Topology awareness of undoubled and doubled lattice fermions

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MITP workshop – 6 March 2023

Genealogy of lattice fermion actions

Classical lattice fermion actions:

- Naive fermions (2^d species in d space-time dimensions)
- Wilson fermions ($N_c 4 N_x N_y N_z N_t \times$ ditto matrix in $d = 4$ dimensions)
- Staggered fermions (reduction by $2^{d/2}$, hence size $N_c N_x N_y N_z N_t \times$ ditto)
- Overlap/domain-wall fermions (unique unitary part of $aD_{\mathrm{W}} - \rho$)

Novel lattice fermion actions:

- Minimally doubled fermions (Karsten-Wilczek, Borici-Creutz, twisted ordering)
- Ameliorated Wilson fermions (Brillouin, hypercube, ...)
- Staggered/naive fermions with lifting (Adams, Hoelbling, ...)

Issues to be considered:

- Nielsen-Ninomya theorem   ($\leftarrow$ this talk)
- suitability for heavy quark phyics (dispersion relation, ...)
- suitability for lattice perturbation theory (LPT)
- computational efficiency (MPI/PGAS, OpenMP/OpenACC/cuda, SIMD)

Introduction: Naive and Wilson fermions

- **Naive fermions**

$$
\begin{aligned}
D_{\mathrm{nai}}(x,y) &= \sum_\mu \gamma_\mu \nabla_\mu(x,y) + m\delta_{x,y} \\
D_{\mathrm{nai}}(p) &= \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + m \\
&= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m \quad \text{with} \quad \bar{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)
\end{aligned}
$$

- **Wilson fermions**

$$
\begin{aligned}
D_{\mathrm{W}}(x,y) &= \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \frac{ra}{2}\sum_\mu \triangle_\mu(x,y) + m\delta_{x,y} \\
D_{\mathrm{W}}(p) &= \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + \frac{r}{a}\sum_\mu \{1-\cos(ap_\mu)\} + m \\
&= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \frac{ra}{2}\sum_\mu \hat{p}_\mu^2 + m \quad \text{with} \quad \hat{p}_\mu \equiv \frac{2}{a}\sin(\frac{ap_\mu}{2})
\end{aligned}
$$

Introduction: Karsten-Wilczek and Borici-Creutz fermions

- **Karsten-Wilczek fermions**

$$D_{\mathrm{KW}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \mathrm{i}\frac{ra}{2}\gamma_4 \sum_{i=1:3} \triangle_i(x,y) + m\delta_{x,y}$$

$$D_{\mathrm{KW}}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + \mathrm{i}\frac{r}{a}\gamma_4 \sum_{i=1:3} \{1-\cos(ap_i)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{ra}{2}\gamma_4 \sum_{i=1:3} \hat{p}_i^2 + m$$

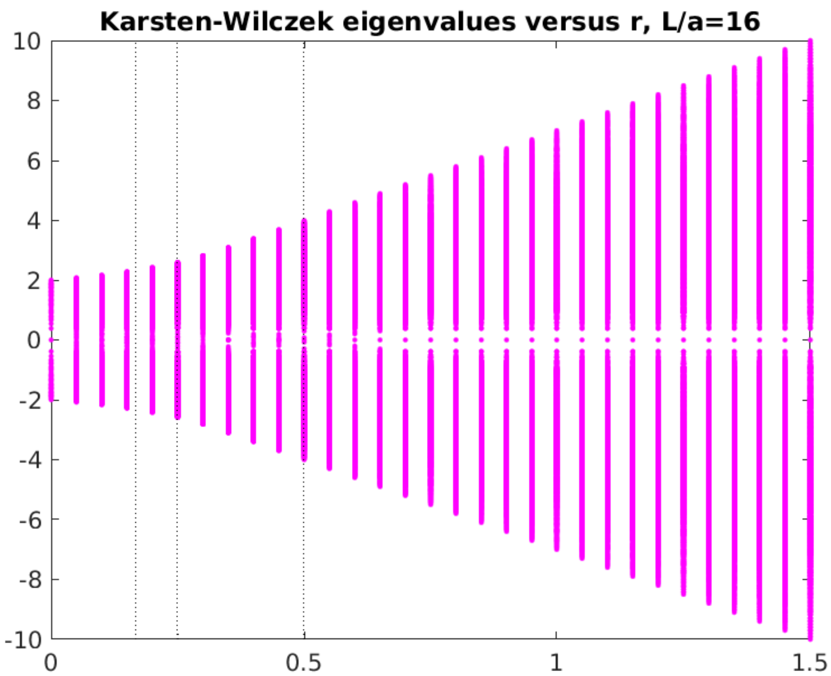
- **Borici-Creutz fermions**

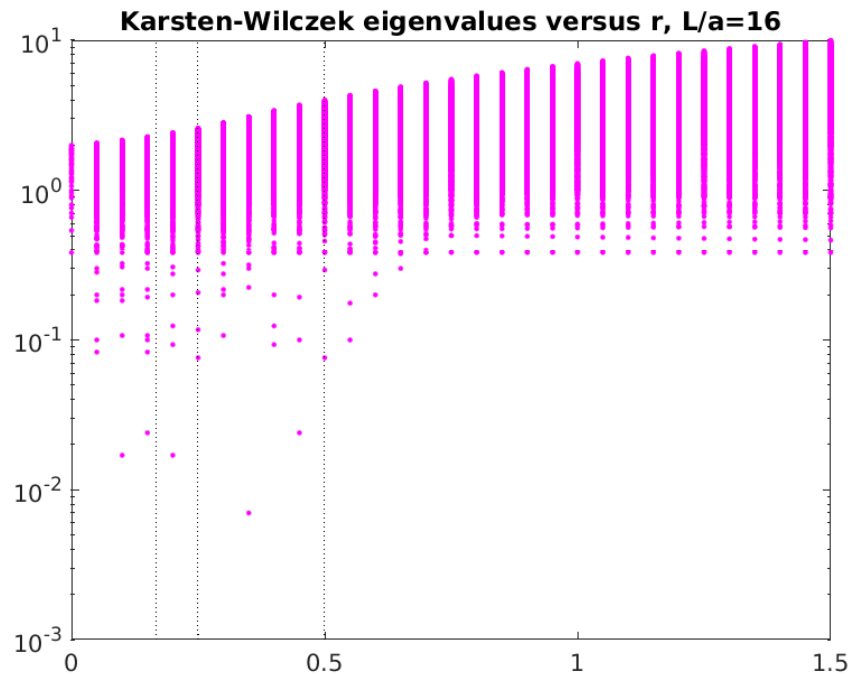
$$D_{\mathrm{BC}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \mathrm{i}\frac{ra}{2}\sum_\mu \gamma'_\mu \triangle_\mu(x,y) + m\delta_{x,y}$$

$$D_{\mathrm{BC}}(p) = \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{r}{a}\sum_\mu \gamma'_\mu \{1-\cos(ap_\mu)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{ra}{2}\sum_\mu \gamma'_\mu \hat{p}_\mu^2 + m \quad \text{with} \quad \gamma'_\mu \equiv \Gamma\gamma_\mu\Gamma, \Gamma \equiv \frac{1}{\sqrt{d}}\sum_\mu \gamma_\mu$$

Karsten-Wilczek free-field eigenvalues versus r in 4D





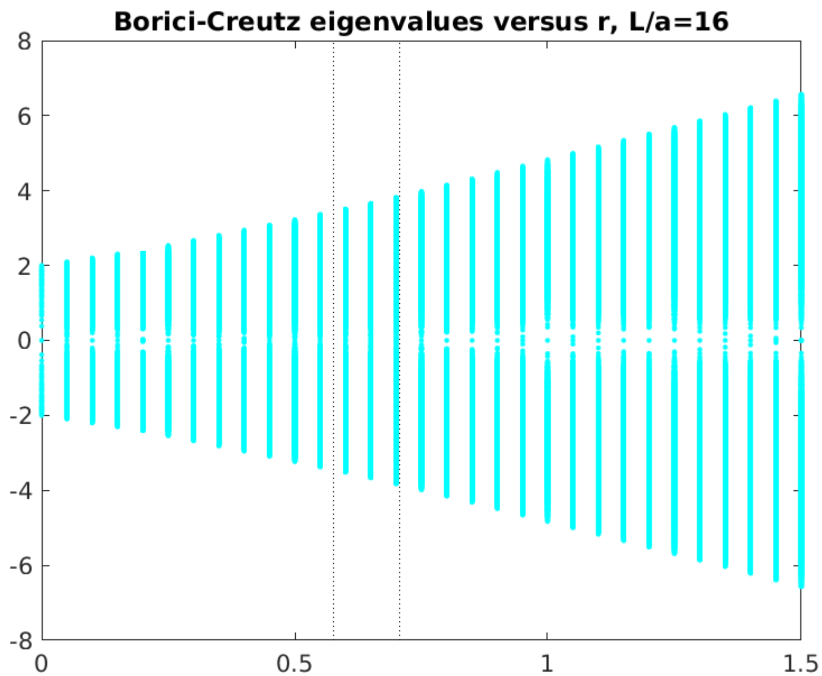
Spectrum at $r = 0$ is naive (i.e. 4-fold staggered) spectrum.

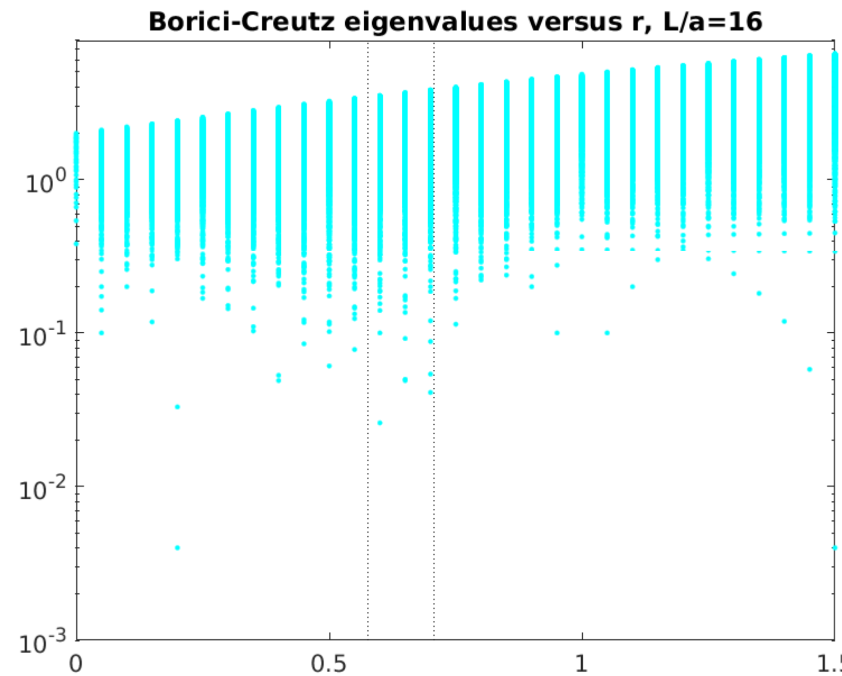
Spectrum at any r is on imaginary axis (chiral symmetry, horizontally displaced).

Spectrum at $r = 1$ covers range $[-7, 7]$ on imaginary axis (worse CN than staggered).

KW species chain is $16 \to 14 \to 8 \to 2$ with transistions at $r = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}$ [SD, JW, 2020].

- ## Borici-Creutz free-field eigenvalues versus r in 4D





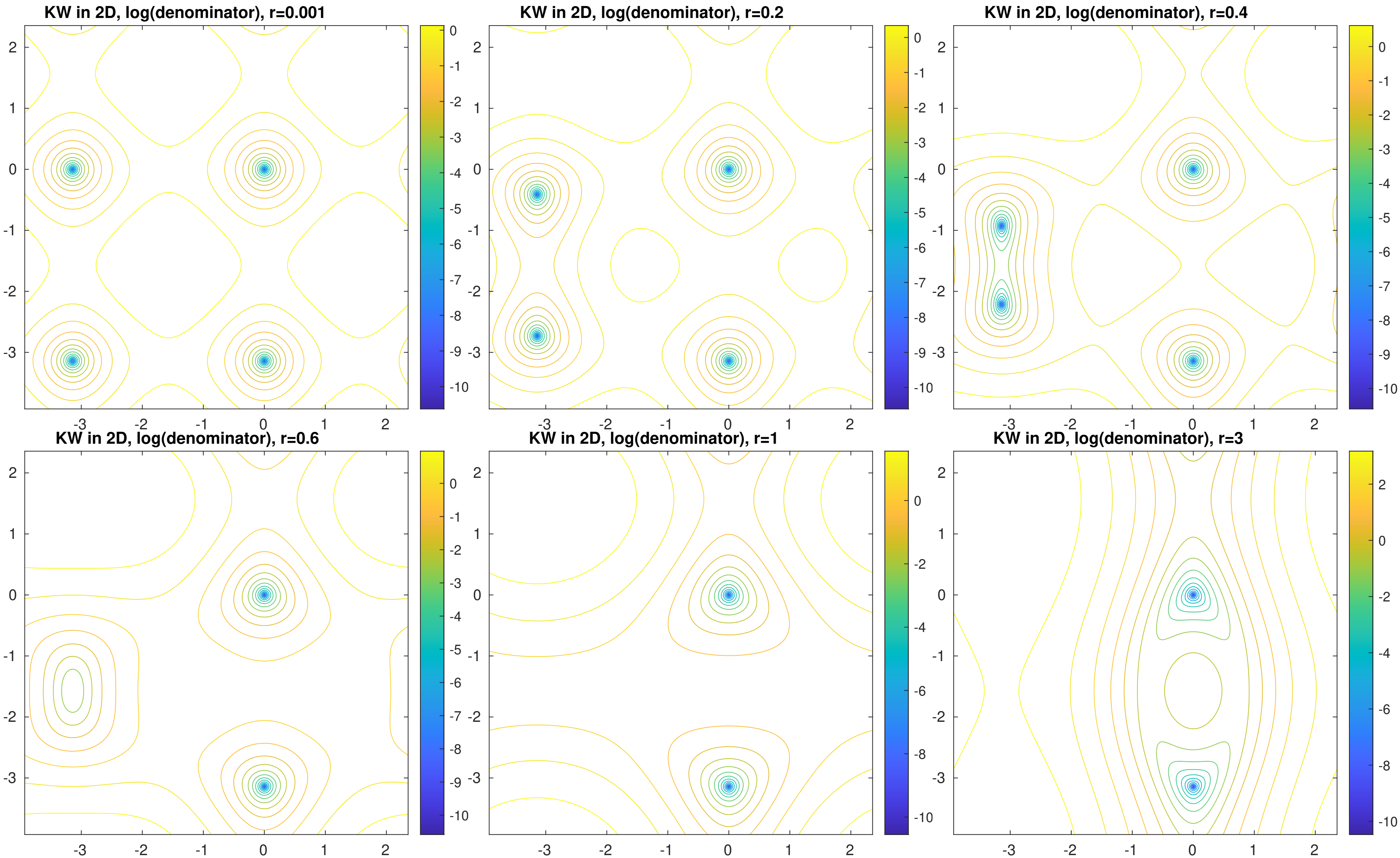
Spectrum at $r = 0$ is naive (i.e. 4-fold staggered) spectrum.

Spectrum at any r is on imaginary axis (chiral symmetry, horizontally displaced).

Spectrum at $r = 1$ covers range $[-4.8284, 2+2\sqrt{2}]$ on imaginary axis (worsened CN).

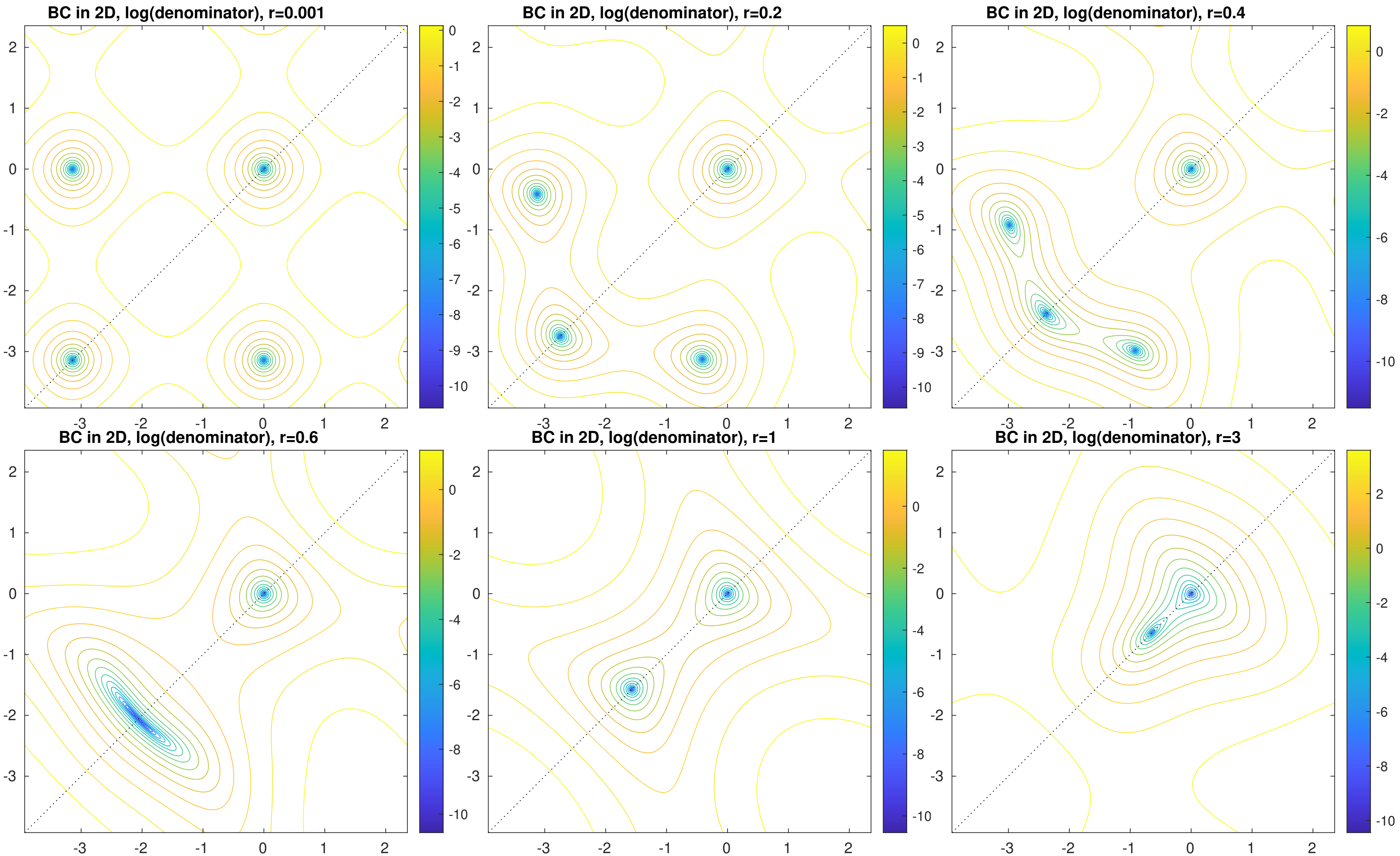
BC species chain is $16 \to 10 \to 2$ with transitions at $r = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}$ [SD, JW 2020].

Pole position drift for KW fermions in 2D (annihilate at r=0.5)



Contour plots of denominator $\sum_{i=1}^{d-1}\bar{p}_i^2+(\bar{p}_d+\frac{ar}{2}\sum_{i=1}^{d-1}\hat{p}_i^2)^2$ [arXiv:2003.10803].

Pole position drift for BC fermions in 2D (merge at r=0.57735)



Contour plots of $\sum_\lambda \bar{p}_\lambda^2 - ar\sum_\lambda \bar{p}_\lambda \hat{p}_\lambda^2 + \frac{a^2r^2}{4}\sum_\lambda \hat{p}_\lambda^4 + \frac{2ar}{d}\sum_{\rho,\sigma}\bar{p}_\rho \hat{p}_\sigma^2$ [2003.10803].

Myths and mysteries concerning topology

"topology: dark side of staggered fermions" (A. Hasenfratz, Lattice,2003, Tsukuba)

Well known (for vector-like gauge theory):

- topology correctly seen by undoubled non-chiral action (Wilson, Bril./hypercube)
- topology correctly seen by undoubled chiral action (overlap, domain-wall)

To be shown (for vector-like gauge theory):

- topology correctly seen by staggered fermions (known for long, see $*$)
- topology correctly seen by Adams fermions (known for long)
- topology correctly seen by naive fermions
- topology correctly seen by Adams-like recipe applied to naive fermion
- topology correctly seen by central-branch fermions (Chowdhury, Misumi, ...)
- topology correctly seen by central-branch-squared (+flipped) fermions
- topology correctly seen by min. doubled fermions (KW and BC)
- topology correctly seen by KW/BC fermion plus lifting term

$(*)$ "figure of merit = chirality" in Hands Teper 1990 and Laursen Smit Vink 1990

Real mystery (to me):

How to put non-vector-like gauge theory on the lattice (S. Catterall on Tue)

Testbed: Schwinger Model (QED in 2D)

Schwinger Model at $N_f = 0$ simulated with Metropolis/overrelax/instanton-hit/P-hit:
$\tau_{\mathrm{int}}(Q_{\mathrm{top}}) \simeq \tau_{\mathrm{int}}(Q^2_{\mathrm{top}}) = O(1)$ at any β [..., arXiv:1203.2560, ...].

