New insights into lattice fermions and topology

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1. Wilson & Domain-wall fermions as SPT

Wilson fermion : species-splitting mass fermion

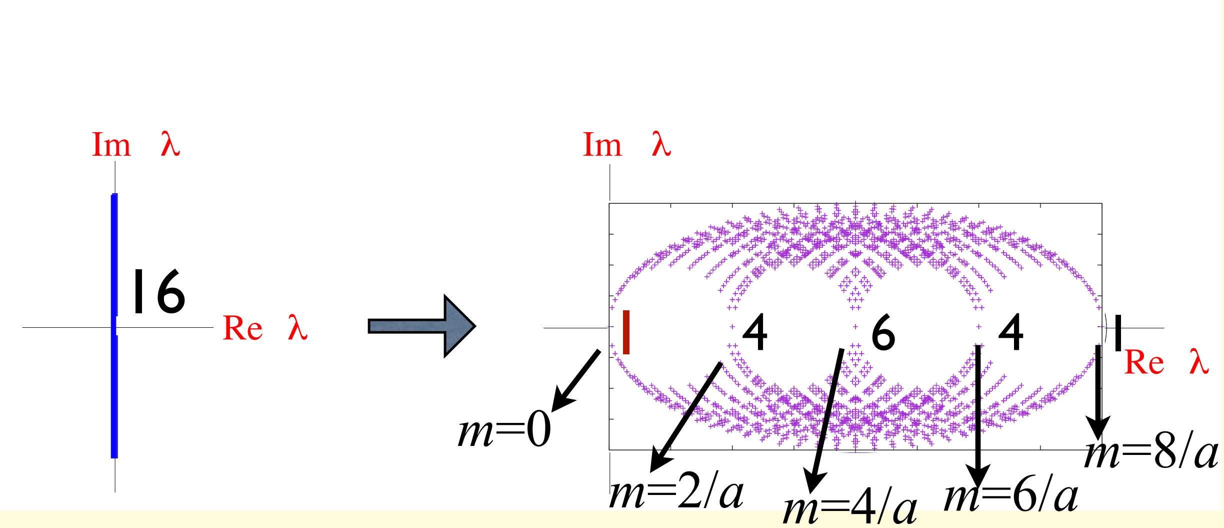
Lattice fermion action with species-splitting term $\sum_{n,\mu} \frac{a^5}{2}\bar{\psi}_n(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu})$

$$\Rightarrow \quad D_W(p) = \frac{1}{a}\sum_\mu [i\gamma_\mu \sin ap_\mu + \underline{(1-\cos ap_\mu)}]$$

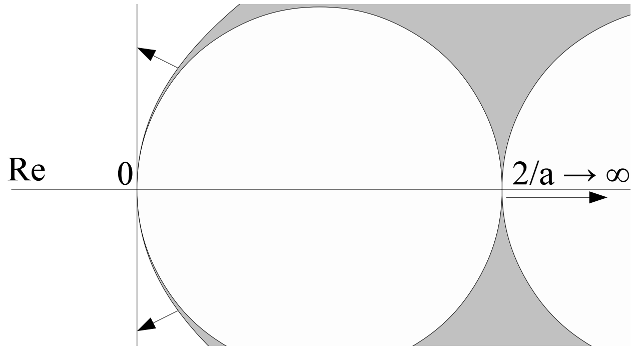
Physical (0,0,0,0) : $D_W(p) = i\gamma_\mu p_\mu + O(a)$

Doubler(π/a,0,0,0) : $D_W(p) = i\gamma_\mu p_\mu + \frac{2}{a} + O(a)$

Only one flavor is massless,
while others have $O(1/a)$ mass.



- 15 species are decoupled → doubler-less
- $1/a$ additive mass renormalization → Fine-tune
- Domain-wall & Overlap fermions → costs



Re-interpret it in terms of Symmetry-Protected-Topological(SPT) order

Symmetry-Protected Topological (SPT) order

- G-Symmetry Protected Topological (SPT) order Wen, et.al., (13)

1. Partition function Z characterized by certain topological charge
2. Unique ground state with trivial gap as long as G is unbroken
3. The Gap is closed when topological charge is changed
4. Gapless modes emerge at boundary btwn two different SPTs
5. 't Hooft anomaly cancelled btwn bulk & boundary with gauged G

All 't Hooft anomalies are (expected to be) classified by SPTs.

