right)$$

Comparing these, we show that this bilinear form results in the Wilson term

$$\frac{1}{2} \bar{\boldsymbol{\psi}} L^{1\mathrm{d}} \boldsymbol{\psi} = S_W^{1\mathrm{d}}$$

Program of proof

- Program of the proof of new conjecture : 1-dim lattice

This fact means that the nullity of the Laplacian matrix corresponding to the Wilson term on 1-dim lattice is equivalent to the 0-th Betti number $\beta_0(T^1) = 1$ for the continuum torus T^1

Specifically,

$$D_W^{1\mathrm{d}} = P_N \otimes \gamma_1 + \frac{1}{2} L \otimes \mathbf{1}_N = \mathbf{0}$$

$$\Longrightarrow \quad D_W^{1\mathrm{d}}(k) = \sum_k \left[i\gamma_1 \sin\left(\frac{2\pi(k-1)}{N}\right) + \mathbf{1}_4 \left\{ 1 - \cos\left(\frac{2\pi(k-1)}{N}\right) \right\} \right] |k\rangle \langle k| = \mathbf{0}$$

$$\Longrightarrow \quad \begin{cases} \sin\left(\dfrac{2\pi(k-1)}{N}\right) = 0 \\ 1 - \cos\left(\dfrac{2\pi(k-1)}{N}\right) = 0 \end{cases} \quad \Longrightarrow \quad k = 1$$

Indeed, # of this solution and the 0-th Betti number $\beta_0(T^1) = 1$ just match!

Program of proof

- Program of the proof of new conjecture : 1-dim lattice

Next, we introduce another Laplacian operator L'

$$L'_{ij} \equiv \begin{cases} -d_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (i,j) \text{ are linked} \\ 0 & \text{if } i \neq j \text{ and } (i,j) \text{ are not linked} \end{cases}$$

e.g. The Laplacian matrix $L'^{1\mathrm{d}}$ of 1-dim naive fermion

$$L'^{1\mathrm{d}} = \begin{pmatrix} -4 & 2 & 0 & & 0 & 0 & -2 \\ -2 & -4 & 2 & \cdots & 0 & 0 & 0 \\ 0 & -2 & -4 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & -4 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -2 & -4 & 2 \\ 2 & 0 & 0 & & 0 & -2 & -4 \end{pmatrix}$$

Program of proof

- Program of the proof of new conjecture : 1-dim lattice

The bilinear form of this operator results in the Wilson term adding a negative mass

$$\frac{1}{2}\bar{\boldsymbol{\psi}}L'^{1\mathrm{d}}\boldsymbol{\psi} = S_W^{1\mathrm{d}} - 2\sum_n \bar{\psi}_n\psi_n$$

This fact means that the nullity of the Laplacian matrix L' corresponding to the Wilson term on 1-dim lattice is equivalent to the 1st Betti number $\beta_1(T^1) = 1$ for the continuum torus T^1

$$D_W'^{1\mathrm{d}} = P_N \otimes \gamma_1 + \frac{1}{2}L' \otimes \mathbf{1}_N = \mathbf{0}$$

$$\Longrightarrow \quad \begin{cases} \sin\left(\dfrac{2\pi(k-1)}{N}\right) = 0 \\ 1 + \cos\left(\dfrac{2\pi(k-1)}{N}\right) = 0 \end{cases} \quad \Longrightarrow \quad k = \frac{N}{2} + 1$$

Indeed, # of this solution and the 1st Betti number $\beta_1(T^1) = 1$ just match!

Program of proof

- Program of the proof of new conjecture : 1-dim lattice

1-dim lattice fermion and the Laplacian operators

- The Laplacian matrix L

  $\Longrightarrow$ The Wilson term

- Nullity of the Laplacian matrix L

  $\Longrightarrow$ The 0-th Betti number $\beta_0 = 1$

- Another Laplacian matrix L'

  $\Longrightarrow$ The Wilson term adding a negative mass

- Nullity of another Laplacian matrix L'

  $\Longrightarrow$ The 1st Betti number $\beta_1 = 1$

Program of proof

- Program of the proof of new conjecture : higher dimensional lattice

For higher dimensional lattice, we utilize Künneth theorem.

Künneth theorem claims that the homology groups of two cellular chain complexes C, C' and $C \times C'$ have the following relation:

$$H_r(C \times C') \cong \bigoplus_{p+q=r} H_p(C) \otimes H_q(C')$$

This theorem means the homology group and its rank (Betti number) of a certain two-dimensional product manifold is obtained from those of the one-dimensional manifolds.

By repeating this, we can obtain the homology groups and Betti numbers for any higher-dimensional manifolds.

Program of proof

- Program of the proof of new conjecture : higher dimensional lattice

Künneth theorem
e.g. $T^4 = S^1 \times S^1 \times S^1 \times S^1$

$$H_0(T^4) \cong H_0(S^1) \otimes H_0(S^1) \otimes H_0(S^1) \otimes H_0(S^1)$$
$$\Longrightarrow \quad \beta_0(T^4) = \beta_0(S^1) \times \beta_0(S^1) \times \beta_0(S^1) \times \beta_0(S^1) = 1$$

$$H_1(T^4) \cong \bigoplus_{p+q+r+s=1} H_p(S^1) \otimes H_q(S^1) \otimes H_r(S^1) \otimes H_s(S^1)$$
$$\Longrightarrow \quad \beta_1(T^4) = \sum_{p+q+r+s=1} \beta_p(S^1) \times \beta_q(S^1) \times \beta_r(S^1) \times \beta_s(S^1)$$
$$= 4$$

where $\beta_0(S^1) = 1$ and $\beta_1(S^1) = 1$.