Wilson gauge action per site:

$s_{\mathrm{wil}}(x) = 1 - \mathrm{Re}(U(x)) = 1 - \cos(\theta(x))$

Plaquette at position $x = (x_1, x_2)$:

$U(x) = U_1(x) U_2(x+e_1) U_1^\dagger(x+e_2) U_2^\dagger(x)$

$U(x) = \exp(\mathrm{i}\theta(x))$

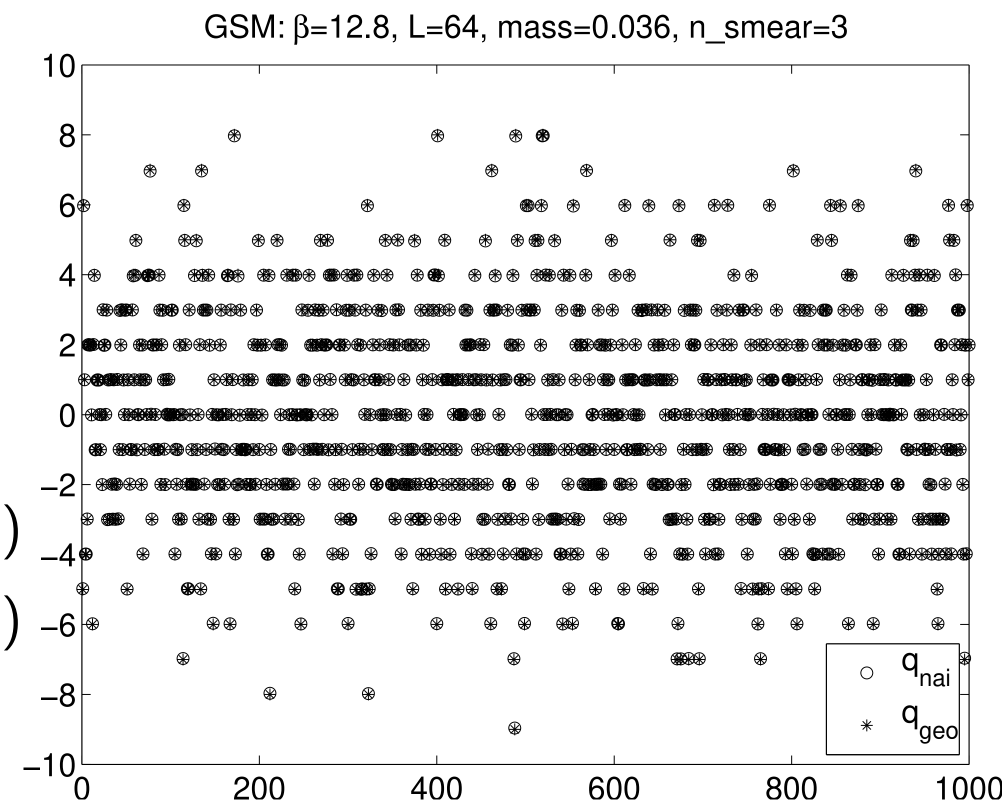
Two gluonic topological charges:

$q^{(3)}_{\mathrm{nai}} = \sum \sin(\theta^{(3)}(x))/(2\pi) \in \mathbf{R}$ ("naive")

$q^{(3)}_{\mathrm{geo}} = \sum \theta^{(3)}(x)/(2\pi) \in \mathbf{Z}$ ("geometric")

$\theta^{(3)}$ is plaquette angle after 3 smearings

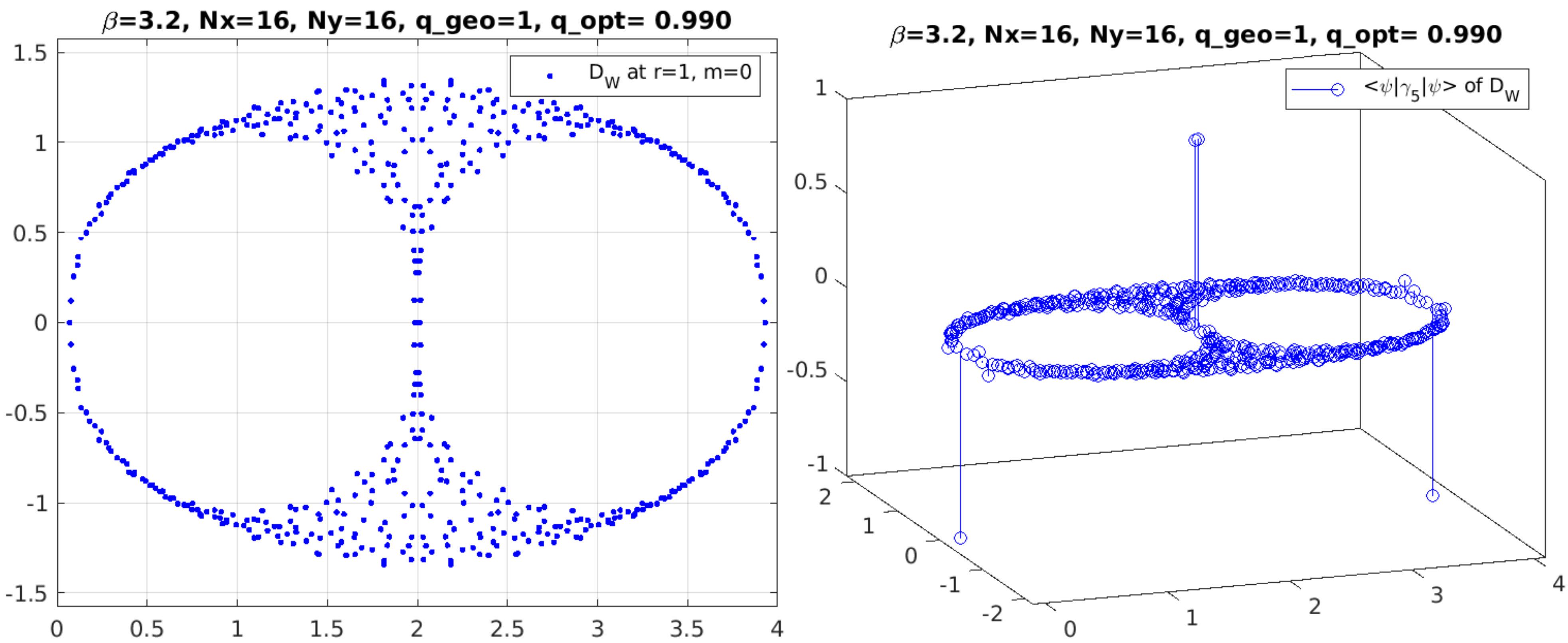
$q_{\mathrm{opt}}(x)$ is clover-leaf version of $q_{\mathrm{nai}}(x)$



All subsequent plots on one 16^2 configuration with $q_{\mathrm{top}} = 1$ at $\beta = 3.2$.

All fermion operators use 1 step of $\rho = 0.25$ stout smearing [Morningstar Peardon 2003].

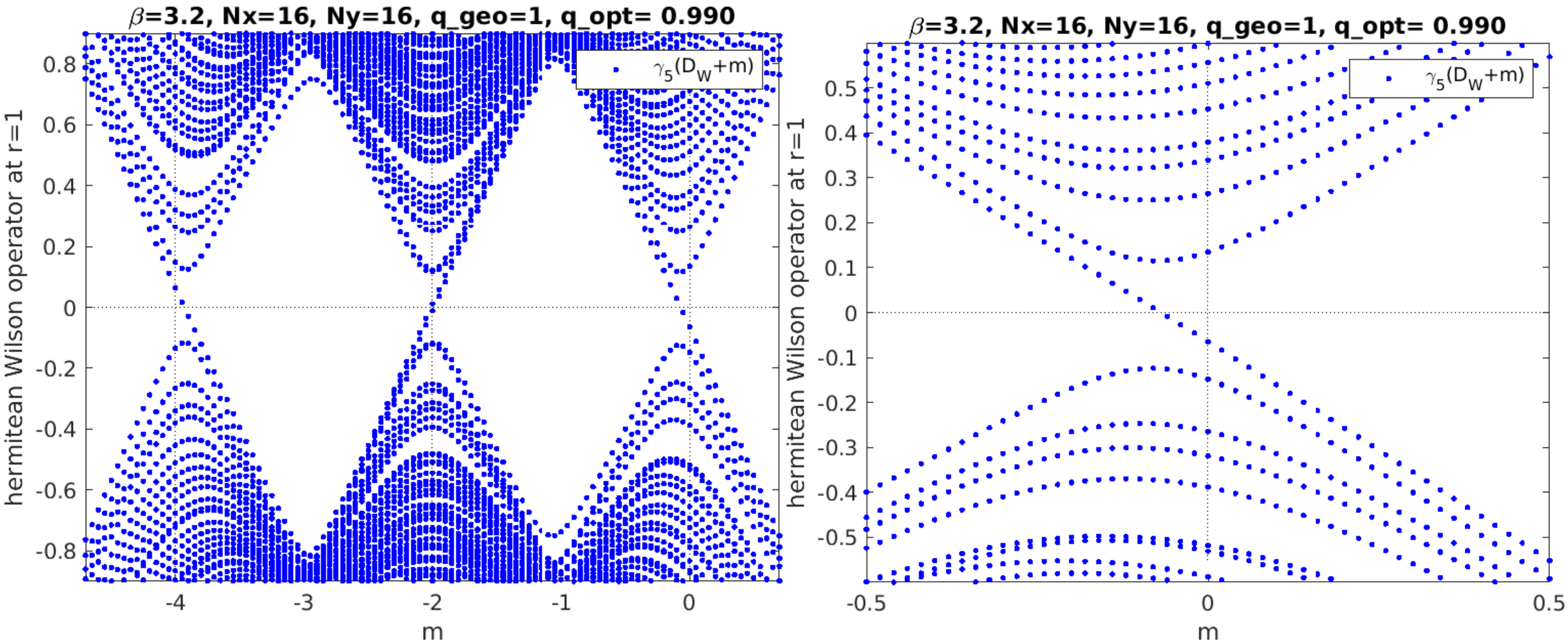
Eigenvalues and topology with Wilson fermions



- $|q|$ would-be zero-modes in physical branch (unchanged in 4D), additive mass shift
- L/R-eigenmode sandwich $\langle\psi|.|\psi\rangle \equiv \langle L|.|R\rangle$ for non-chiral D [Hip et al 2001]
- all subsequent plots taken from [arXiv:2203.15699] with J. Weber

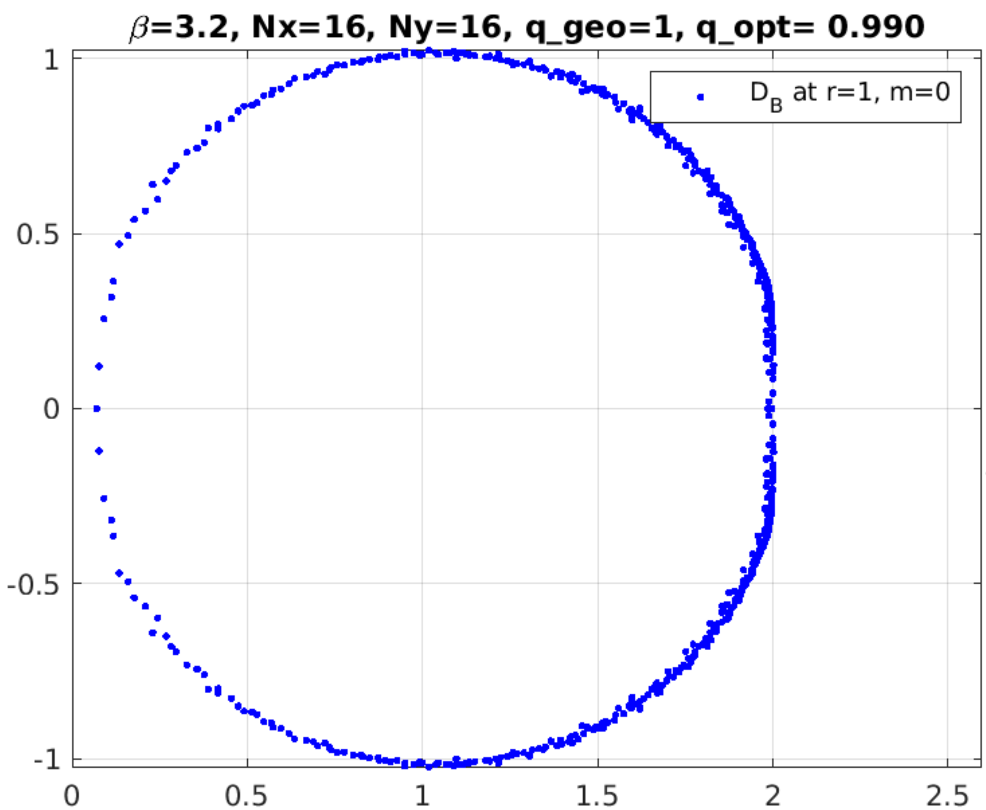
Proposal: central-branch fermions realize 2 species in 2D (6 in 4D) with same chirality and without additive mass shift [Misumi Yumoto 2020].

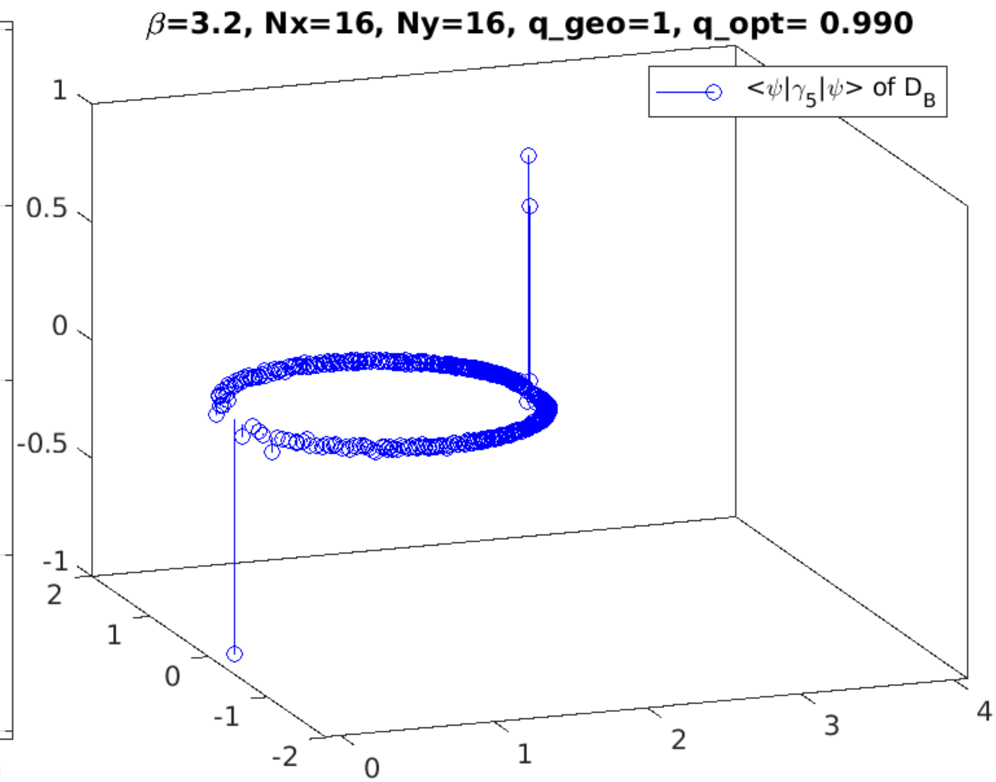
• Spectral flow with Wilson fermions



- eigenvalues of $H_\mathrm{W}(m) \equiv \gamma_5(D_\mathrm{W}+m)$ at $r=1$ versus am
- one species at $am \simeq 0$, two at $am \simeq -2$, one at $am \simeq -4$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down/up)

Eigenvalues and topology with Brillouin fermions



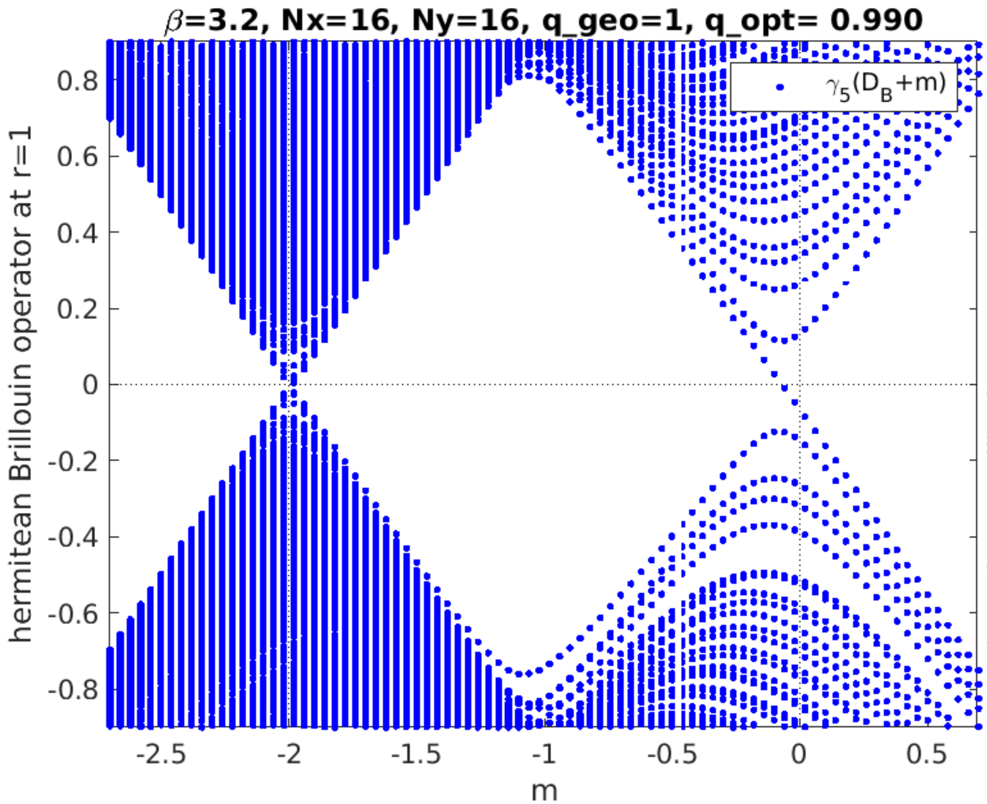


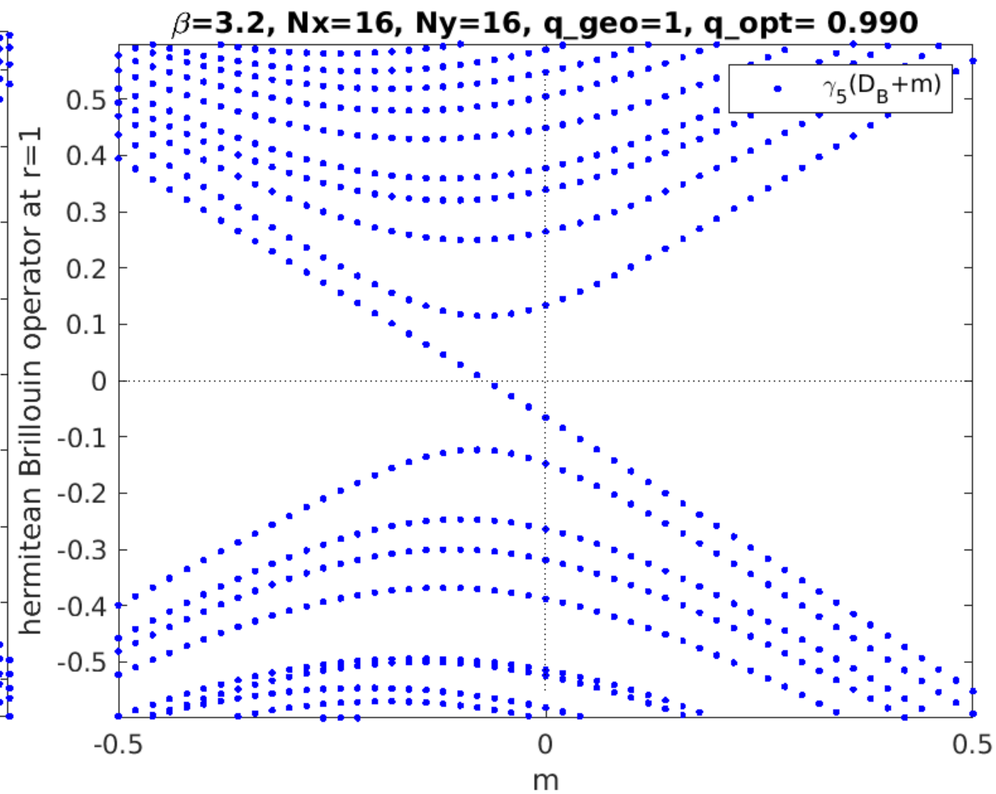
$D_{\mathrm{B}} = \sum_\mu \gamma_\mu \nabla_\mu - \frac{r}{2}\triangle$ like Wilson but ∇_μ and $\triangle$ with hypercubic stencil (3^d-points)

- $|q|$ would-be zero-modes in physical branch (unchanged in 4D), additive mass shift
- L/R-eigenmode sandwich $\langle\psi|.|\psi\rangle \equiv \langle L|.|R\rangle$ for non-chiral D (compare 1302.0773)

Proposal: use as overlap-kernel, already close to shifted-unitary [arXiv:1701.00726].

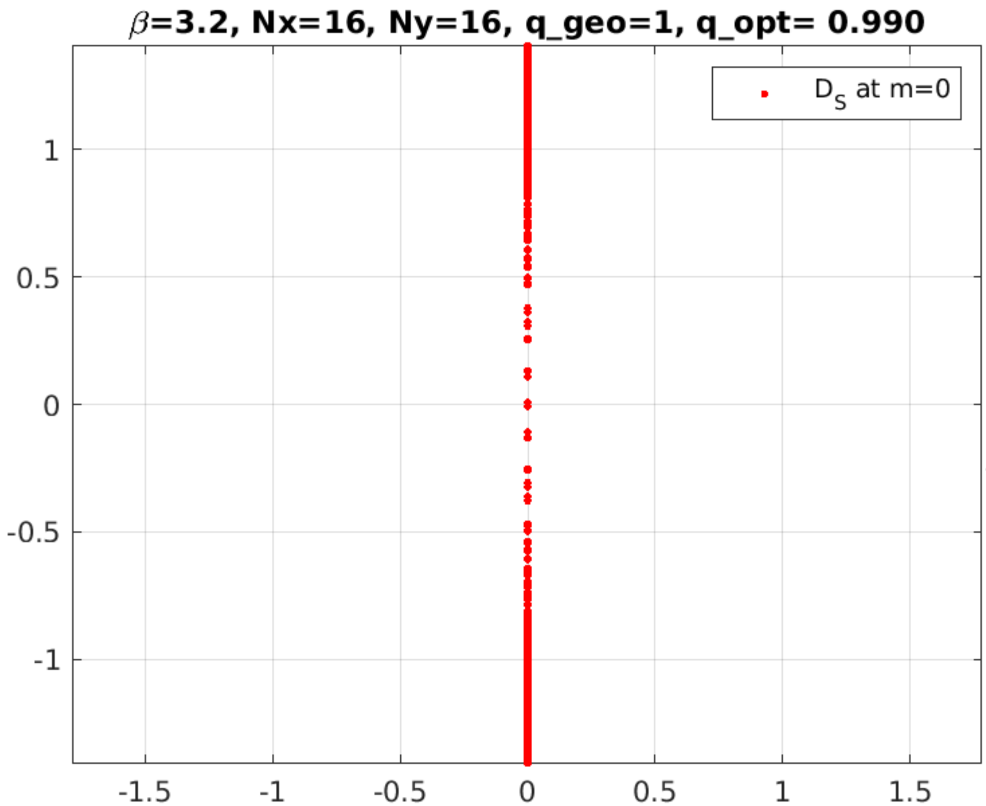
Spectral flow with Brillouin fermions

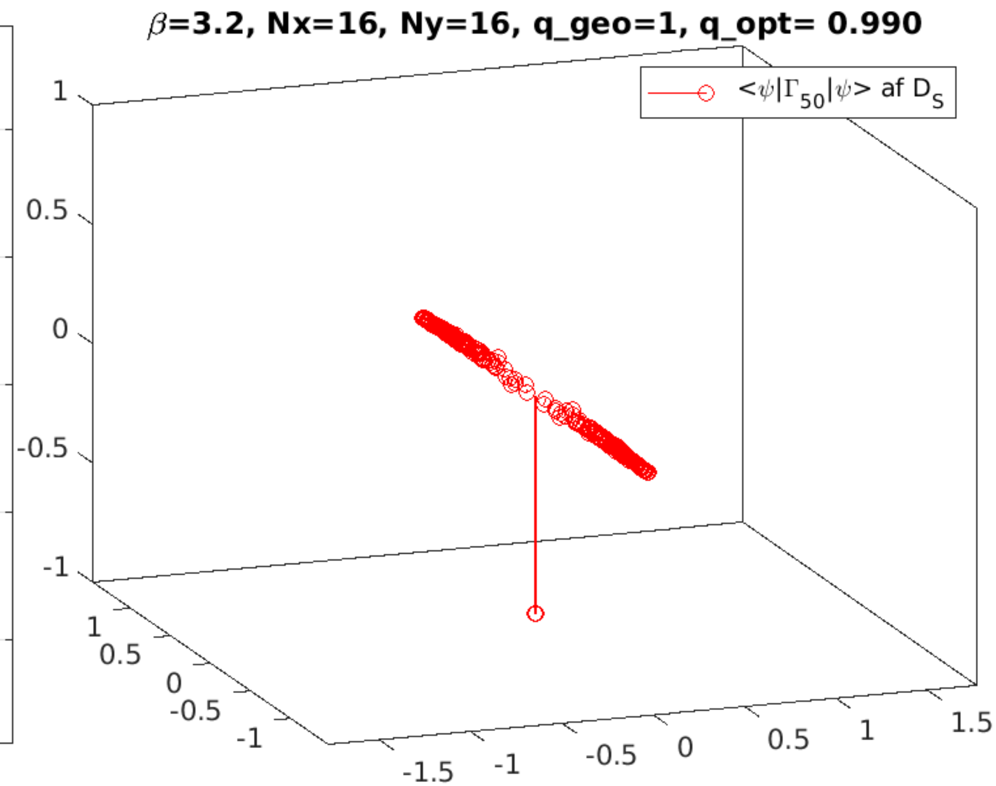




- eigenvalues of $H_\mathrm{B}(m) \equiv \gamma_5(D_\mathrm{B}+m)$ at $r=1$ versus am
- one species at $am \simeq 0$, three at $am \simeq -2$ (competing chiralities)
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- large-scale structure different (small-scale structure similar) to Wilson case

