

The Theta parameter with minimal doubling

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- $F\tilde{F}$ and Theta
- $\eta = i\bar{\psi}\gamma_5\psi$ and Theta
- Wilson fermions and Theta
- minimal doubling and Theta

QCD

$$L = \frac{1}{2} \text{Tr}(F_{\mu\nu} F_{\mu\nu}) + \bar{\psi}(\not{D} + M)\psi$$

- renormalizable quantum field theory
- describes protons, neutrons, pions, etc.
- only parameters m_q/Λ_{qcd}
 - α_s not a parameter, adjust to get m_p right

Famous possible addition

$$L \rightarrow L + \Theta \frac{g^2}{16\pi^2} \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$$

- remains renormalizable
- formally $F_{\mu\nu}\tilde{F}_{\mu\nu}$ is a total divergence
- integral depends on fields at the boundary
- for **smooth** fields: counts a winding at infinity
 - $\nu = \frac{g^2}{16\pi^2} \int d^4x \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$ integer

Classical configurations of non-trivial winding exist

- “instantons”
- weight configurations in path integral by $e^{i\nu\Theta}$
- Physics depends non-trivially on Theta
- Monte Carlo sign problem, but still well defined

$$G = \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$$

- a pseudoscalar gluon field
- creates the lightest pseudoscalar glueball
- addition to L gives the field an expectation value
- $\langle \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) \rangle \propto \Theta$
- vacuum violates CP

What else can we add to L

- how about some other pseudoscalar field
- $L \rightarrow L + Ci\bar{\psi}\gamma_5\psi = L + C\eta'$
- remains renormalizable
- gives the eta prime an expectation
- also violates CP

Change variables $\psi \rightarrow e^{i\phi\gamma_5}\psi$

- multiply left handed states by $e^{i\phi}$
- multiply right handed states by $e^{-i\phi}$
- mixes new term with the mass
 - $\bar{\psi}\psi \rightarrow \cos(\phi) \bar{\psi}\psi + i \sin(\phi) \bar{\psi}\gamma_5\psi$
- can remove η' term with some mass shifts

$$\psi \rightarrow e^{i\phi\gamma_5} \psi$$

- measure not symmetric when topology present

Index theorem:

- Dirac operator has chiral zero modes
 - $n_+ - n_- = \nu$
- zero modes insert $e^{\pm i\phi}$ in path integral

Topology: weights configurations in path integral by $e^{i\nu\Theta}$

Addition of η : weights configurations by $e^{i\nu N_f \phi}$

$$\Theta \longleftrightarrow N_f \phi$$

Fujikawa (1979): These are equivalent modifications!!!

Adding flavored pseudoscalar fields does nothing

- $L \rightarrow L + C\pi_i$
- $\psi \rightarrow e^{i\phi\tau_i\gamma_5}\psi$
- $\bar{\psi}\psi \rightarrow \cos(\phi) \bar{\psi}\psi + i \sin(\phi) \bar{\psi}\tau_i\gamma_5\psi$
- $\text{Tr } \tau_i = 0 \longrightarrow \det(e^{i\phi\tau_i\gamma_5}) = 1$
- quark flavors cancel

Not exact on the lattice: twisted mass

- ETMC use to reduce lattice artifacts

Symmetry of measure

- $\psi \rightarrow e^{i\phi\tau_i\gamma_5}\psi$
- pion is a Goldstone boson
 - $m_\pi^2 \propto m_q$

Not a symmetry of measure

- $\psi \rightarrow e^{i\phi\gamma_5}\psi$
- eta prime is not a Goldstone boson
 - $m_{\eta'}^2 \propto \Lambda_{qcd}^2$

Classical instanton action

- $S_i = 8\pi^2/g_0^2$ suggests instanton suppression $e^{-\frac{8\pi^2}{g_0^2}}$

But η' mass goes as

- $m_{\eta'} \propto \Lambda_{qcd} \propto \frac{1}{a} e^{-1/(2\beta_0 g_0^2)} g_0^{-\beta_1/\beta_0^2}$