Kapustin (14), Witten (15), Yonekura (16), Yonekura, Witten (19)

Symmetry-protected topological phase

ex.) (2+1)-dim free massive Dirac fermion = U(1) SPT = IQHS

$m < 0$

$$Z = e^{-2\pi i \eta}$$

(η: APS η-invariant $\equiv \sum_i \mathrm{sgn}[\lambda_i]$)

$m > 0$ $(m \to +\infty)$

$$Z = 1$$

2-dim chiral fermion Z_{bndry}

Symmetry-protected topological phase

ex.) (2+1)-dim free massive Dirac fermion = U(1) SPT = IQHS

$m < 0$

$$Z = e^{-2\pi i \eta}$$

APS index theorem $= e^{\frac{i}{4\pi}\int AdA}$

$m > 0 \ \ (m \to +\infty)$

$Z = 1$

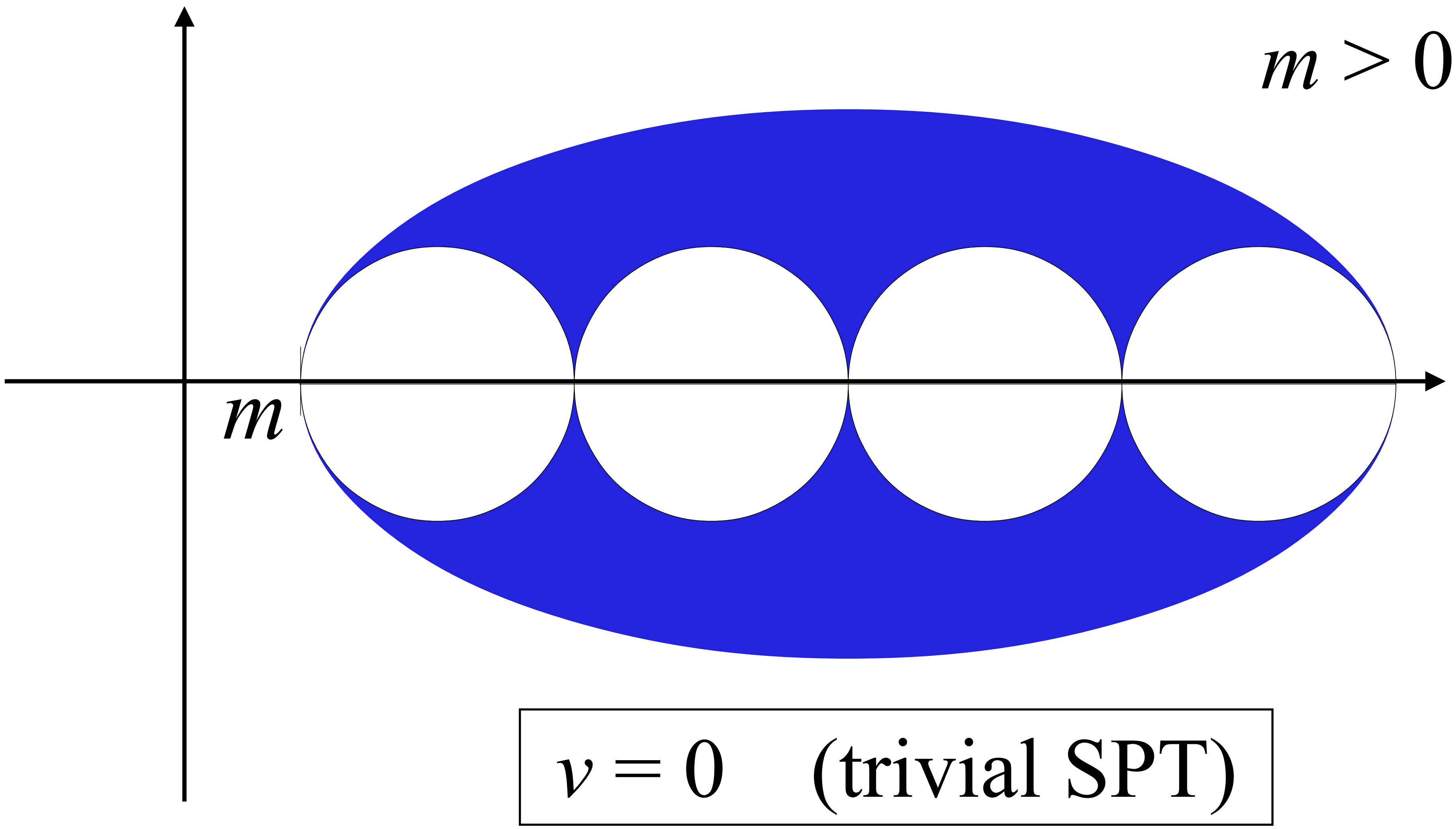
2-dim chiral fermion Z_{bndry}

$$Z_{\mathrm{total}} = Z_{\mathrm{bulk}} \cdot Z_{\mathrm{bndry}} \longrightarrow Z_{\mathrm{bulk}} e^{\frac{i}{4\pi}\int F} \cdot Z_{\mathrm{bndry}} e^{-\frac{i}{4\pi}\int F} = Z_{\mathrm{total}}$$