Eigenvalues and topology with staggered fermions





$2|q|$ would-be zero-modes (changes to $4|q|$ in 4D), remnant chiral symmetry $U(1)_\epsilon$

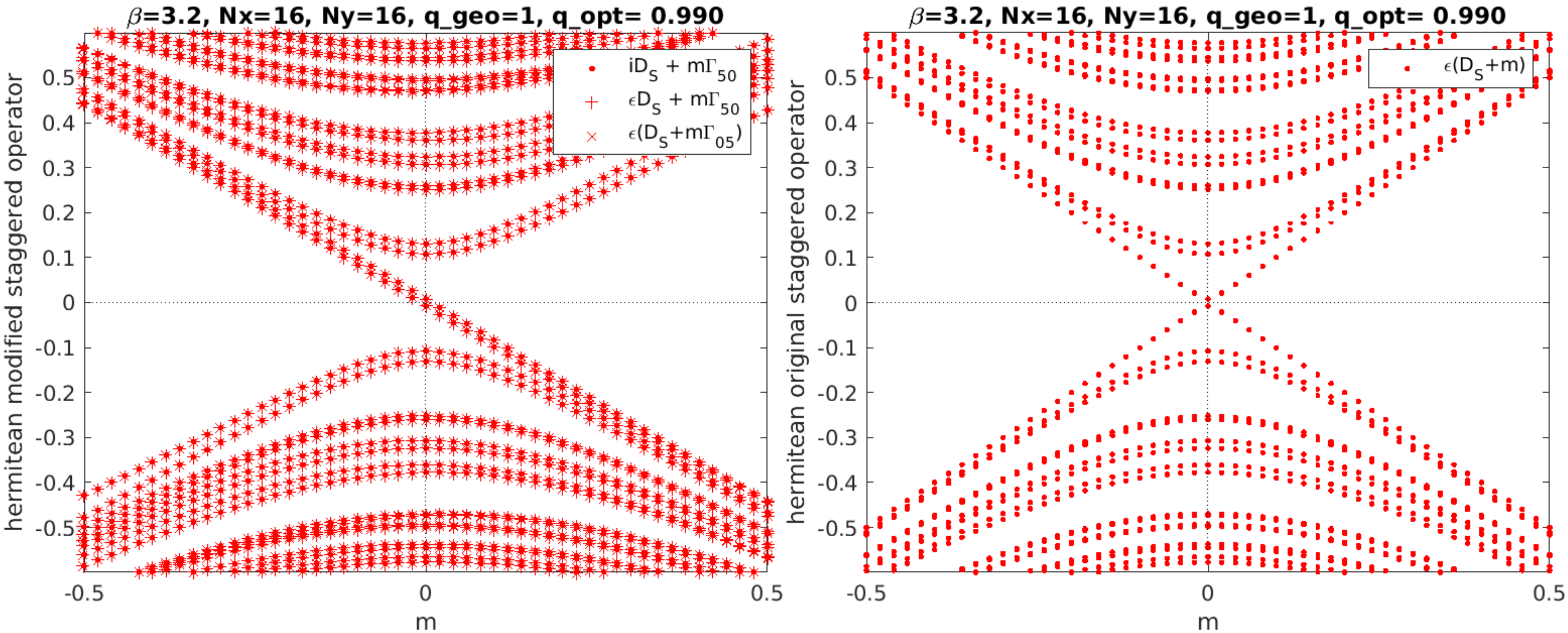
$\epsilon \equiv \gamma_5 \otimes \xi_5$ not sensitive to topology (see "backup pages" for meaning of $\gamma_\mu \otimes \xi_\nu$)

$\Gamma_5 \simeq \gamma_5 \otimes 1$ crafted to "turn around" chirality of second mode (both point down)

$\Xi_5 \simeq 1 \otimes \xi_5$ not sensitive to topology (Γ_5 and Ξ_5 depend on gauge-field U)

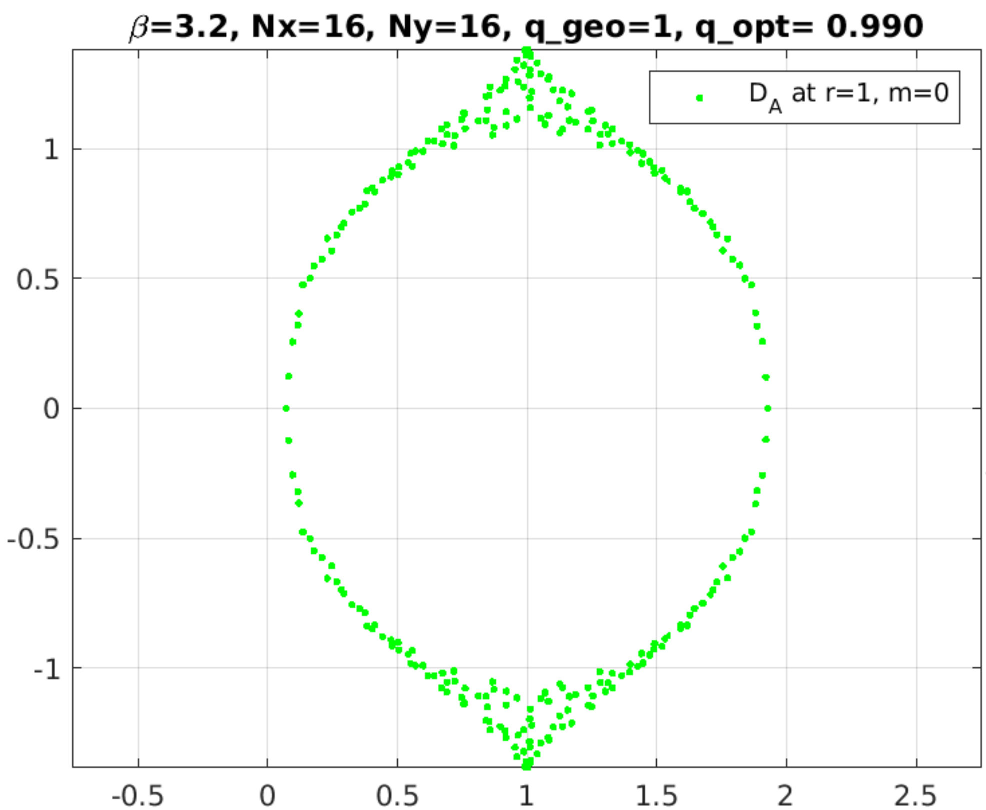
$1 \equiv 1 \otimes 1$ not sensitive to topology (like ϵ not shown)

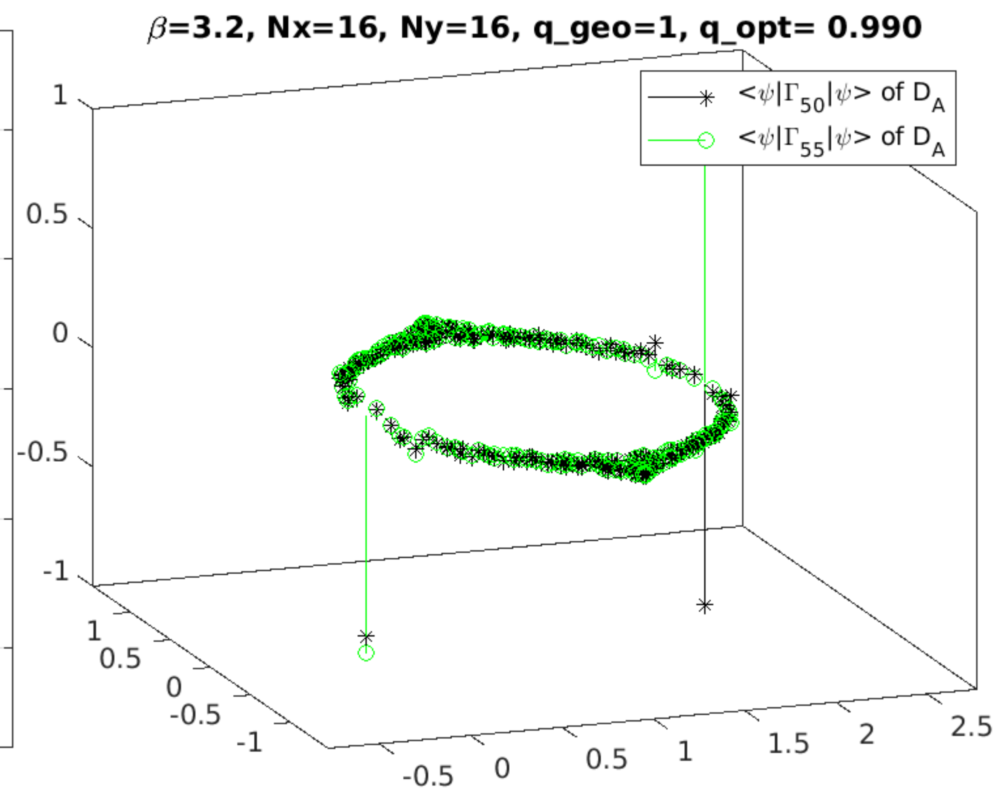
• Spectral flow with staggered fermions



- eigenvalues of $H_\mathrm{S} \equiv \epsilon(D_\mathrm{S} + m\Gamma_{05})$ versus am
- $\epsilon\Gamma_{05} = \Gamma_{50}$ and $\mathrm{spec}(\mathrm{i}D_\mathrm{S}) = \mathrm{spec}(\epsilon D_\mathrm{S})$ allow for reformulations
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- dull choice $\epsilon(D_\mathrm{S} + m)$ amounts to wrong chirality operator

Eigenvalues and topology with Adams fermions





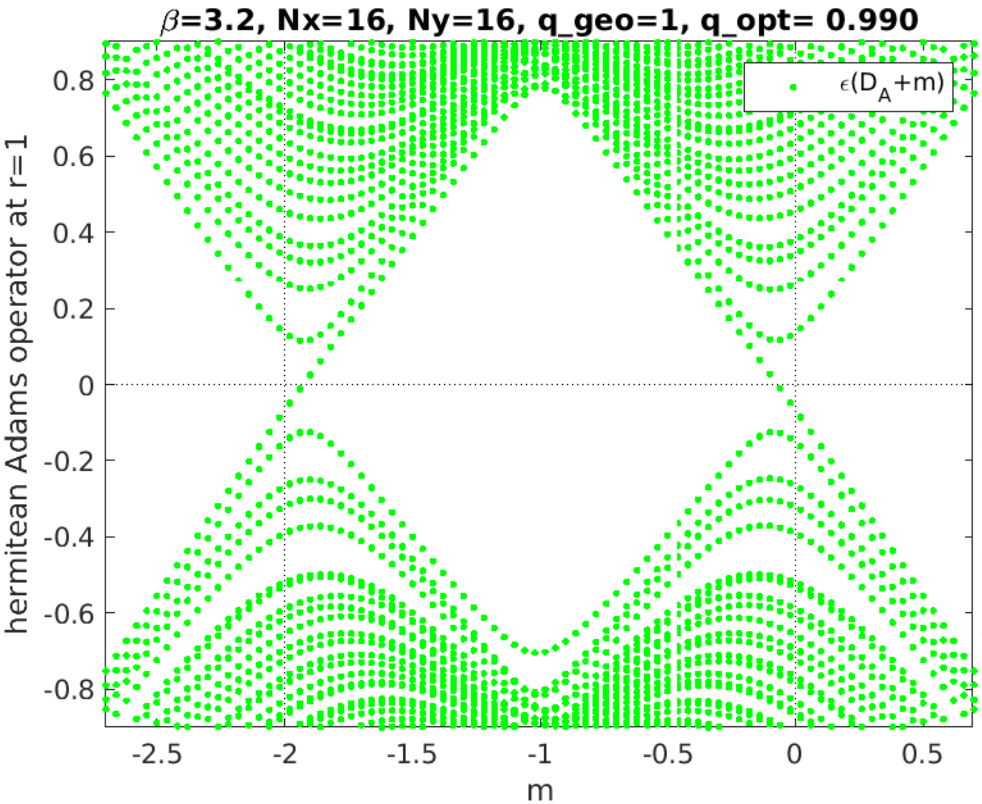
$$D_{\mathrm{Adams}} = \sum_\mu \eta_\mu \nabla_\mu + r[1\otimes\xi_5 + 1\otimes 1] \qquad \begin{pmatrix}\text{respects } \epsilon\text{-hermiticity} \\ \text{breaks chiral symmetry}\end{pmatrix}$$

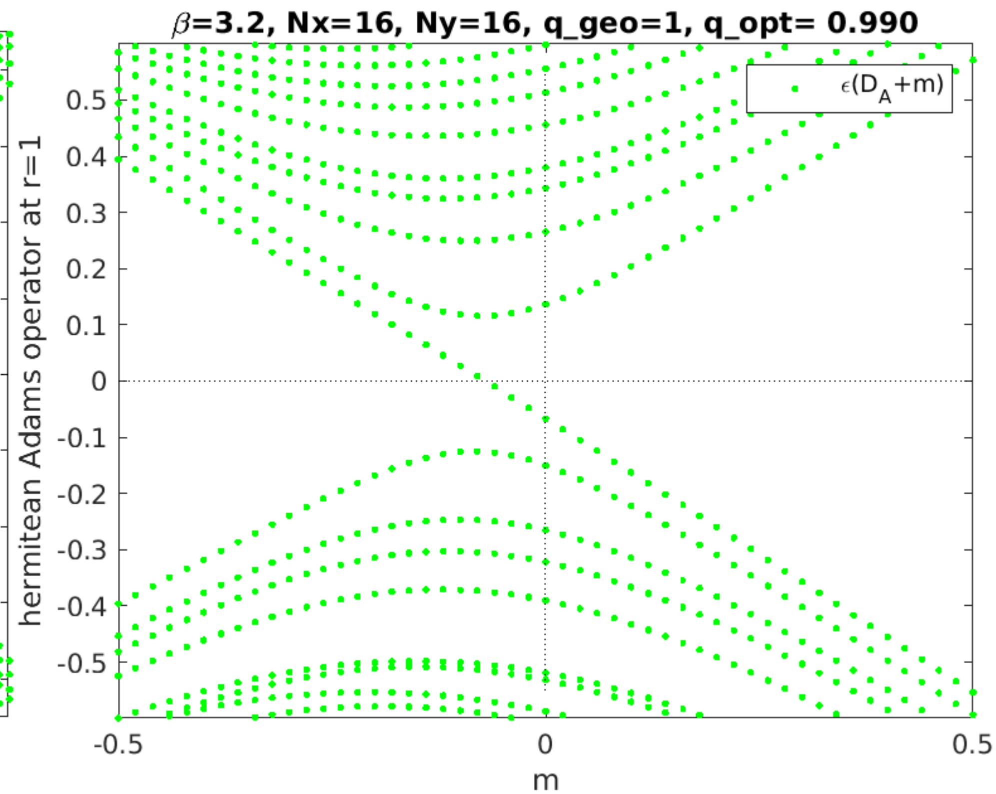
- $|q|$ would-be zero-modes in physical branch (become $2|q|$ in 4D) [Adams 2009]

$\Gamma_5 \simeq \gamma_5 \otimes 1$ produces downward pointing modes (1 physical, 1 doubler)

$\epsilon \equiv \gamma_5 \otimes \gamma_5$ produces oppositely oriented modes (1 physical, 1 doubler)

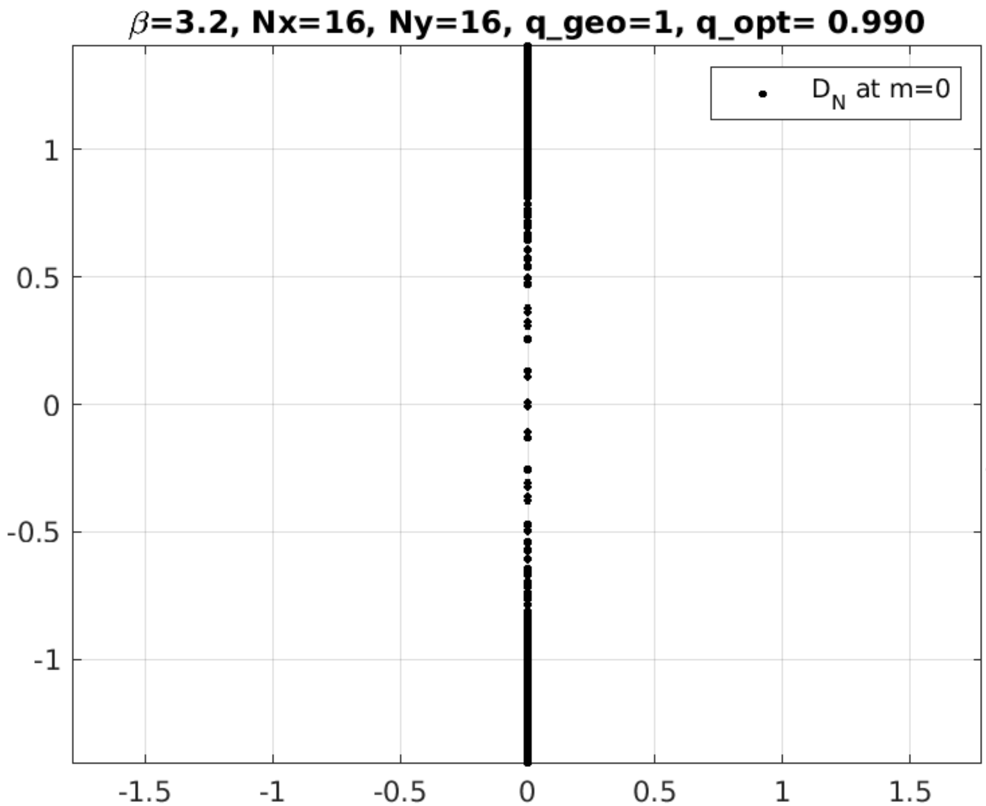
- ## Spectral flow with Adams fermions

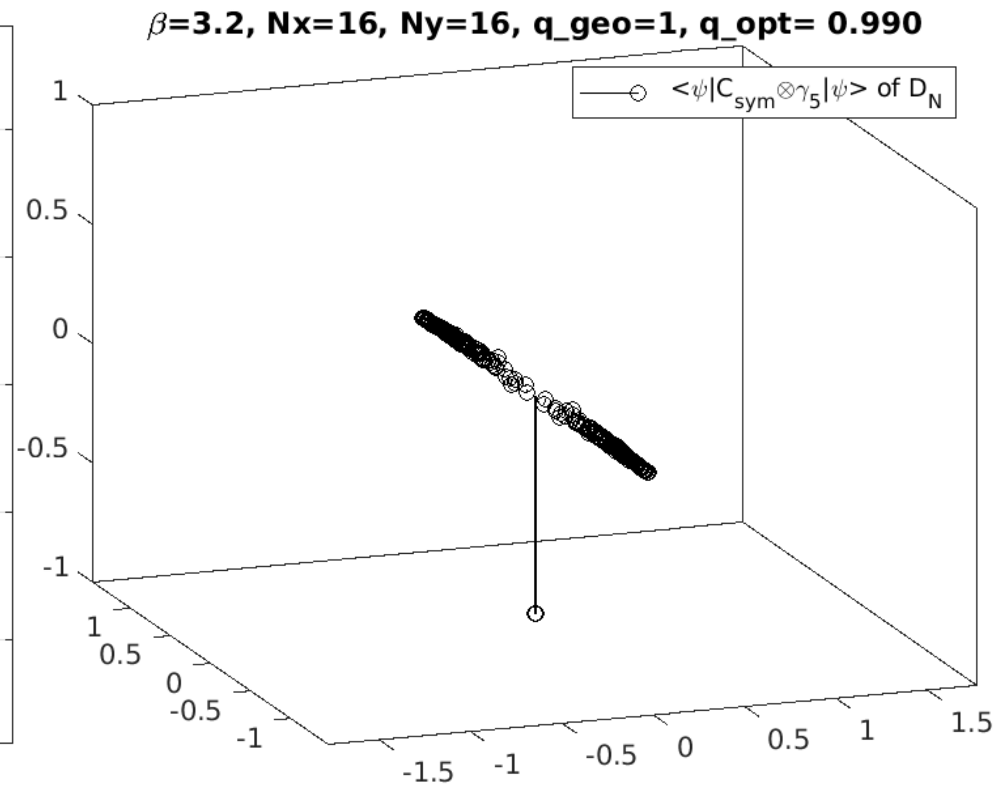




- eigenvalues of $H_{\rm A} \equiv \epsilon(D_{\rm A}+m)$ versus am
- one species at $am \simeq 0$, one at $am \simeq -2$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down/up)
- ingenuity of Adams construction allows $\epsilon(D_{\rm A}+m)$ with standard chirality operator