Classical suppression seriously overly estimated

- $\frac{1}{2\beta_0 g_0^2} = \frac{8\pi^2}{(11-2n_f/3)g_0^2} \ll \frac{8\pi^2}{g_0^2}$

Topology more likely on quantum vacuum

Lattice

Limit defines the continuum theory

- fully non-perturbative

Topology tricky to define

- space of lattice fields simply connected
- typical gauge fields are non-differentiable

Cooling can remove roughness

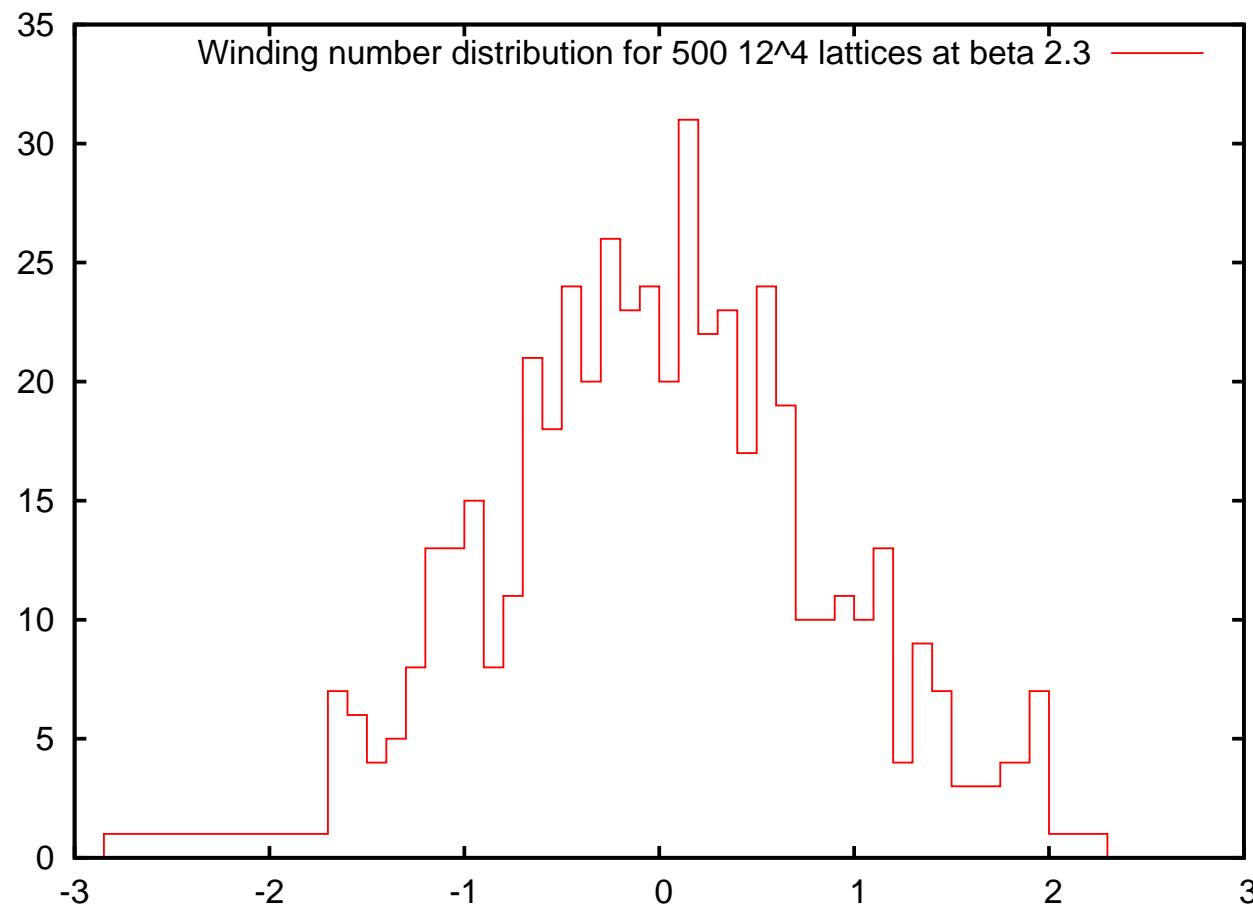
- Teper (1988)
- Narayanan and Neuberger (1994)
- Luscher (2010) “gradient flow”