Program of proof

- Program of the proof of new conjecture : higher dimensional lattice

Based on the Künneth theorem, the Laplacian operators giving Betti numbers β_r are expressed as the sum of tensor products of the Laplacians L, L'.

e.g. 4-dim lattice fermion like $T^4 = S^1 \times S^1 \times S^1 \times S^1$

- A Laplacian operator giving $\beta_0(T^4) = 1$

$$\mathcal{L}^{4\mathrm{d}}_{r=0} = \Big(L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes L \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes L \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes L \Big) \otimes \mathbf{1}_4$$

# of nullity of Dirac matrix adding this matrix is **1!**

- A Laplacian operator giving $\beta_4(T^4) = 1$

$$\mathcal{L}^{4\mathrm{d}}_{r=4} = \Big(L' \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes L' \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes L' \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes L' \Big) \otimes \mathbf{1}_4$$

# of nullity of Dirac matrix adding this matrix is **1!**

Program of proof

- Program of the proof of new conjecture : higher dimensional lattice

e.g. 4-dim lattice fermion like $T^4 = S^1 \times S^1 \times S^1 \times S^1$

- 4 Laplacian operators giving $\beta_1(T^4) = 4$

$$\begin{aligned}
\mathcal{L}^{4\mathrm{d}}_{r=4,1} &= \Big(L' \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \\
&\qquad + \mathbf{1}_N \otimes \mathbf{1}_N \otimes L \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes L\Big) \otimes \mathbf{1}_4 \\
\mathcal{L}^{4\mathrm{d}}_{r=4,2} &= \Big(L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes L' \otimes \mathbf{1}_N \otimes \mathbf{1}_N \\
&\qquad + \mathbf{1}_N \otimes \mathbf{1}_N \otimes L \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes L\Big) \otimes \mathbf{1}_4 \\
\mathcal{L}^{4\mathrm{d}}_{r=4,3} &= \Big(L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \\
&\qquad + \mathbf{1}_N \otimes \mathbf{1}_N \otimes L' \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes L\Big) \otimes \mathbf{1}_4 \\
\mathcal{L}^{4\mathrm{d}}_{r=4,4} &= \Big(L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N + \mathbf{1}_N \otimes L \otimes \mathbf{1}_N \otimes \mathbf{1}_N \\
&\qquad + \mathbf{1}_N \otimes \mathbf{1}_N \otimes L \otimes \mathbf{1}_N + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes L'\Big) \otimes \mathbf{1}_4
\end{aligned}$$

Sum of # of the nullity of these Laplacians just is **4!**

Program of proof

- Program of the proof of new conjecture : higher dimensional lattice

At least for the higher dimensional torus T^D, hyperball B^D, and a product space $T^D \times B^D$, we believe it is possible to prove our new conjecture.

For the manifold such as S^D, we have to develop a more generic way of generalization to higher-dimensions.

We comment on another avenue toward proof of the conjecture.

Squaring the free naive Dirac matrix leads to another Laplacian operator.

If we can prove that the number of zero-modes of this Laplacian is the sum of the Betti numbers of the continuum manifold, we can easily give a generic proof for any kind of manifolds including S^D

Summary

Summary

Lattice field theory and Spectral graph theory

Lattice field theory	$\Longrightarrow$	Spectral graph theory
Lattice fermion	$\Longrightarrow$	Directed and Weighted spectral graph
# of species	$\Longrightarrow$	Nullity of spectral matrix

We can use known theorems to study lattice field theory.

Summary

New conjecture on species doubling of lattice fermions

Our conjecture claims that the maximal number of fermion species on the lattice-discretized D-dimensional manifold is equal to the summation of Betti numbers $\beta_r(\mathcal{M})$ over $0 \leq r \leq D$ for the continuum manifold $\mathcal{M}$. It is expressed as

$$\max[\mathcal{N}({}^*\mathcal{M})] = \sum_{r=0}^{D} \beta_r(\mathcal{M}) \,, \tag{2}$$

where $\mathcal{N}({}^*\mathcal{M})$ is the number of fermion species on the lattice-discretized manifold ${}^*\mathcal{M}$.

Summary

Program of the proof of new conjecture

Introduce r-th Laplacian operator from topology and Hodge theory.
Prove each of Betti numbers ($\beta_0 = 1$ and $\beta_1 = 1$) is equivalent to the nullity of Laplacian operators L, L' on 1-dim torus or 1-dim ball by regarding lattice fermion.

$$\Delta_r \equiv \partial_{r+1}\partial_{r+1}^* + \partial_r^*\partial_r$$

$$\downarrow$$

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4-dim Torus and Hyperball.

$$H_r(C \times C') \cong \bigoplus_{p+q=r} H_p(C) \otimes H_q(C')$$

$$\downarrow$$

Classify necessary conditions and complete proof