Eigenvalues and topology with naive fermions

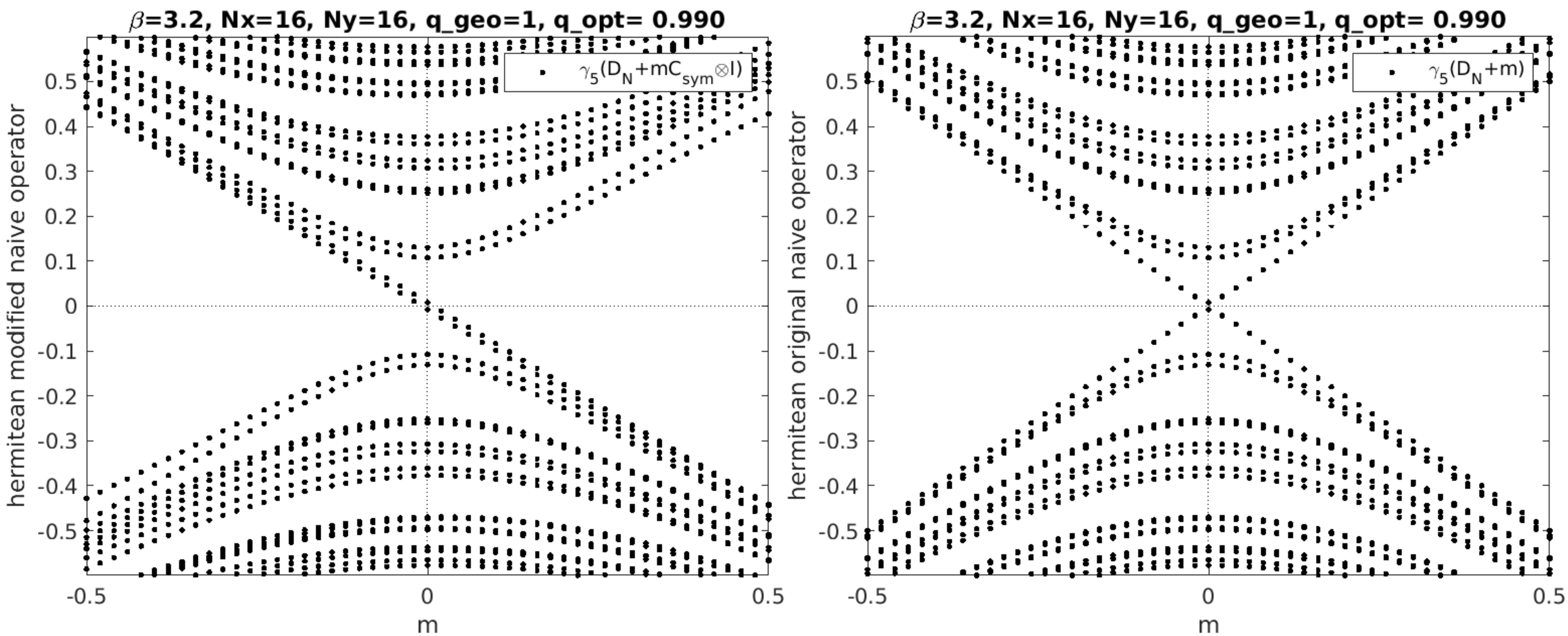




$$D_{\text{naive}} = \sum_\mu \eta_\mu \nabla_\mu$$

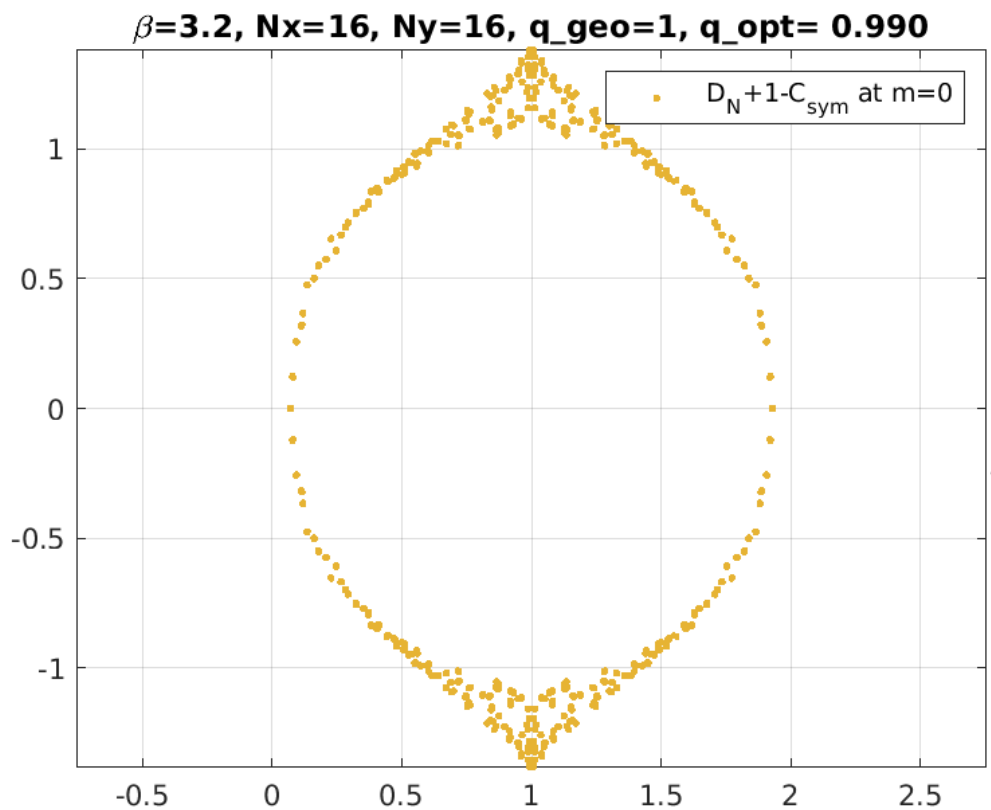
- eigenvalue spectrum like staggered, but 2-fold extra degeneracy (4-fold in 4D)
- γ_5-chiralities exactly zero (like ϵ-chiralities for staggered)
- suitable chirality operator is $X = C_{\text{sym}} \otimes \gamma_5$ (2 needles down)

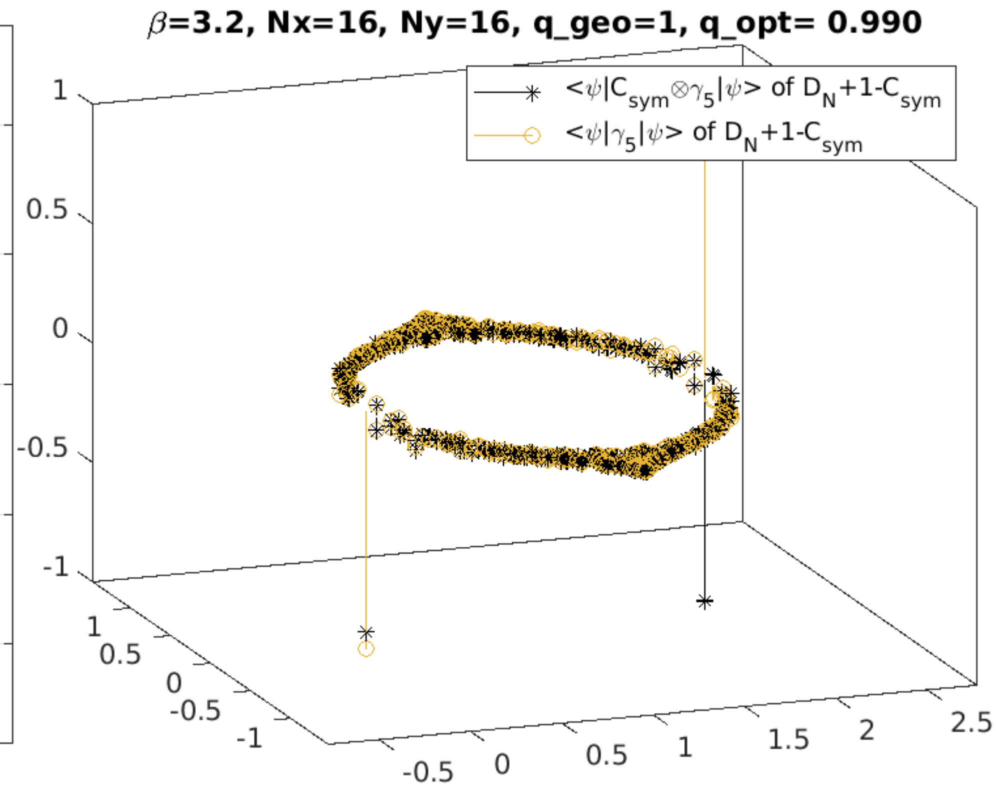
- **Spectral flow with naive fermions**



- eigenvalues of $H_\mathrm{N} \equiv \gamma_5(D_\mathrm{N} + mC_\mathrm{sym} \otimes 1)$ versus am
- choice matches $X = C_\mathrm{sym} \otimes \gamma_5$ being a good chirality operator for D_N
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- dull choice $\gamma_5(D_\mathrm{N} + m)$ amounts to wrong chirality operator

Eigenvalues and topology with naive+lifting fermions





$$D_{\mathrm{A-like}} = \sum_\mu \eta_\mu \nabla_\mu + r[1 - C_{\mathrm{sym}}]$$

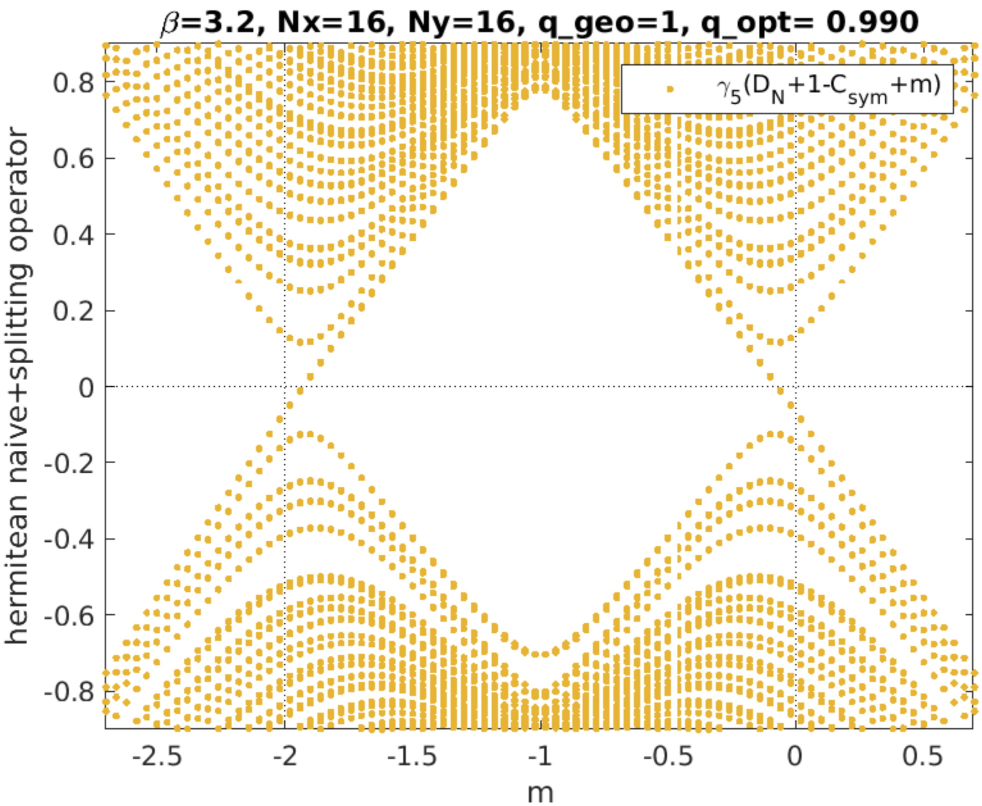
(respects γ_5-hermiticity
breaks chiral symmetry)

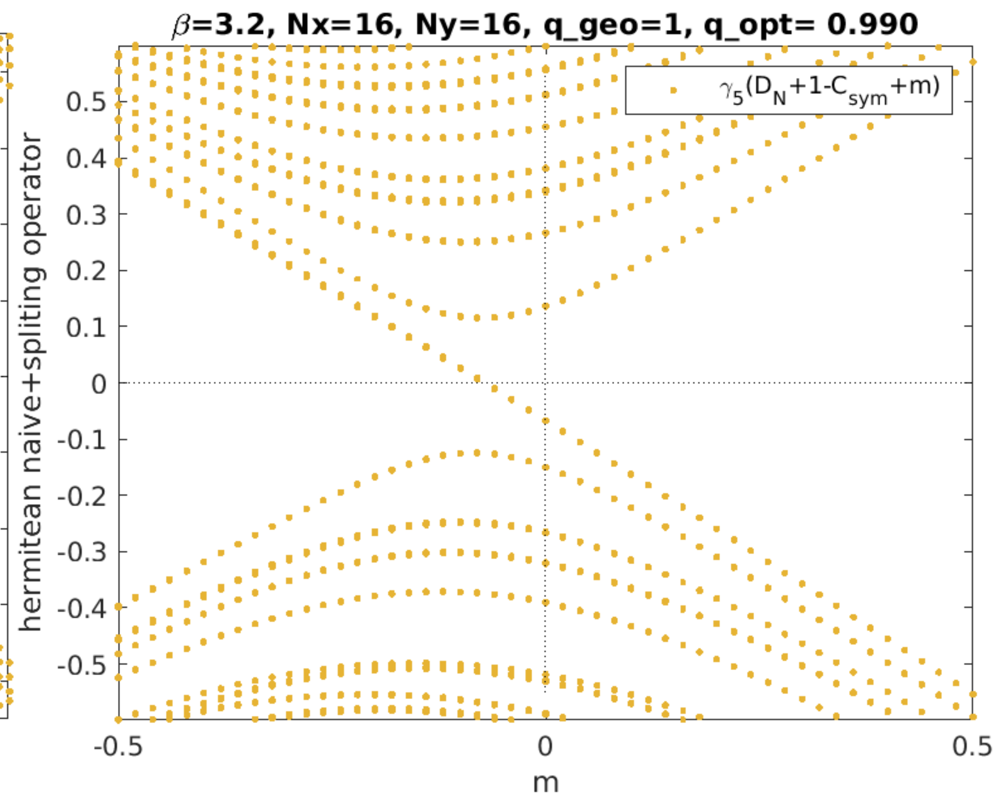
- $2|q|$ would-be zero-modes in physical branch (become $8|q|$ in 4D)

$C_{\mathrm{sym}} \otimes \gamma_5$ produces downward pointing modes (2 physical, 2 doubler)

$1 \otimes \gamma_5$ produces oppositely oriented modes (2 physical, 2 doubler)

- Spectral flow with naive+lifting (Adams-like) fermions

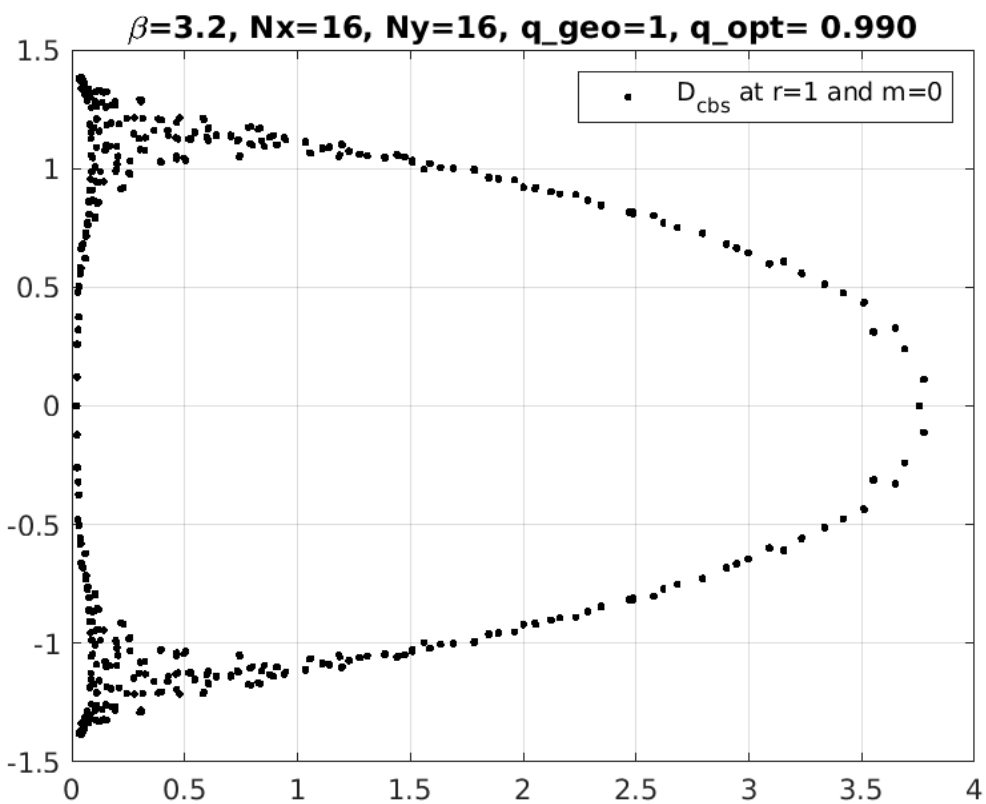


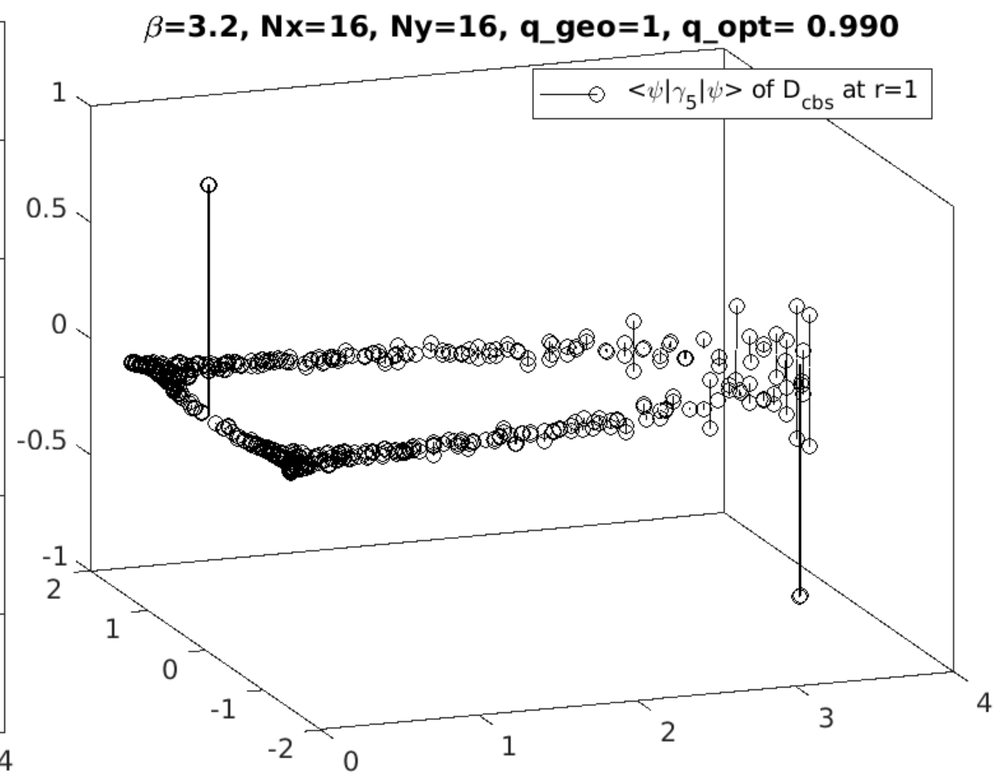


- eigenvalues of $H_{\mathrm{A-like}} \equiv \gamma_5(D_{\mathrm{A-like}} + m)$ versus am
- two species at $am \simeq 0$, two at $am \simeq -2$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down/up)
- ingenuity of Adams construction allows $\gamma_5(D_{\mathrm{A-like}}+m)$ with standard γ_5-operator

Eigenvalues and topology with cbs fermions

Aid to memory: "central branch squared" (only sum of $C_\mu = \frac{1}{2}\triangle_\mu + 1$ is squared)

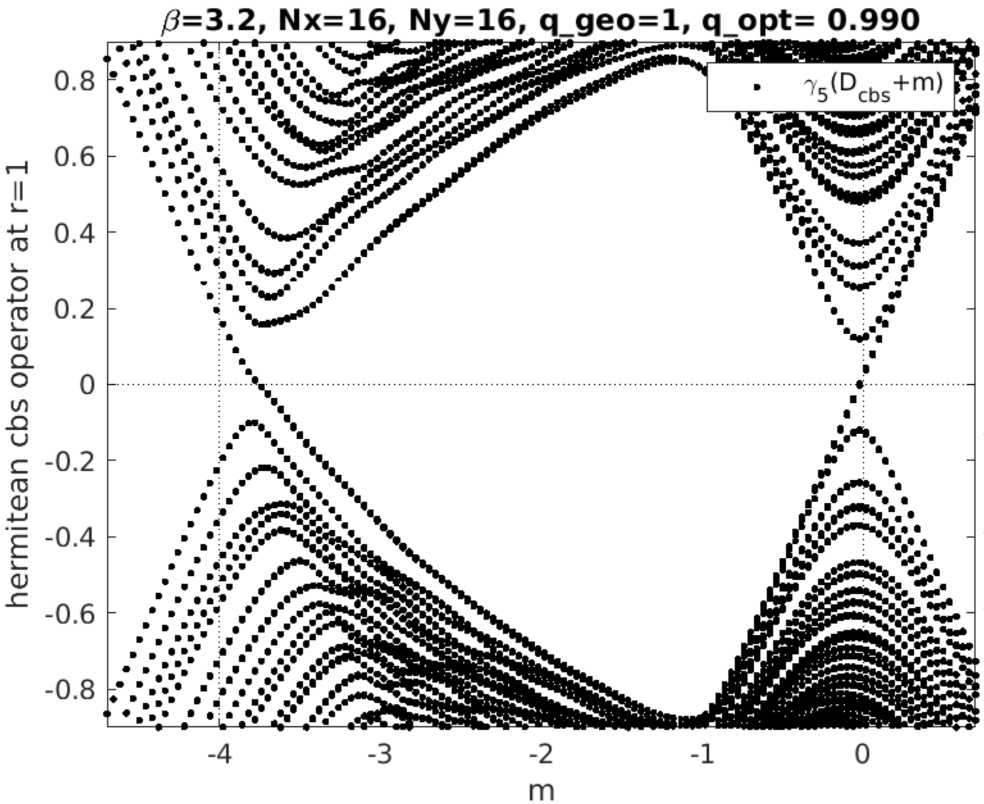


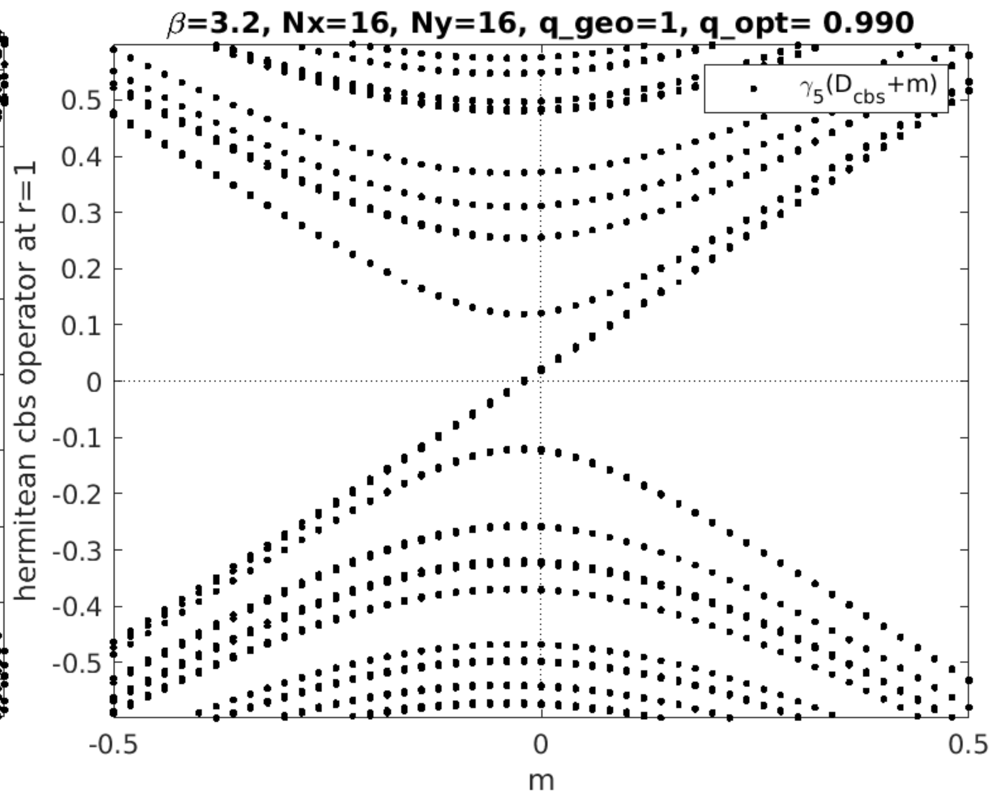


$$D_{\mathrm{cbs}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) + \frac{r}{a}\Big[-\frac{a^2}{2}\sum_\mu \triangle_\mu - dI\Big]^2_{x,y}$$