Cooling algorithm not unique

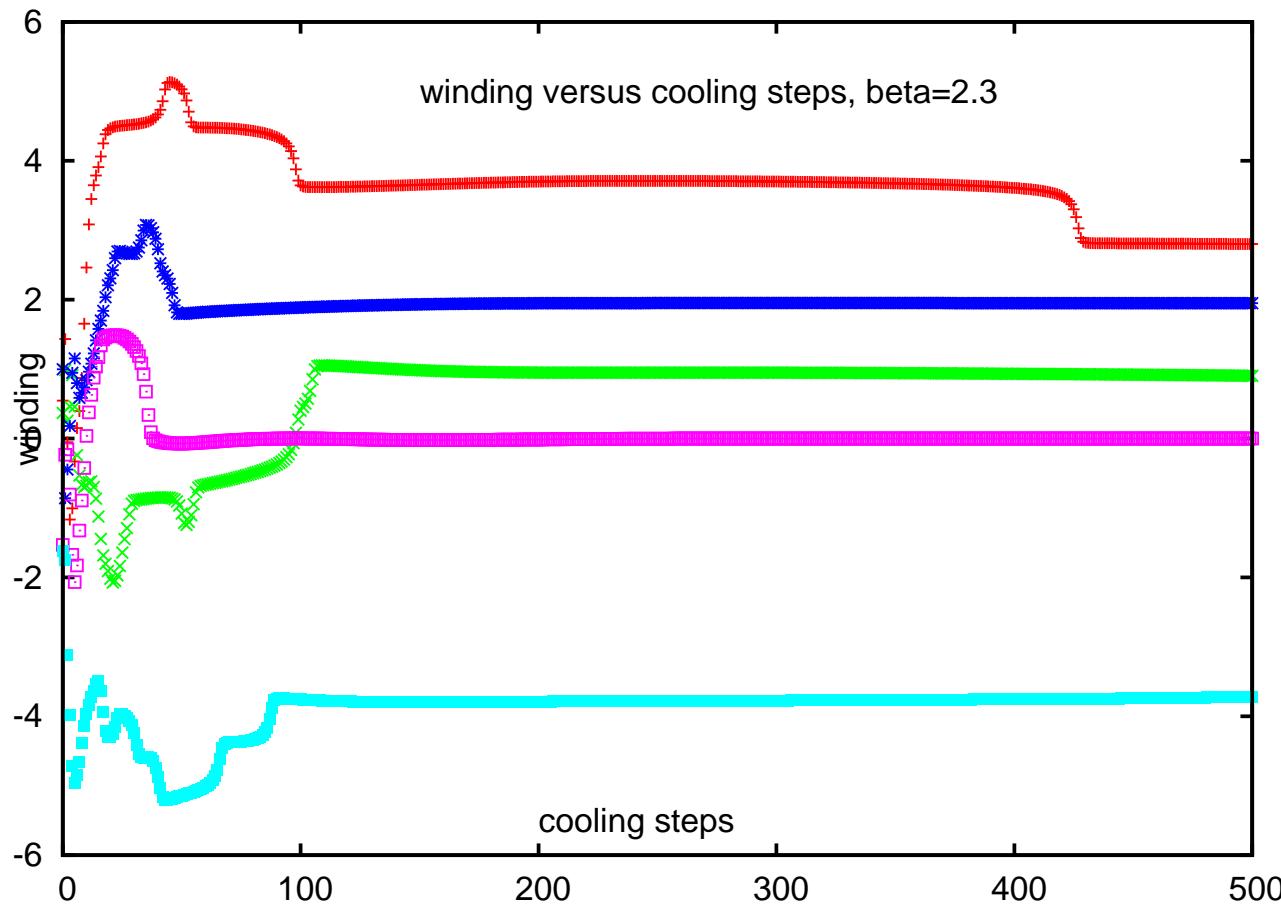
- how to direct the flow
 - don't always reach the same winding
- how long to cool
 - too much and instantons can collapse