- from central-branch perspective, left-doubler and right-doubler collapse
- $2|q|$ would-be zero-modes in physical branch (become $6|q|$ in 4D)

• Spectral flow with cbs fermions

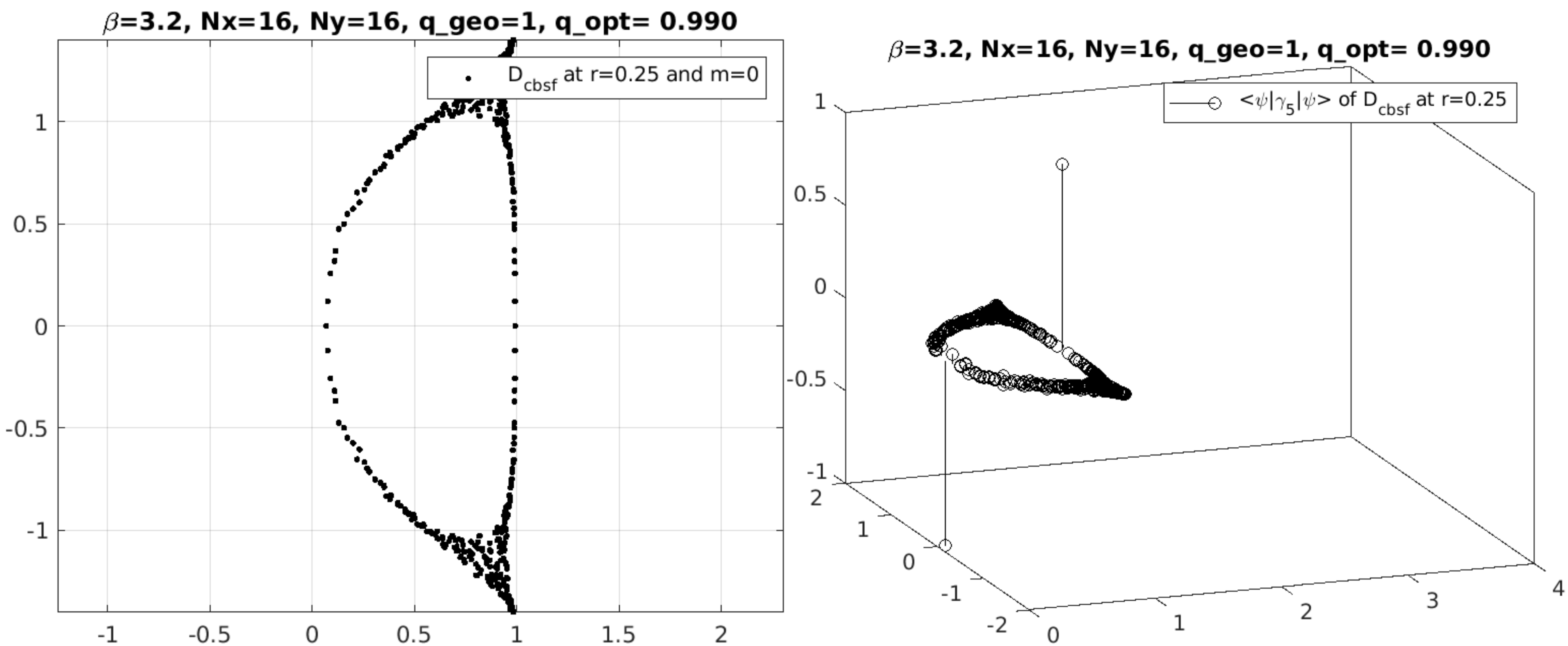




- eigenvalues of $\gamma_5(D_{\text{cbs}} + m)$ versus am
- two species at $am \simeq 0$, two at $am \simeq -2$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle up/down)
- chiralities interchanged w.r.t. Wilson et al (no problem)
- additive mass shift significantly smaller than for Wilson/Brillouin fermions

Eigenvalues and topology with cbsf fermions

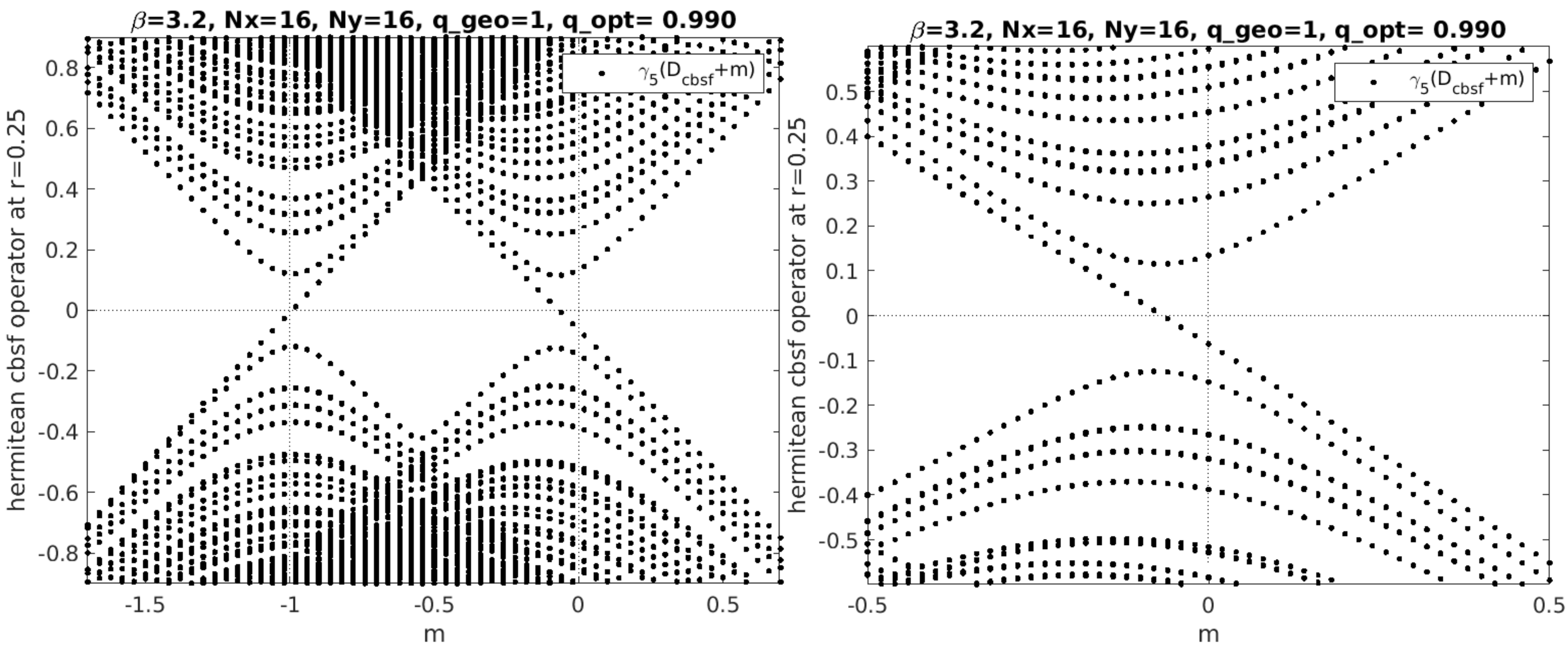
Aid to memory: "central branch squared and flipped" (only $C_\mu = \frac{1}{2}\triangle_\mu + 1$ is squared)



$$D_{\mathrm{cbsf}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) + \frac{r}{a} d^2 I_{x,y} - \frac{r}{a}\Big[-\frac{a^2}{2}\sum_\mu \triangle_\mu - dI\Big]^2_{x,y}$$

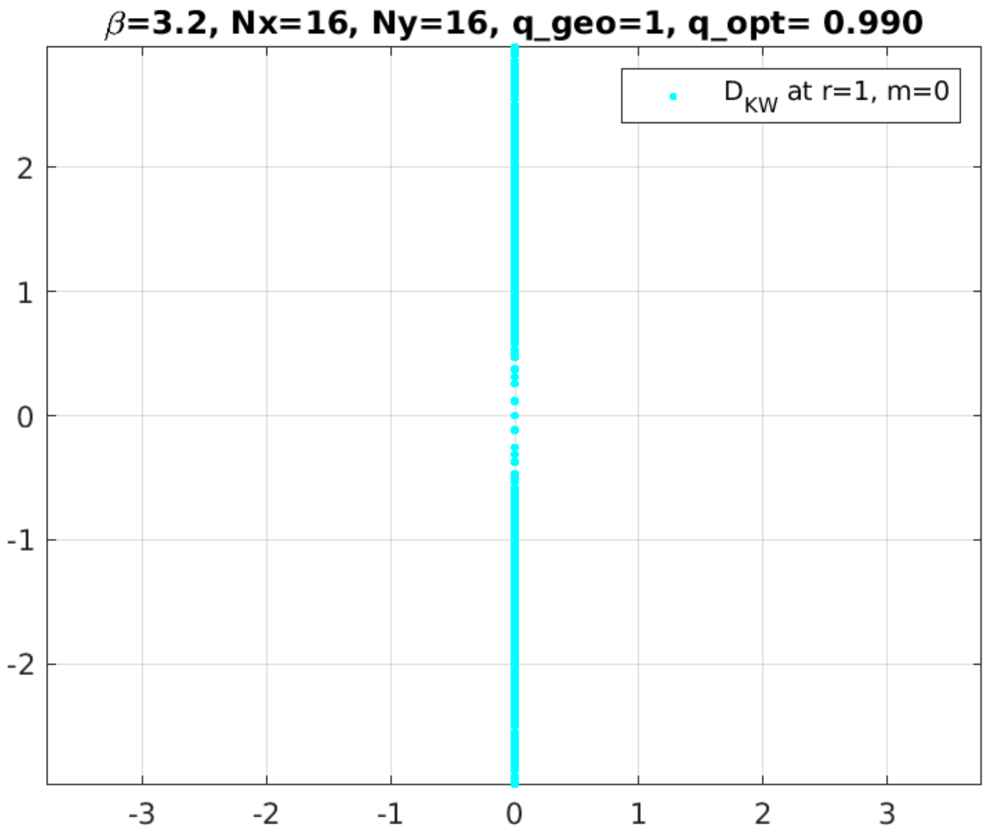
- same as central-branch, just left/right branches flipped
- $2|q|$ would-be zero-modes in physical branch (remain $2|q|$ in 4D)

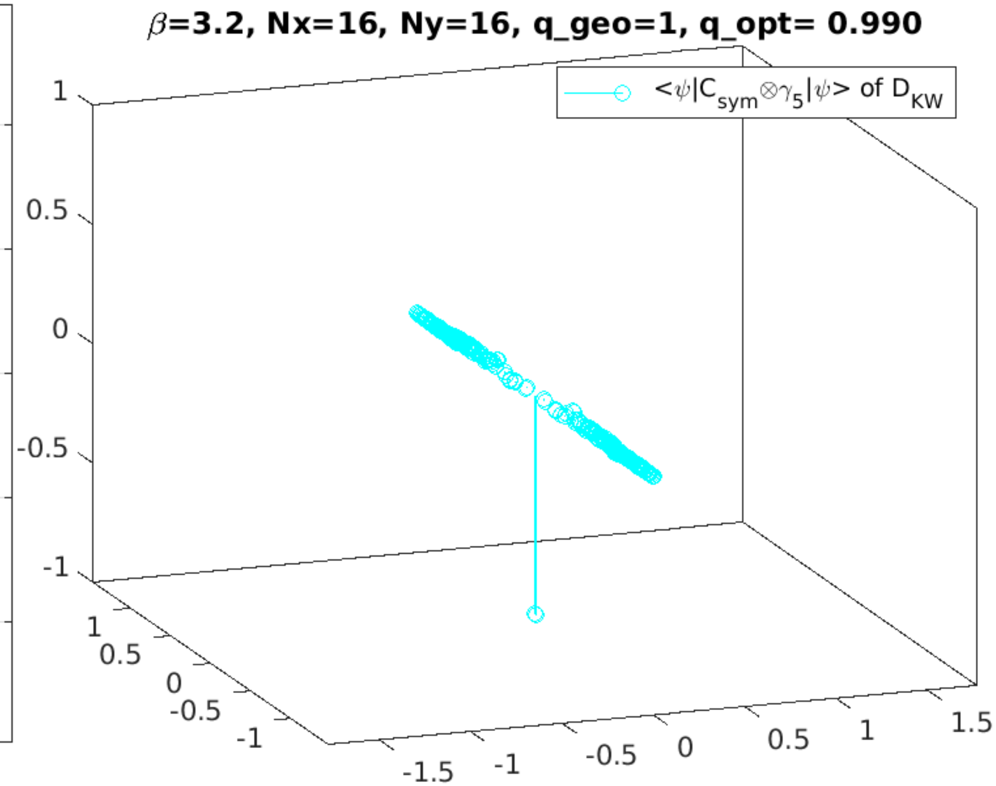
- **Spectral flow with cbsf fermions**



- eigenvalues of $\gamma_5(D_{\text{cbsf}} + m)$ versus am
- two species at $am \simeq 0$, two at $am \simeq -2$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down/up)
- chiralities back to Wilson et al
- additive mass shift comparable to Wilson/Brillouin fermions

Eigenvalues and topology with Karsten-Wilczek fermions



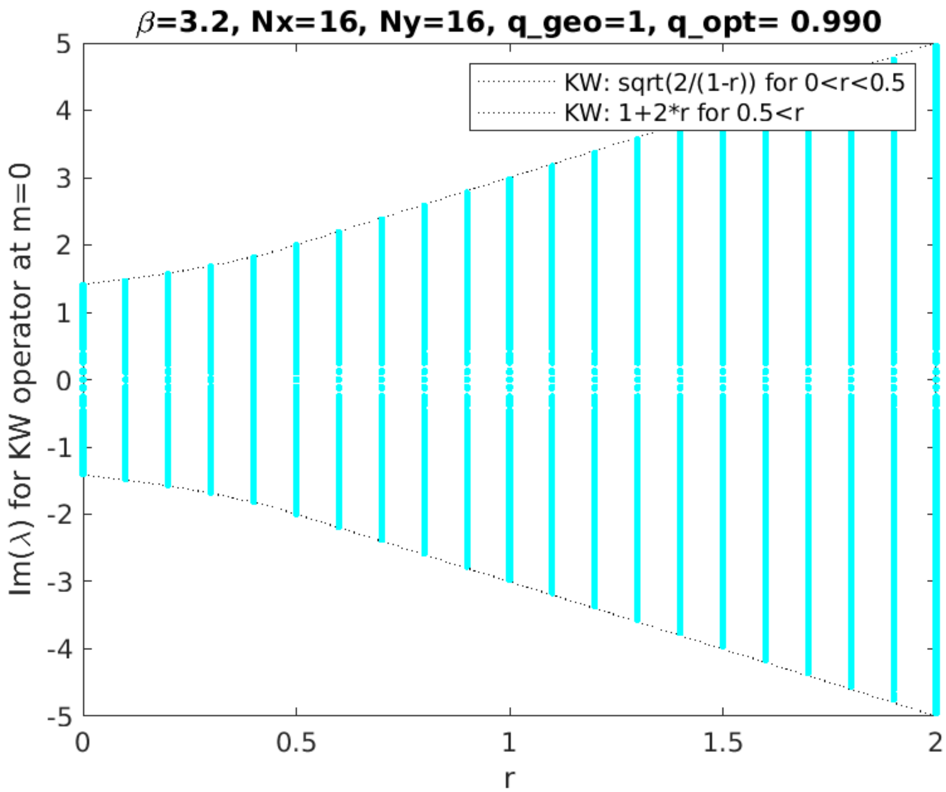


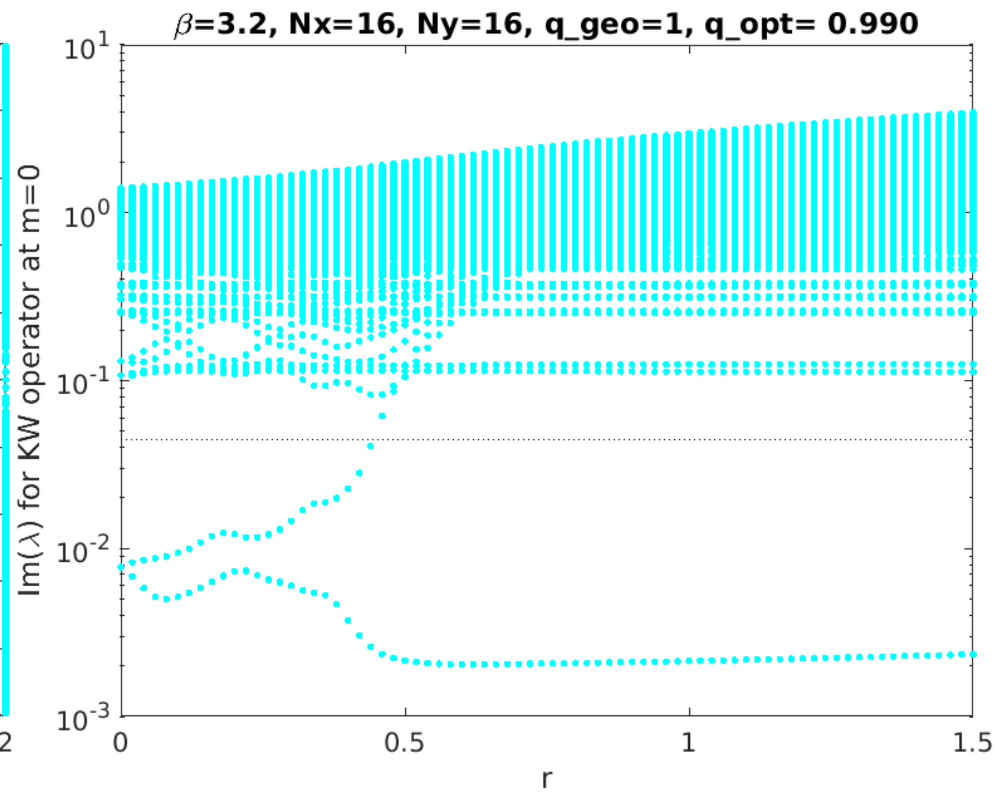
- $2|q|$ would-be zero-modes at $r = 1$ (unchanged in 4D), remnant chiral symmetry
- pertinent L/R-eigenmodes of $D_{\rm KW}$ not sensitive to γ_5

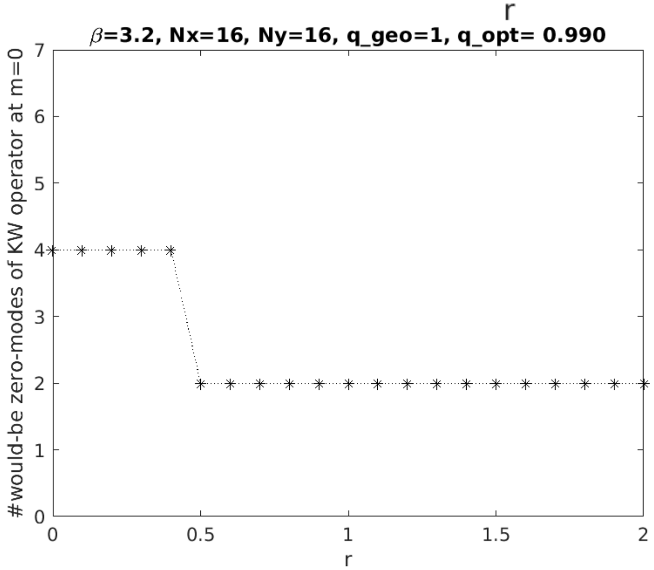
Operator X can be crafted to have $\langle L|X|R\rangle \neq 0$ with L/R-eigenmodes of $D_{\rm KW}$

Options are $X = \frac{1}{2}(C_1 + C_2)^2 \otimes \gamma_5$ and $X = C_{\rm sym} \otimes \gamma_5$ with $C_\mu \equiv \frac{1}{2}\triangle_\mu + 1$

- **Transition $D_{\text{naive}} \to D_{\text{KW}}$ as a function of r on $|q|=1$ configuration**







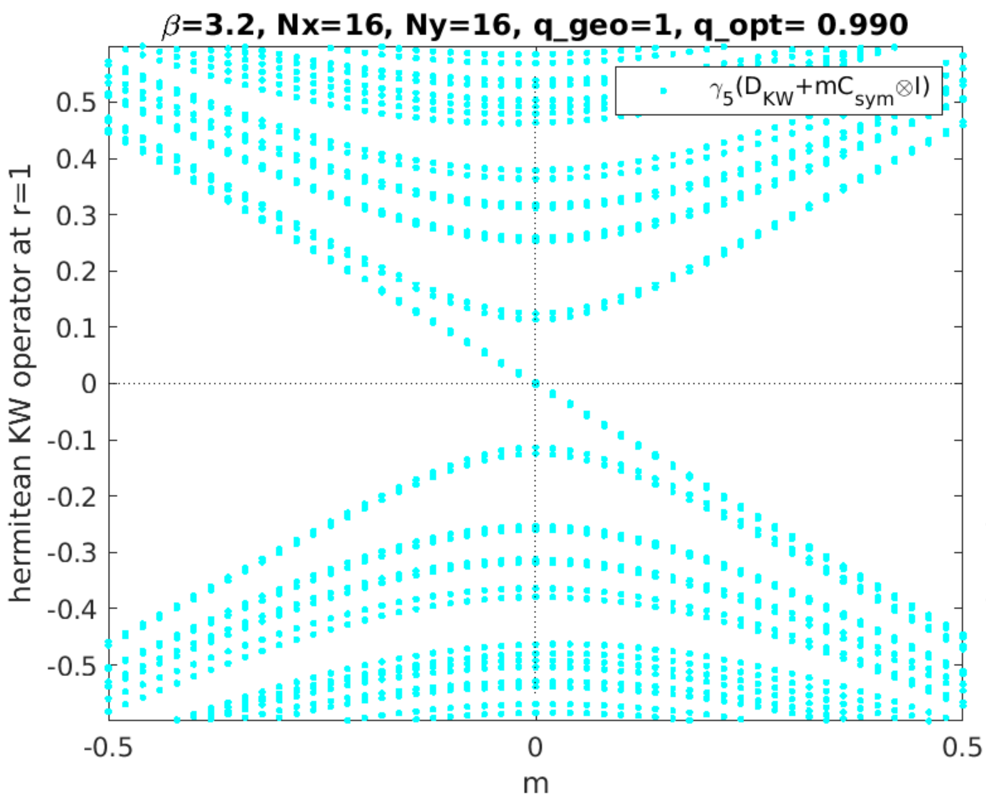
Findings in [2203.15699]:

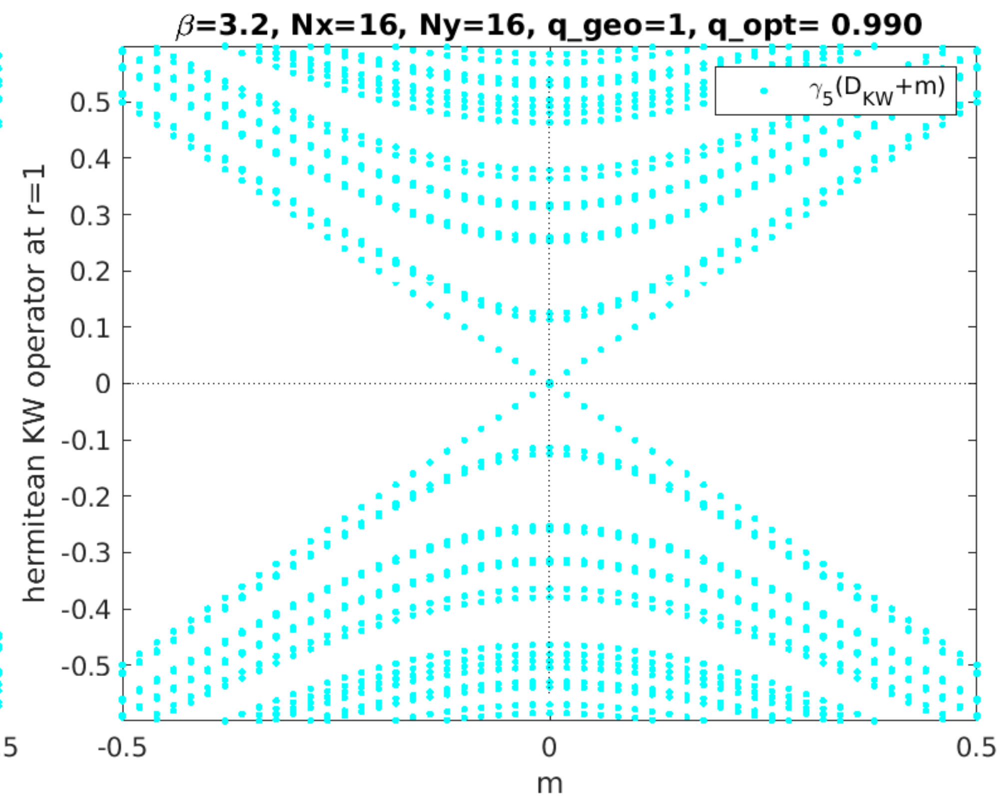
$\mathrm{Im}(\lambda_{\mathrm{KW}}(r))$ nearly saturates KW free-field bound

KW species chain in 2D is $4 \to 2$ (transition at $r = 0.5$)

number of would-be zero-modes evolves as $4|q| \to 2|q|$

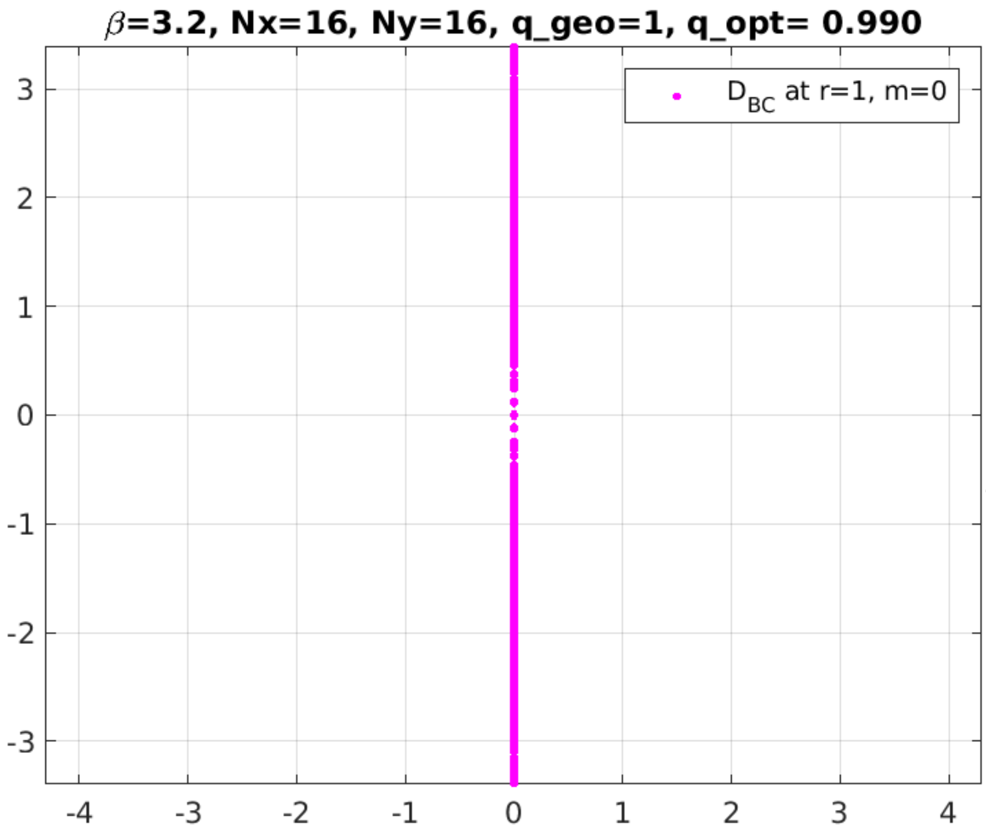
• Spectral flow with Karsten-Wilczek fermions

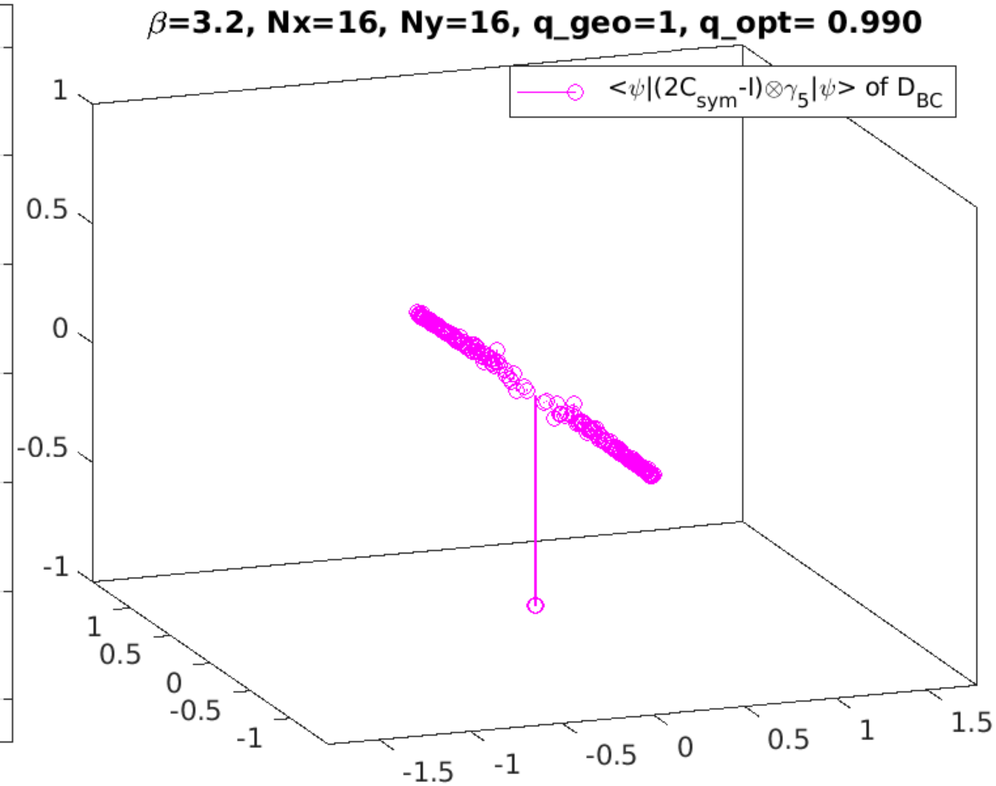




- eigenvalues of $H_{\mathrm{KW}} \equiv \gamma_5(D_{\mathrm{KW}} + mC_{\mathrm{sym}} \otimes 1)$ versus am
- choice matches $X = C_{\mathrm{sym}} \otimes \gamma_5$ being a good chirality operator for D_{KW}
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- near-degeneracy much better than for BC fermions (cf. below)
- dull choice $\gamma_5(D_{\mathrm{KW}} + m)$ amounts to wrong chirality operator

Eigenvalues and topology with Borici-Creutz fermions



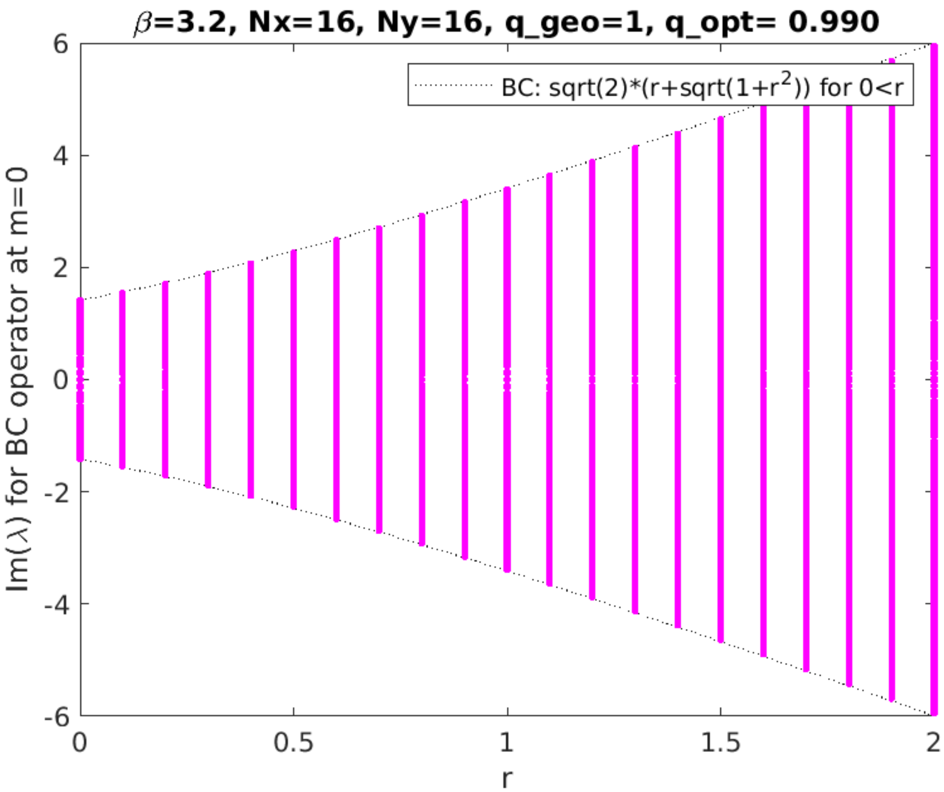


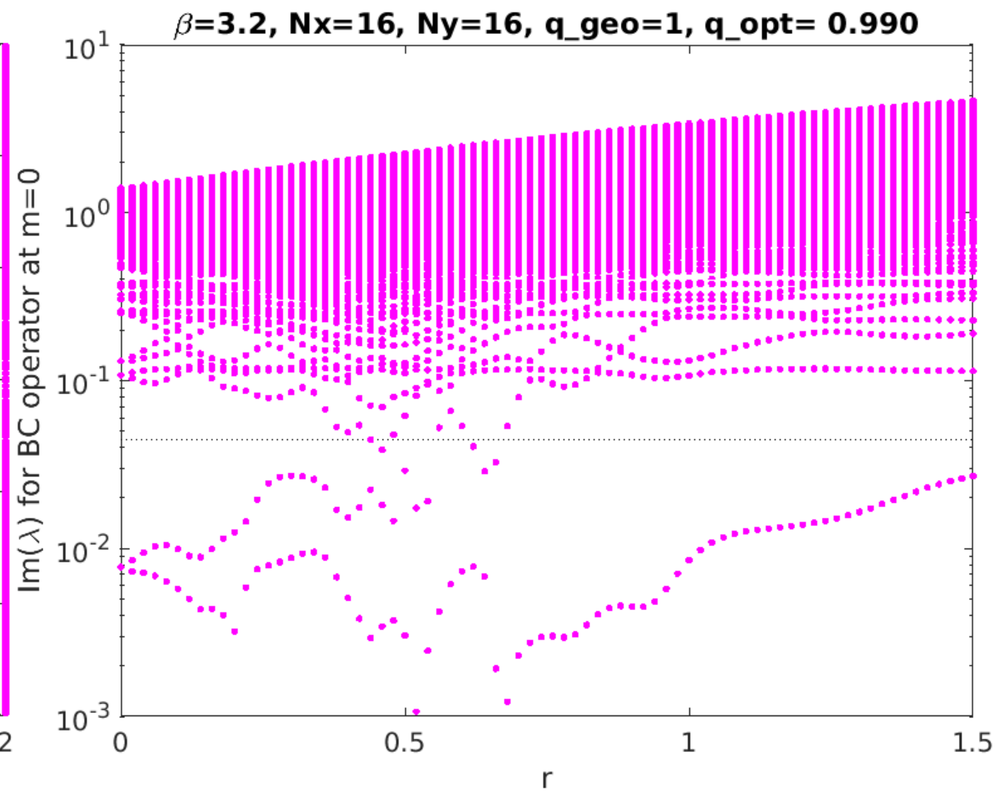
- $2|q|$ would-be zero-modes at $r=1$ (unchanged in 4D), remnant chiral symmetry
- pertinent L/R-eigenmodes of $D_{\rm BC}$ not sensitive to γ_5 (not shown)

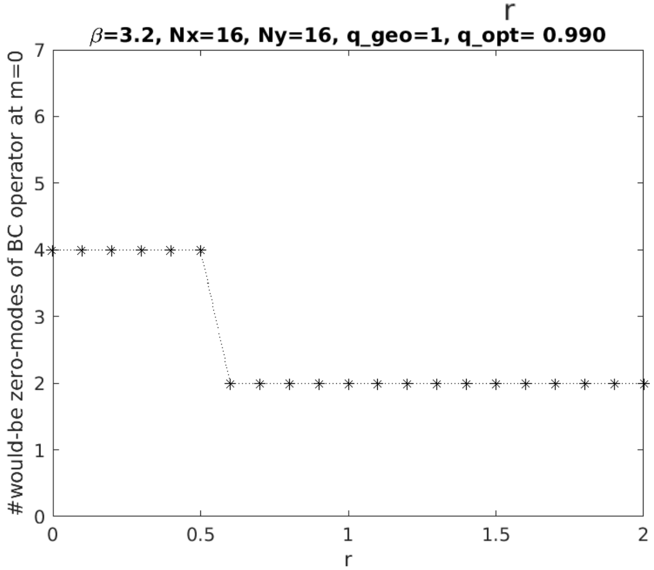
Operator X can be crafted to have $\langle L|X|R\rangle \neq 0$ with L/R-eigenmodes of $D_{\rm BC}$

Options are $X = \frac{1}{2}(C_1 + C_2)^2 \otimes \gamma_5$ and $X = (2C_{\rm sym} - 1) \otimes \gamma_5$ with $C_\mu \equiv \frac{1}{2}\triangle_\mu + 1$

- **Transition $D_{\text{naive}} \to D_{\text{BC}}$ as a function of r on $|q|=1$ configuration**







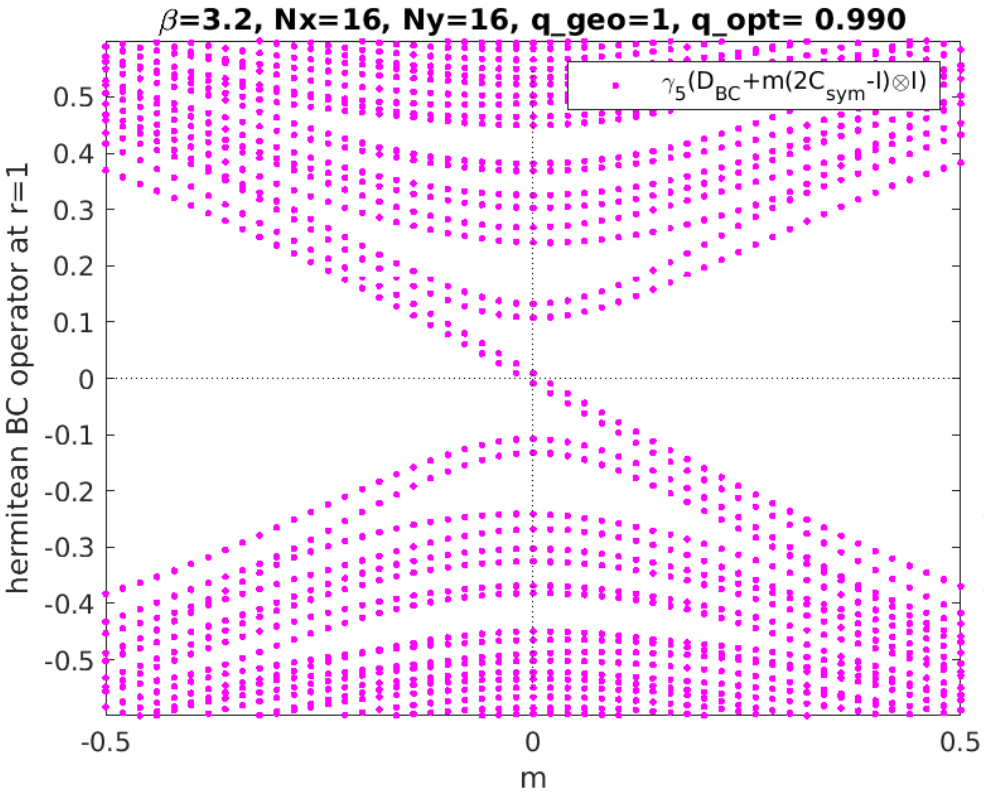
Findings in [2203.15699]:

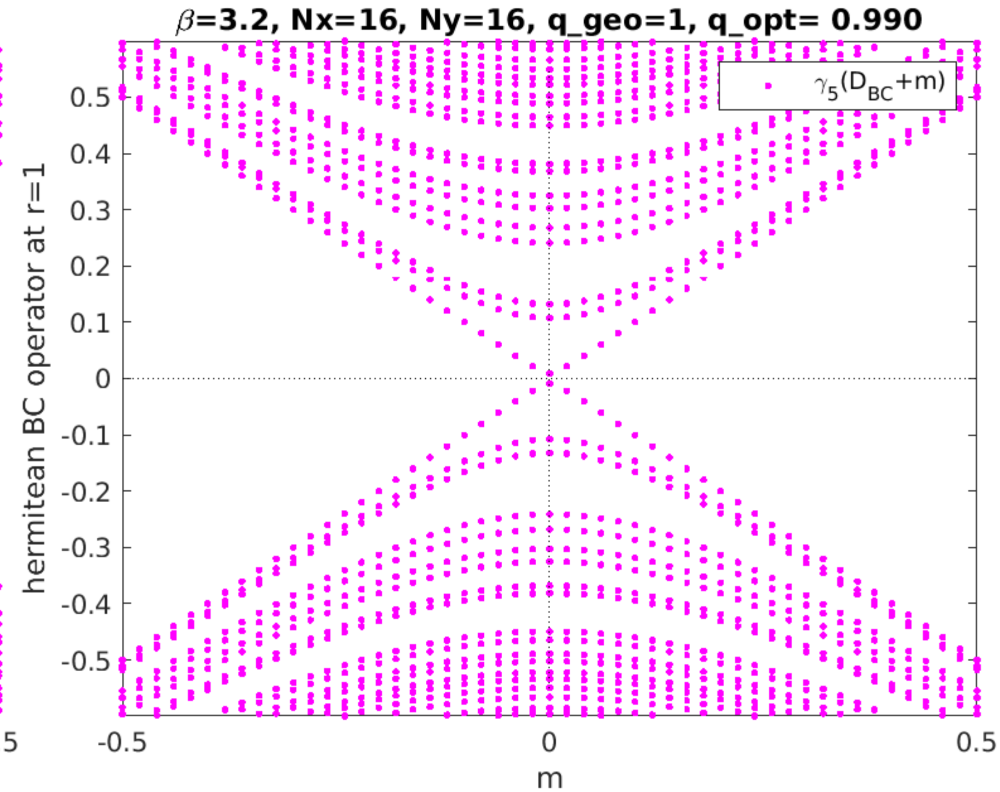
$\mathrm{Im}(\lambda_{\mathrm{KW}}(r))$ nearly saturatess BC free-field bound

KW species chain in 2D is $4 \to 2$ (transition at $r = \frac{1}{\sqrt{3}}$)

number of would-be zero-modes evolves as $4|q| \to 2|q|$

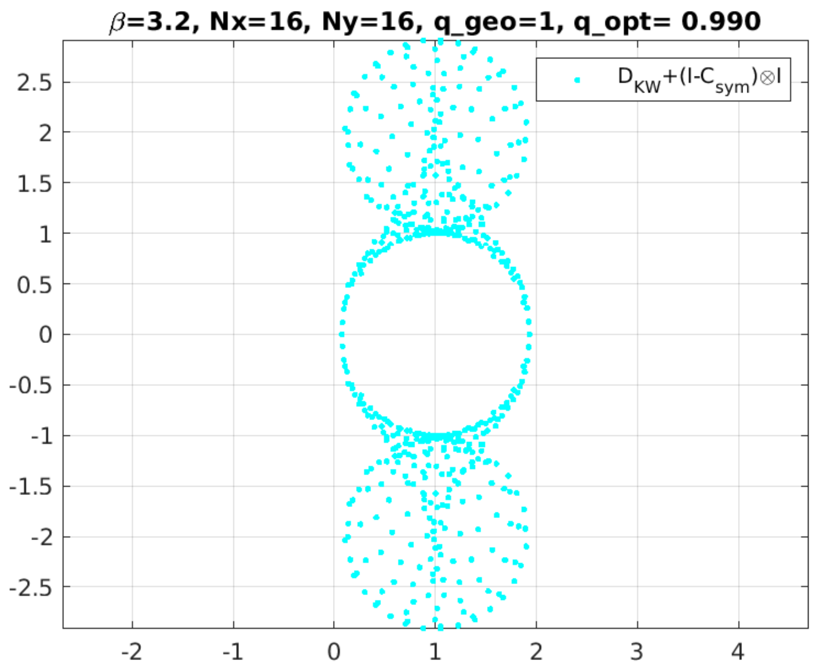
• Spectral flow with Borici-Creutz fermions