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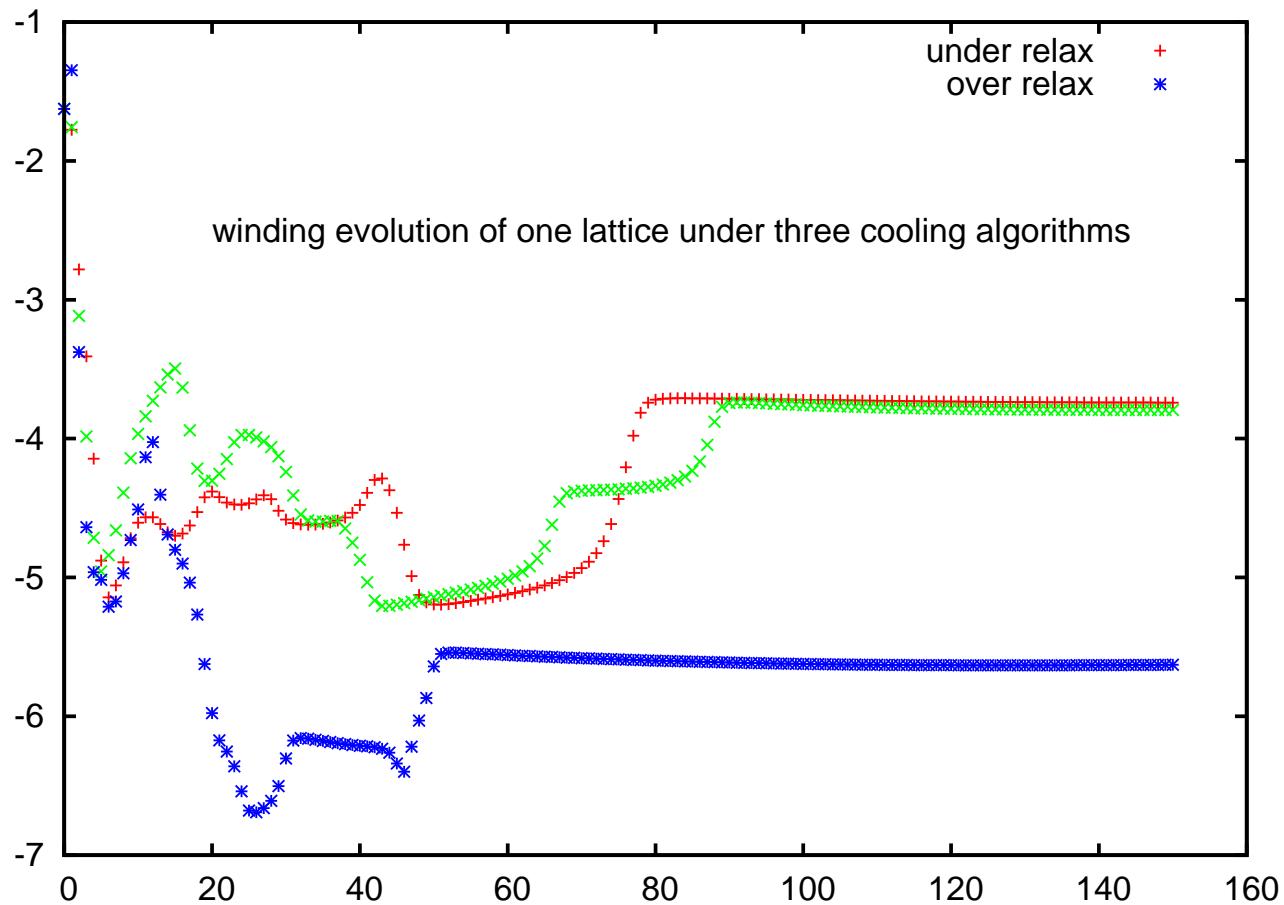
- arXiv:1007.5502[hep-lat] (2010)
- SU(2) pure gauge, define $F\tilde{F}$ on individual hypercubes



Cooling brings winding to integer values



Value can depend on cooling algorithm



Including $i\bar{\psi}\gamma_5\psi$ term in L straightforward

- still have a sign problem
- $\langle \int d^4x \ \eta'(x)\eta'(0) \rangle \propto 1/m_{\eta'}^2$
- as with winding, may have long correlation times