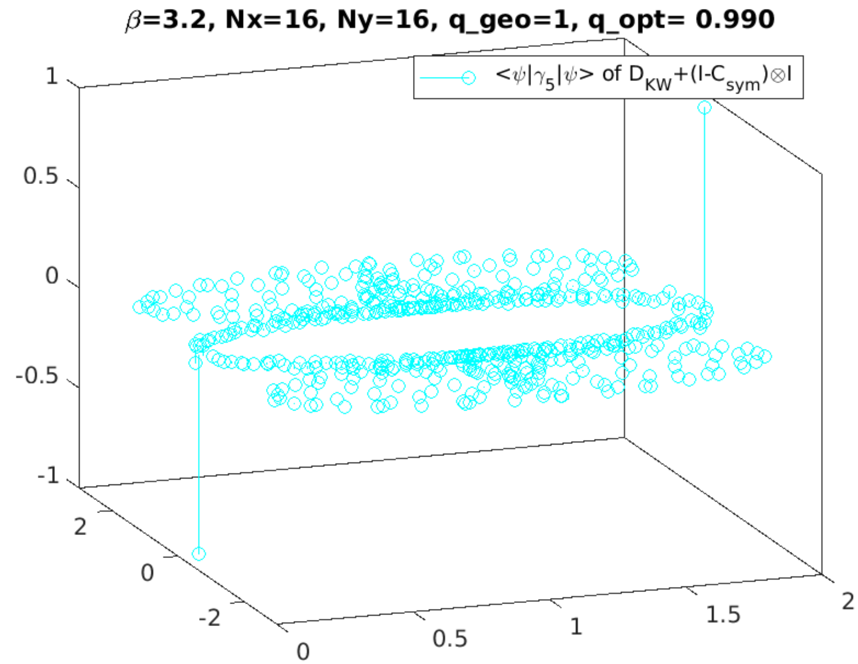


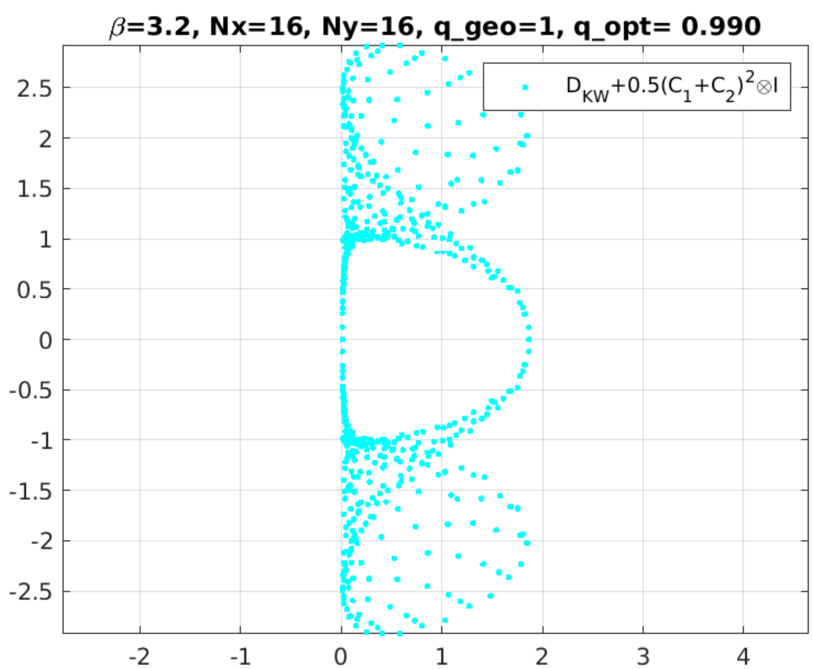


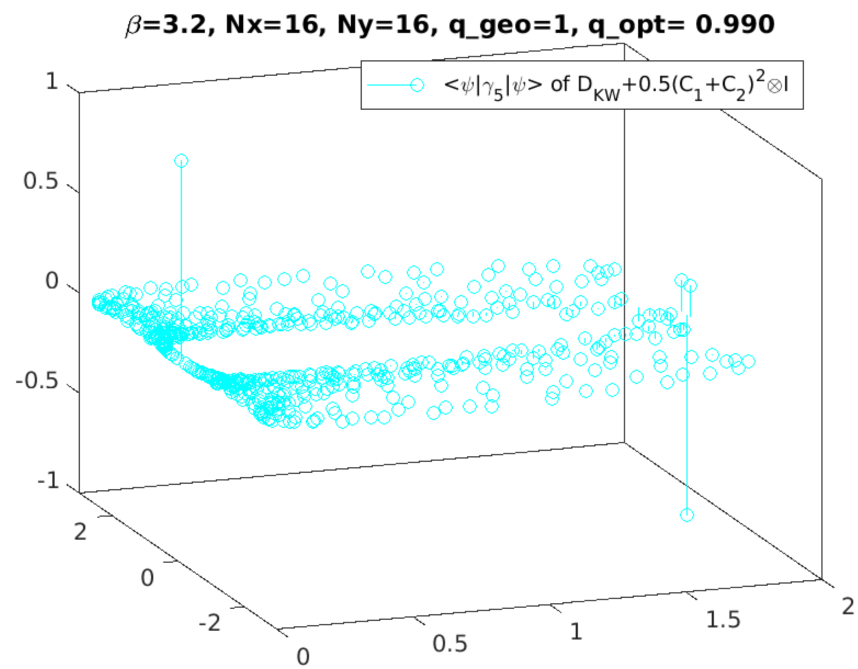
- eigenvalues of $H_{\rm BC} \equiv \gamma_5(D_{\rm BC} + m[2C_{\rm sym} - 1] \otimes 1)$ versus am
- choice matches $X = [2C_{\rm sym} - 1] \otimes \gamma_5$ being a good chirality operator for $D_{\rm BC}$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- near-degeneracy much worse than for KW fermions (cf. above)
- dull choice $\gamma_5(D_{\rm BC} + m)$ amounts to wrong chirality operator

Eigenvalues and topology with KW fermions plus lifting-term

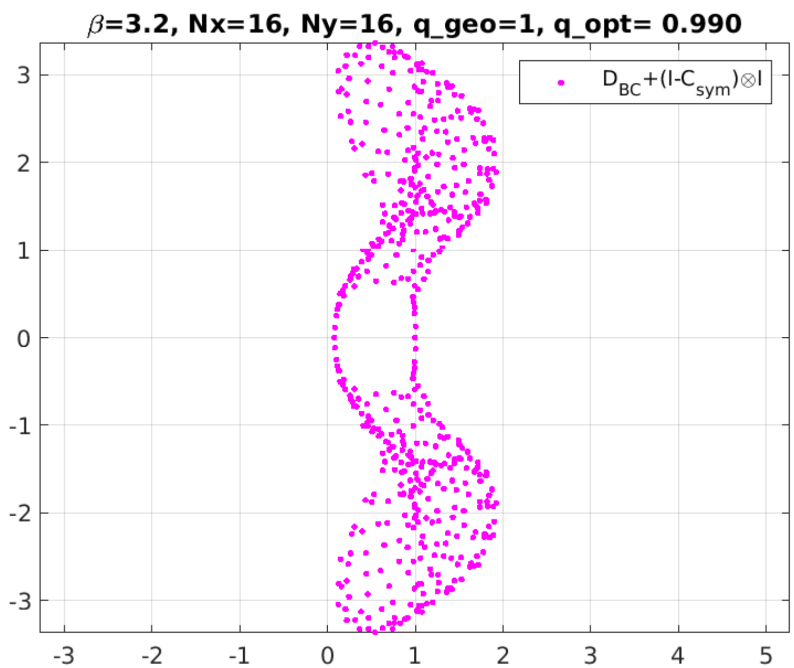


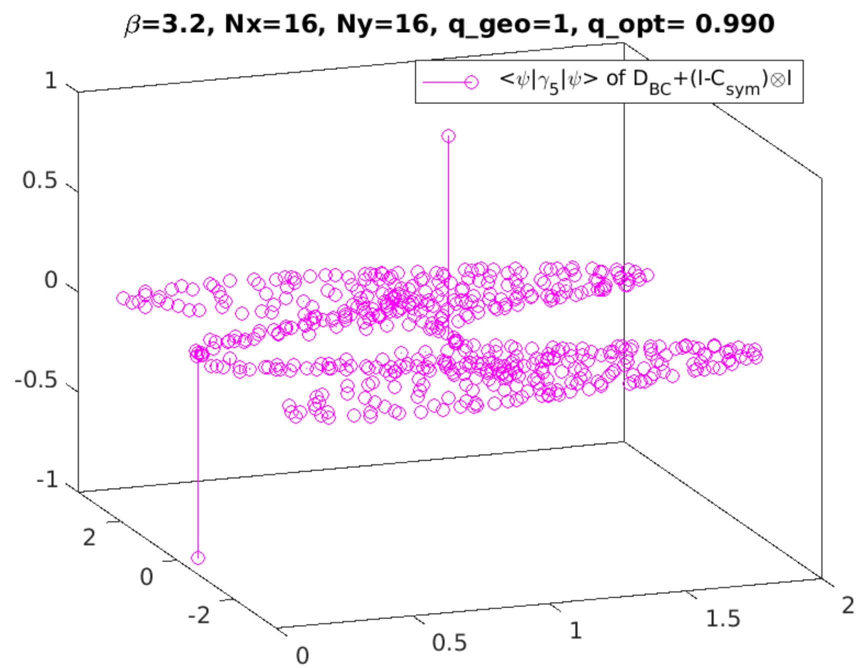


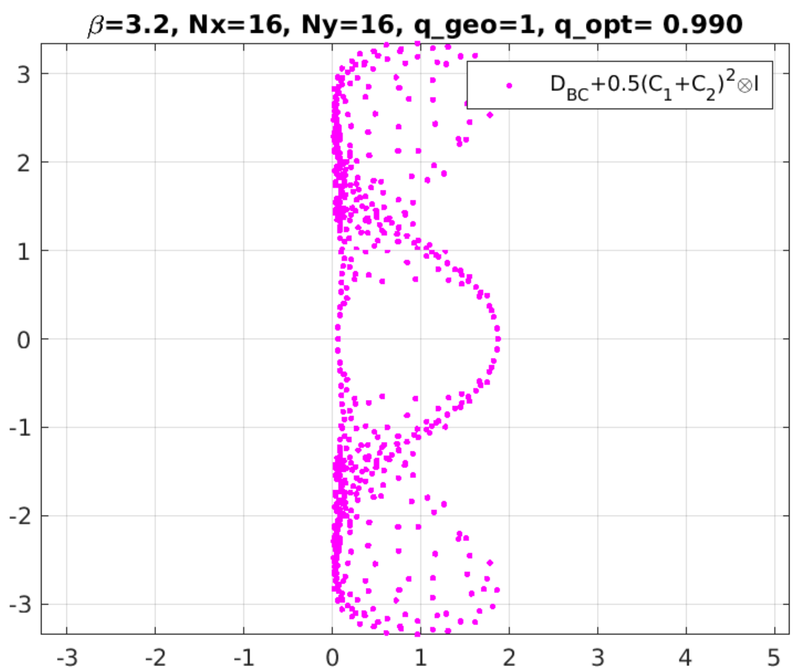


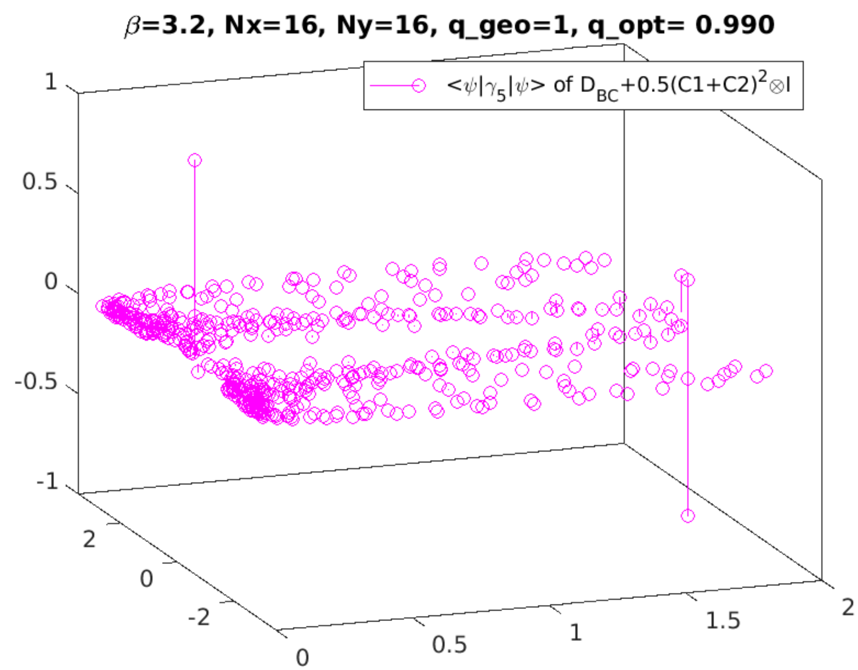


Eigenvalues and topology with BC fermions plus lifting-term









Summary (so far)

Mir wird von alledem so dumm, als ging mir ein Mühlrad im Kopf herum. [Goethe]

I feel like a mill wheel is turning in my head. [unprofessional translation by SD]

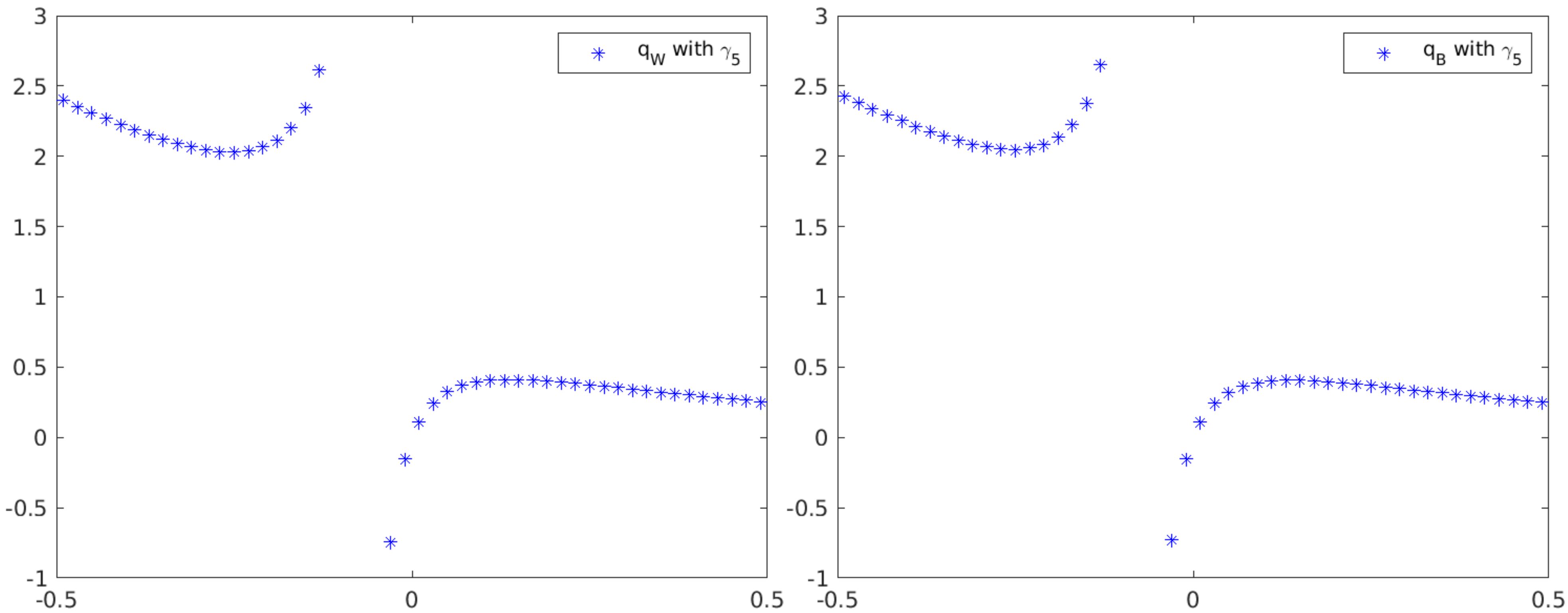
Summary (so far)

KW and BC fermions have matrix size like Wilson fermions ($N_c 4 N_x N_y N_z N_t$ in 4D).
KW and BC fermions have exact chiral symmetry (eigenvalues on imaginary axis).
KW and BC fermions have condition number less favorable than staggered fermions.

They have $2|q|$ would-be zero-modes with *opposite chiralities* (as staggered) in 2D.
This figure remains $2|q|$ in 4D (while staggered fermions have $4|q|$ in 4D).

With an appropriate chirality operator all lattice fermions perceive $q_{\mathrm{top}}[U]$ correctly.

Topological charge via Wilson/Brillouin fermion



$$q_{\mathrm{W}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{W}}+m)^{-1} I\otimes\gamma_5] \quad , \qquad q_{\mathrm{B}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{B}}+m)^{-1} I\otimes\gamma_5]$$

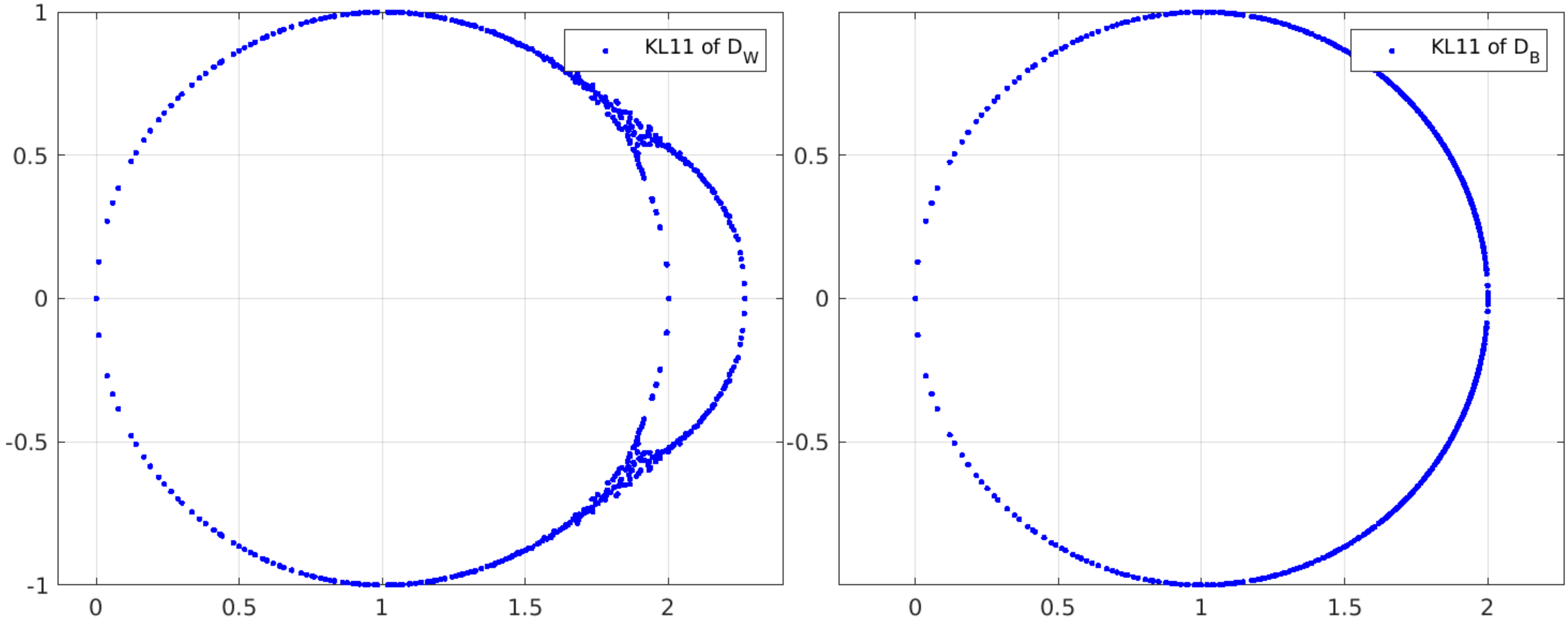
- apparent pole structure plausible (see App. C of arXiv:2203.15699) from

$$q_{\mathrm{W}} \simeq m\frac{2(2r+m_{\mathrm{crit}})^2}{(m-m_{\mathrm{crit}})(2r+m_{\mathrm{crit}})(4r+2m_{\mathrm{crit}})} = \frac{m}{m-m_{\mathrm{crit}}}$$

Kenney-Laub version of Wilson/Brillouin overlap fermion

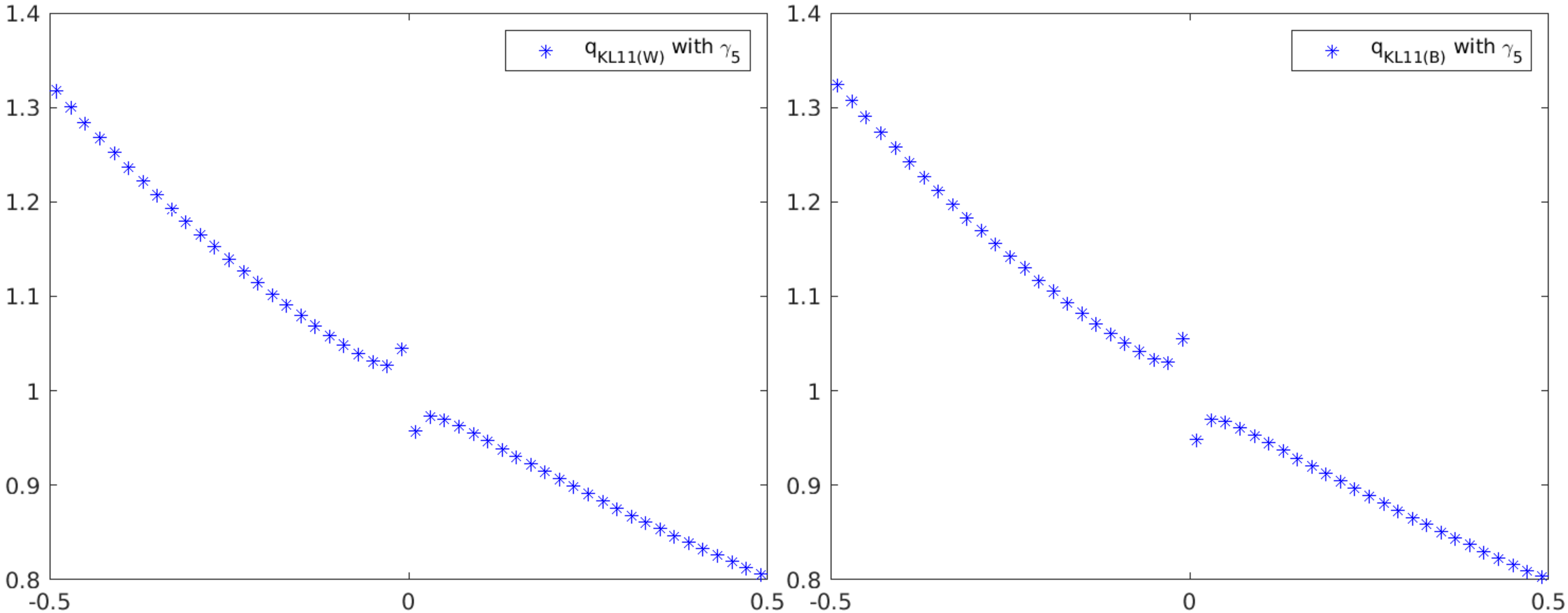
$D_\mathrm{W}^\mathrm{KL11} = (D_\mathrm{W}-1)\frac{A_\mathrm{W}+3}{3A_\mathrm{W}+1}+1$ with $A_\mathrm{W} \equiv (D_\mathrm{W}-1)^\dagger(D_\mathrm{W}-1)$ approximate overlap

$D_\mathrm{B}^\mathrm{KL11}$ defined accordingly with Brillouin kernel



- additive mass shift much reduced
- near-unitarity of $D_\mathrm{kernel}^\mathrm{KL11}-1$ particularly striking for $D_\mathrm{kernel}=D_\mathrm{B}$
- fixed-order rational approximation conceptually similar to domain-wall formulation