Can move η' term to Wilson term with chiral rotation

- much like η' equivalence to $F\tilde{F}$ terms

Wilson fermions

Hopping:

- $x_i \rightarrow x_{i+1} + i\gamma_\mu - 1$
- $x_i \leftarrow x_{i+1} - i\gamma_\mu - 1$
- $am_q \rightarrow am_q + 2$

Free limit

- $D(p) \sim \gamma_\mu \sin(p_\mu) + 1 - \cos(p_\mu) \rightarrow p + O(p^2)$
- doublers at $p \sim \pi$ get large mass

$$\psi \rightarrow e^{i\phi\gamma_5} \psi$$

- move CP violation between mass and Wilson term
 - $am_q e^{i\phi\gamma_5} + 2 \rightarrow am_q + 2e^{-i\phi\gamma_5}$
 - $x_i \longrightarrow x_{i+1} + i\gamma_\mu - e^{-i\phi\gamma_5}$
 - $x_i \longleftarrow x_{i+1} - i\gamma_\mu - e^{-i\phi\gamma_5}$

Seiler and Stamatescu (1981)

- Theta: relative phase of Wilson and mass terms

For the lattice

- η' is a simple local operator
- no cooling required
- intuitive source of CP violation $\langle \eta' \rangle$
- “susceptibility” $\sim 1/m_{\eta'}^2$
- still a sign problem
- disconnected diagrams must be included

Connection with $F\tilde{F}$

- at fourth order of the hopping expansion
- fermion hops around hypercubes

Parallel with continuum index theorem derivation

- $\nu = \text{Tr } \gamma_5 e^{D_f^2/\Lambda^2} = \frac{g^2}{16\pi^2} \int d^4x \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$
- regulate large eigenvalues with $e^{D_f^2/\Lambda}$
 - “heat kernel” (in my new book)

Minimal Doubling

Consider Karsten-Wilczek formalism

$$D(p) = i \sum_{i=1}^3 \gamma_i \sin(p_i) + \frac{i\gamma_4}{\sin(\alpha)} \left(\cos(\alpha) + 3 - \sum_{\mu=1}^4 \cos(p_\mu) \right)$$

Propagator has two poles at $\vec{p} = 0, p_4 = \pm \alpha$

Chiral symmetry $[D, \gamma_5]_+ = 0$

Two species have different chiral rotations $\gamma'_5 = -\gamma_5$

Exact chiral symmetry is “flavored”

To bring in the Theta parameter

- adding $i\bar{\psi}\gamma_5\psi$ does not work
- need to rotate two species oppositely

Point splitting in arXiv:1010.0110 solves this

$$u_x = \frac{1}{2} e^{i\alpha x_4} \left(\psi_x + i \frac{U_{x,x-e_4} \psi_{x-e_4} - U_{x,x+e_4} \psi_{x+e_4}}{2 \sin(\alpha)} \right)$$
$$d_x = \frac{1}{2} \Gamma e^{-i\alpha x_4} \left(\psi_x - i \frac{U_{x,x-e_4} \psi_{x-e_4} - U_{x,x+e_4} \psi_{x+e_4}}{2 \sin(\alpha)} \right)$$

From this we construct the physical η' field

$$\begin{aligned}
\eta'(x) &= \frac{i}{2}(\bar{u}_x \gamma_5 u_x + \bar{d}_x \gamma_5 d_x) \\
&= \frac{1}{8} \left(\bar{\psi}_{x-e_4} U \gamma_5 \psi_x - \bar{\psi}_x U \gamma_5 \psi_{x-e_4} \right. \\
&\quad \left. - \bar{\psi}_{x+e_4} U \gamma_5 \psi_x + \bar{\psi}_x U \gamma_5 \psi_{x+e_4} \right)
\end{aligned}$$

- all terms connect even with odd parity sites

Theta parameter equivalent to this η' in the action

Summary

Topological term in QCD Lagrangean

- equivalent to

Linear term in eta prime

- simpler for the lattice
- slow topology \longrightarrow slow η' correlator
- real part up quark mass irrelevant
- minimal doubling requires point splitting