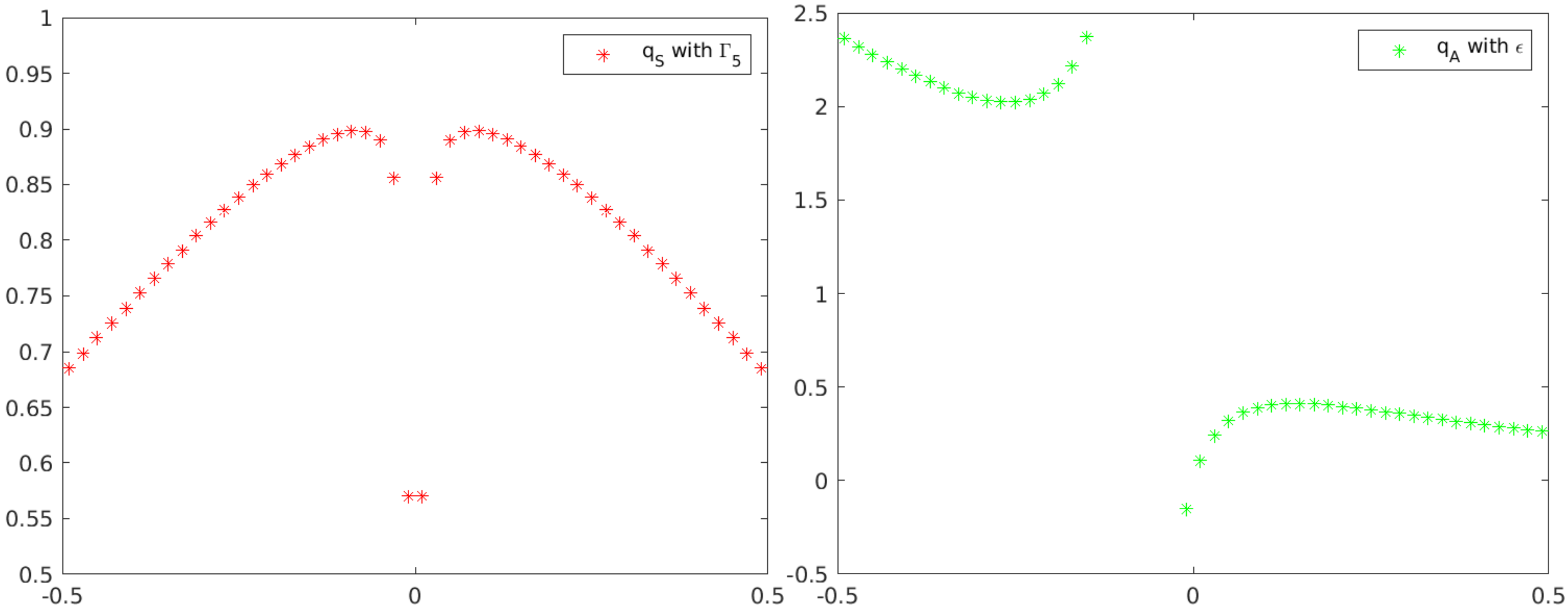
Topological charge via Wilson/Brillouin overlap fermion



$$q_{\mathrm{W}}^{\mathrm{KL11}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{W}}^{\mathrm{KL11}}+m)^{-1} I\otimes\gamma_5]\;, \quad q_{\mathrm{B}}^{\mathrm{KL11}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{B}}^{\mathrm{KL11}}+m)^{-1} I\otimes\gamma_5]$$

- "continuum-like" definition $q_{\mathrm{fer}}[U] = -m\,\mathrm{tr}[D_m^{-1}\gamma_5]$ works for $m \neq m_{\mathrm{crit}}$
- apparent pole structure with reduced additive mass and smaller amplitude
- no visble difference between W-kernel and B-kernel

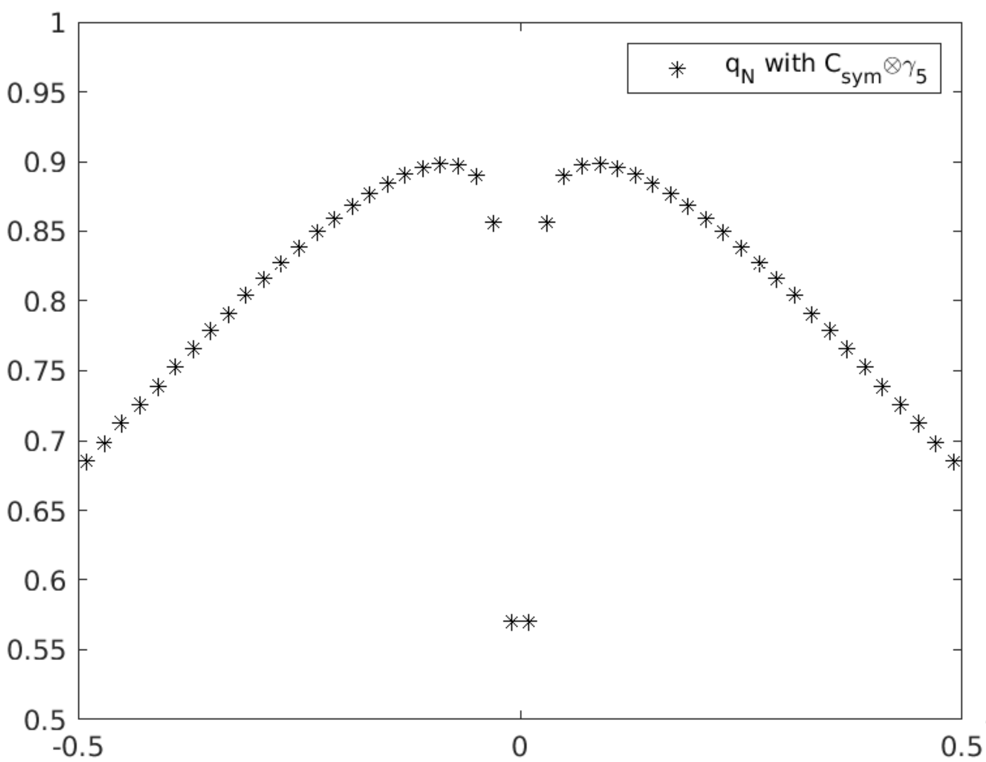
Topological charge via staggered/Adams fermion

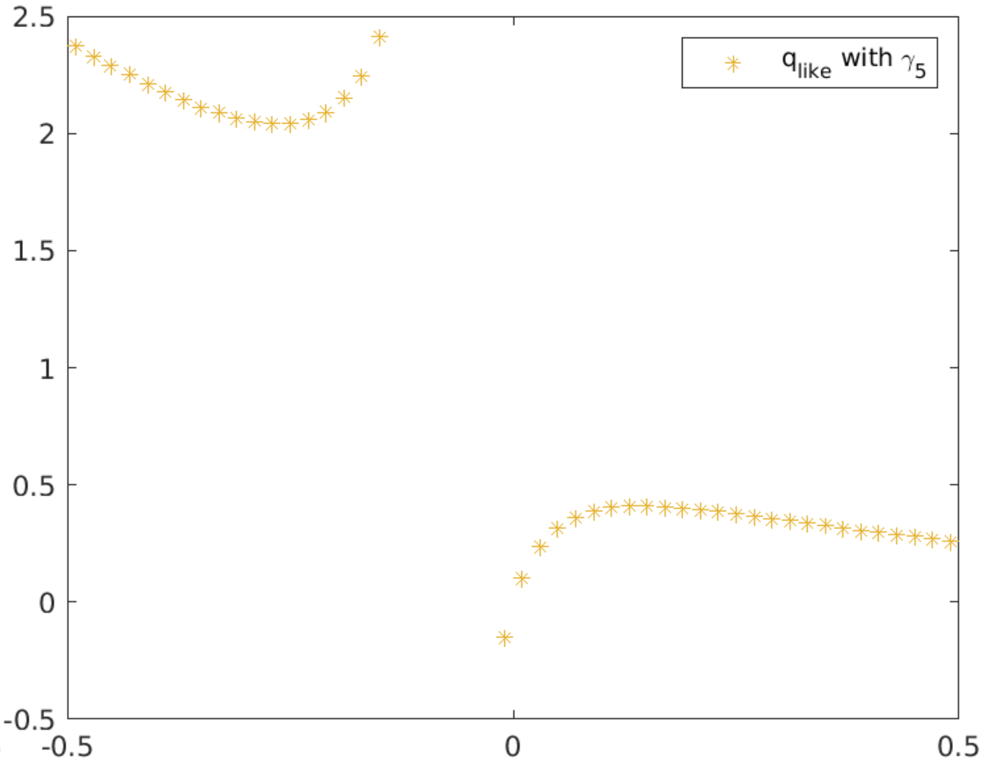


$$q_{\mathrm{S}}[U] = -\frac{m}{2}\,\mathrm{tr}[(D_{\mathrm{S}}+m)^{-1}\Gamma_{50}] \quad , \qquad q_{\mathrm{A}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{S}}+1-\Gamma_{05}+m)^{-1}\Gamma_{55}]$$

- two-species formulation requires factor $\frac{1}{2}$ [staggered]
- no additive mass shift with $q_{\mathrm{S}}[U]$ (chiral symmetry)
- additive mass shift of Adams comparable to Wilson case

Topological charge via naive/Adams-like fermion

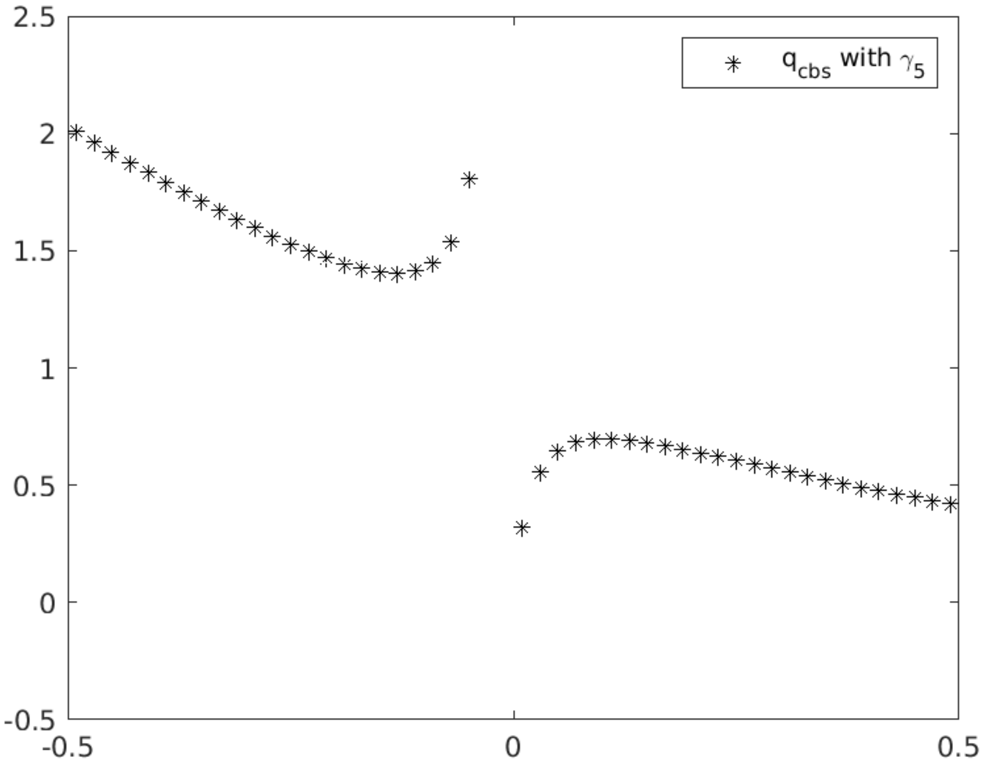


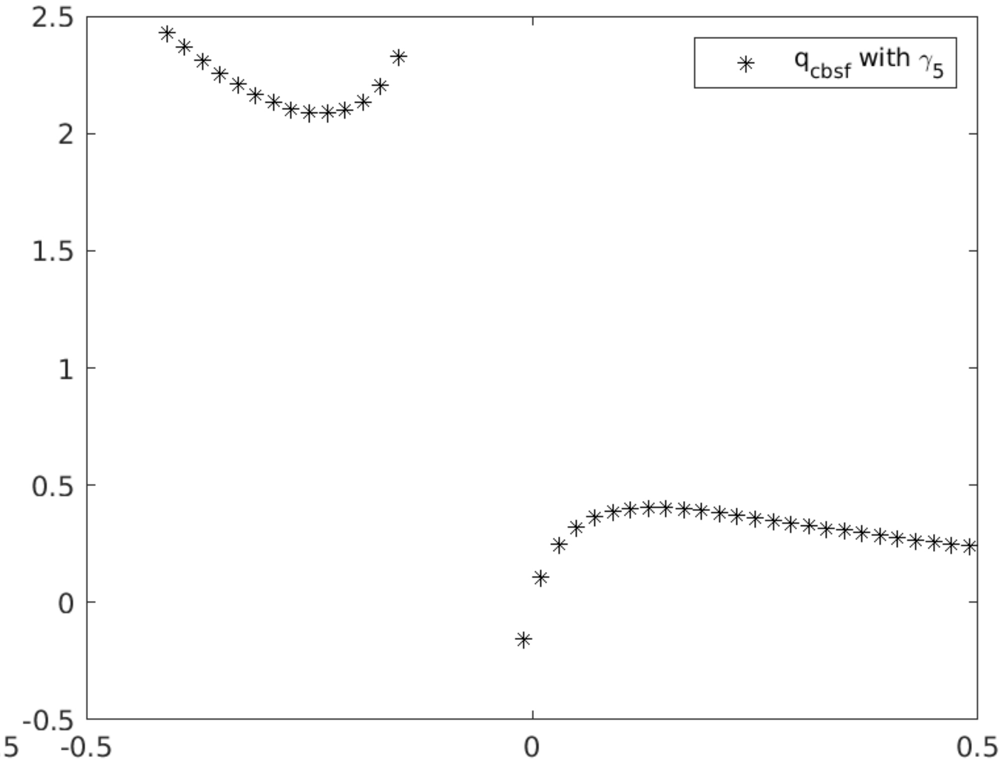


$$q_{\mathrm{N}}[U] = -\frac{m}{4}\,\mathrm{tr}[(D_{\mathrm{N}}+m)^{-1}C_{\mathrm{sym}}\otimes\gamma_5],\;\; q_{\mathrm{like}}[U] = -\frac{m}{2}\,\mathrm{tr}[(D_{\mathrm{N}}+1-C_{\mathrm{sym}}+m)^{-1}I\otimes\gamma_5]$$

- four-species formulation requires factor $\frac{1}{4}$ [naive]
- two-species formulation requires factor $\frac{1}{2}$ [A-like]
- additive mass shift of A-like comparable to Wilson and Adams cases

Topological charge via cbs/cbsf fermion

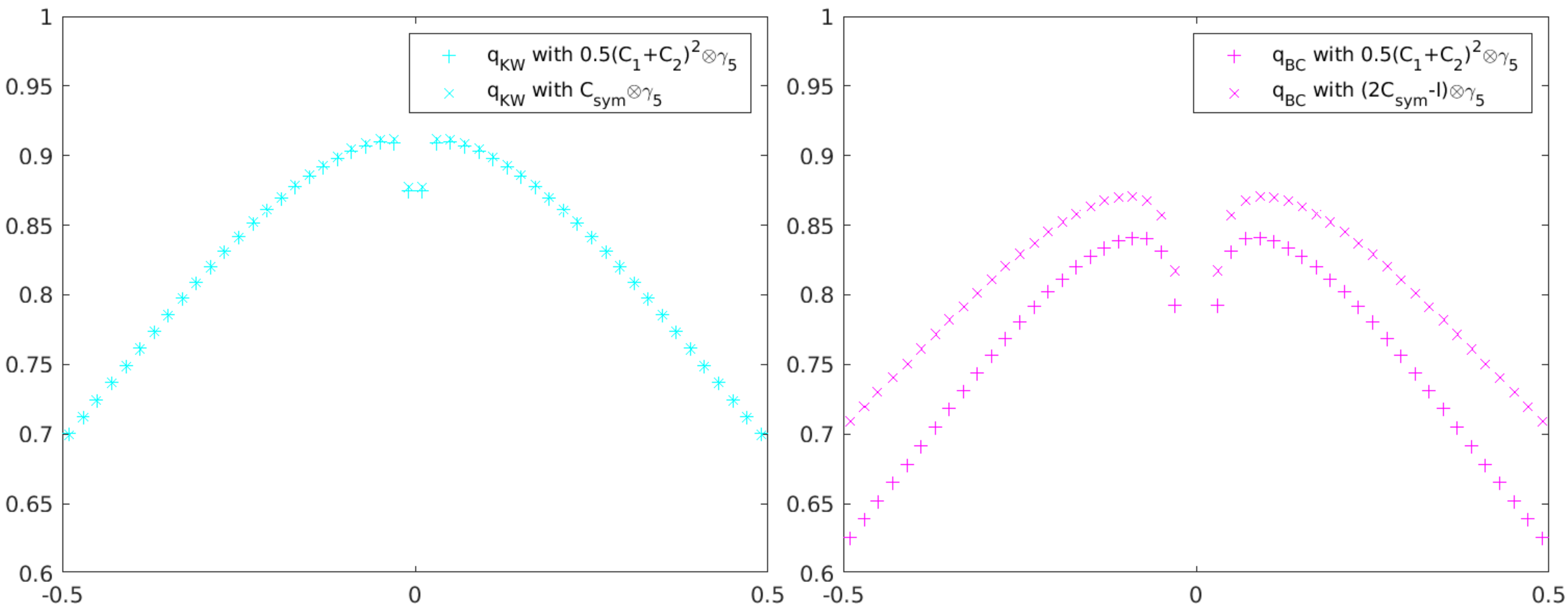




$$q_{\rm cbs}[U] = +\frac{m}{2}\,\mathrm{tr}[(D_{\rm cbs}+m)^{-1} I \otimes \gamma_5]\ , \qquad q_{\rm cbsf}[U] = -\frac{m}{2}\,\mathrm{tr}[(D_{\rm cbsf}+m)^{-1} I \otimes \gamma_5]$$

- two-species formulation requires factor $\frac{1}{2}$ [cbs and cbsf]
- non-zero mass shifts in both cases, more virulent for $q_{\rm cbsf}$
- opposite sign for cbs, normal sign for cbsf (normal signs in 4D)

Topological charge via KW/BC fermion



$$q_{\rm KW}[U] = -\frac{m}{2}\,\mathrm{tr}[(D_{\rm KW}+m)^{-1}X_{\rm KW}]\,, \quad q_{\rm BC}[U] = -\frac{m}{2}\,\mathrm{tr}[(D_{\rm BC}+m)^{-1}X_{\rm BC}]$$

$$X_{\rm KW} = \left\{ \begin{array}{l} \frac{1}{2}(C_1+C_2)^2\otimes\gamma_5 \\ C_{\rm sym}\otimes\gamma_5 \end{array} \right. , \quad X_{\rm BC} = \left\{ \begin{array}{l} \frac{1}{2}(C_1+C_2)^2\otimes\gamma_5 \\ (2C_{\rm sym}-1)\otimes\gamma_5 \end{array} \right.$$

- two-species formulation requires factor $\frac{1}{2}$ [KW and BC]
- no additive mass shift for both KW an BC (chiral symmetry)

BACKUP PAGES

Flavored mass/lifting terms

$$C_\mu(x,y) = \frac{1}{2}[U_\mu(x)\delta_{x+\hat\mu,y} + U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}] = \frac{1}{2}a^2\triangle_\mu(x,y) + \delta_{x,y}$$

$M_S = 1$	Scalar	(0-link)	1
$M_V = \sum_{\mathrm{sym}} C_\mu$	Vector	(1-link)	$\frac{1}{4}[C_1+C_2+C_3+C_4]$
$M_T = \sum_{\mathrm{sym}}\sum_{\mathrm{per}} C_\mu C_\nu$	Tensor	(2-link)	see detail
$M_A = \sum_{\mathrm{sym}}\sum_{\mathrm{per}} C_\mu C_\nu C_\rho$	Axial	(3-link)	see detail
$M_P = \sum_{\mathrm{per}} C_\mu C_\nu C_\rho C_\sigma$	Pseudo	(4-link)	$\frac{1}{24}[C_1C_2C_3C_4 + \mathrm{perms}] = C_{\mathrm{sym}}$

detail T: $\frac{1}{12}[C_1C_2 + \mathrm{perm}] + ... + \frac{1}{12}[C_3C_4 + \mathrm{perm}]$
6 square brackets [...] each of which contains 2 terms

deatil A: $\frac{1}{24}[C_2C_3C_4 + \mathrm{perms}] + ...$
4 square brackets [...] each of which contains 6 terms

Brillouin fermion: dim=5 term (Laplacian) is $M_V+M_T+M_A+M_P$
Brillouin fermion: dim=4 term is $\sum_\mu \gamma_\mu \nabla_\mu^{\mathrm{iso}}$ instead of $\sum_\mu \gamma_\mu \nabla_\mu^{\mathrm{std}}$

Creutz, Kimura, Misumi (10, 11)

Review of staggered mass/lifting terms

The $(\gamma_\mu \otimes 1)$ and $(\gamma_5 \otimes 1)$ “taste singlet” operators are defined by

$$\begin{aligned}
\Gamma_\mu(x,y) &\equiv \Gamma_{\mu 0}(x,y) &=& \frac{1}{2}\eta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat\mu,y} + U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}\Big] \\
\Gamma_5(x,y) &\equiv \Gamma_{50}(x,y) &=& \frac{1}{4!}\sum_{\mathrm{perm}} \epsilon_{\mathrm{perm}}\Gamma_1\Gamma_2\Gamma_3\Gamma_4
\end{aligned}$$

and the $(1 \otimes \xi_\mu)$ and $(1 \otimes \xi_5)$ “spinor singlet” operators are defined by

$$\begin{aligned}
\Xi_\mu(x,y) &\equiv \Gamma_{0\mu}(x,y) &=& \frac{1}{2}\zeta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat\mu,y} + U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}\Big] \\
\Xi_5(x,y) &\equiv \Gamma_{05}(x,y) &=& \frac{1}{4!}\sum_{\mathrm{perm}} \epsilon_{\mathrm{perm}}\Xi_1\Xi_2\Xi_3\Xi_4
\end{aligned}$$

with the consequence that both Γ_{50} and Γ_{05} are 4-hop operators. Furthermore, the latter two operators relate to each other by a simple Γ_{55} operation (from left or right).

Acceptable mass terms are proportional to $(1 \otimes 1)$ or $(1 \otimes \xi_5)$ or possibly $(1 \otimes \xi_\mu\xi_\nu)$.

Adams species lifting

In practice it is advantageous to introduce the commutators in spinor and taste space

$$\begin{aligned}
\Gamma_{\mu\nu}(x,y) &\equiv \frac{\mathrm{i}}{2}\Big(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu\Big) \qquad \longleftrightarrow \qquad \gamma_{\mu\nu}\otimes 1 \\
\Xi_{\mu\nu}(x,y) &\equiv \frac{\mathrm{i}}{2}\Big(\Xi_\mu\Xi_\nu - \Xi_\nu\Xi_\mu\Big) \qquad \longleftrightarrow \qquad 1\otimes\xi_{\mu\nu}
\end{aligned}$$

respectively, with $\gamma_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$ a.k.a. $\sigma_{\mu\nu}$ and $\xi_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\xi_\mu, \xi_\nu]$, which yields

$$\begin{aligned}
\Gamma_{50}(x,y) &\simeq -\frac{1}{6}\Big(\Gamma_{12}\Gamma_{34} - \Gamma_{13}\Gamma_{24} + \Gamma_{14}\Gamma_{23} + \Gamma_{23}\Gamma_{14} - \Gamma_{24}\Gamma_{13} + \Gamma_{34}\Gamma_{12}\Big) \\
\Gamma_{05}(x,y) &\simeq -\frac{1}{6}\Big(\Xi_{12}\Xi_{34} - \Xi_{13}\Xi_{24} + \Xi_{14}\Xi_{23} + \Xi_{23}\Xi_{14} - \Xi_{24}\Xi_{13} + \Xi_{34}\Xi_{12}\Big)
\end{aligned}$$

Adams: Promote 2 of the 4 tastes of D_{stag} to doublers by $\Gamma_{05} = \Xi_5 \simeq (1\otimes\xi_5)$.
Key observation is that the remaining 2 species share *one chirality*.

Corollary: It makes sense to apply overlap construction to shifted kernel $X = D_{\mathrm{A}} - \rho$.
The resulting operator will be doubled, but the two species will be chiral.