't Hooft anomaly is cancelled between bulk and boundary

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ sum of chiral charges of species with $m < 0$

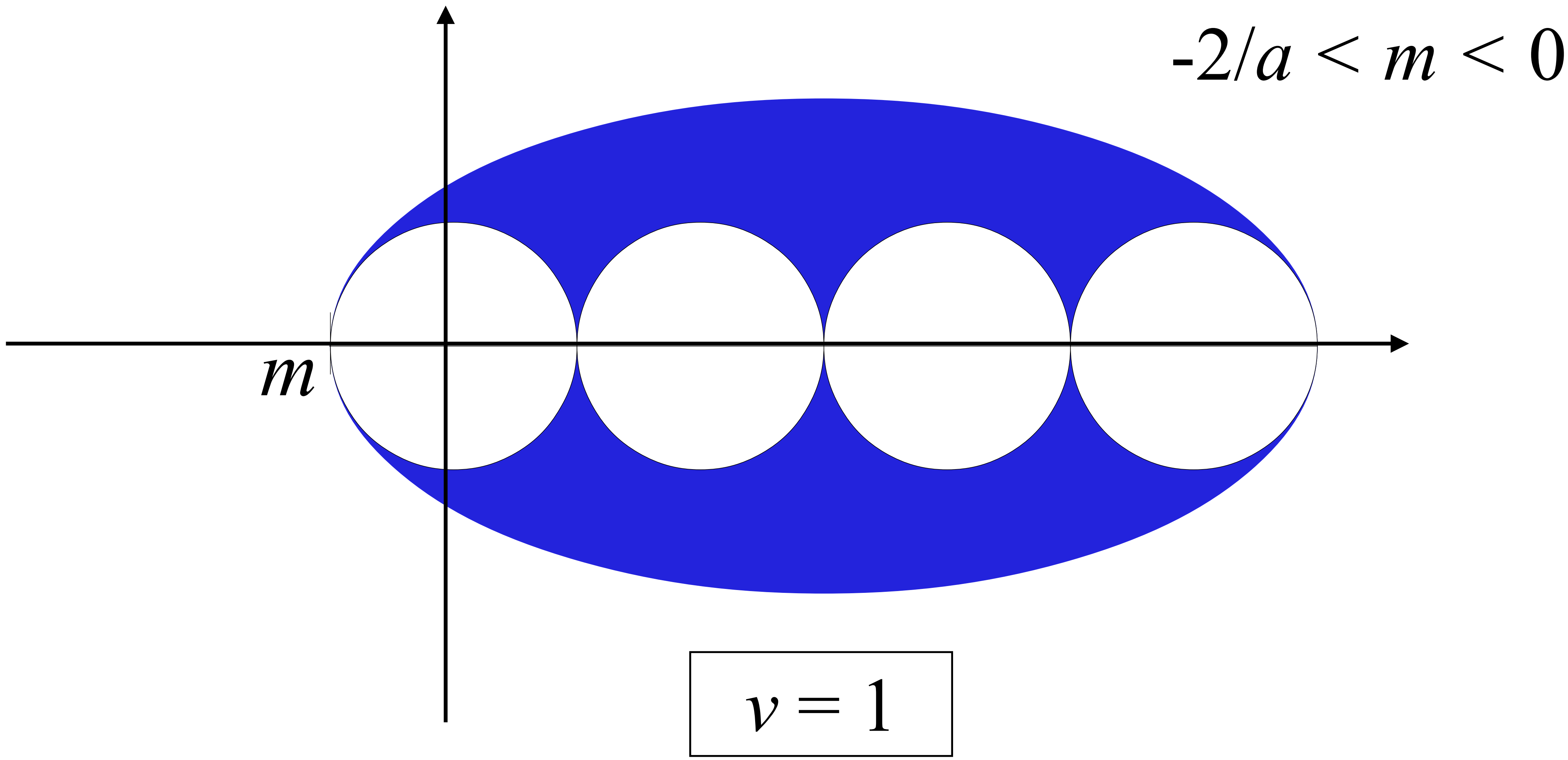


The topological charge is defined by Berry connection for free fermion

$$\nu_{4\mathrm{D}} = -\frac{1}{16\pi^2}\int_{\mathrm{BZ}} d^4p \,\mathrm{Tr}\, F * F$$

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ sum of chiral charges of species with $m < 0$

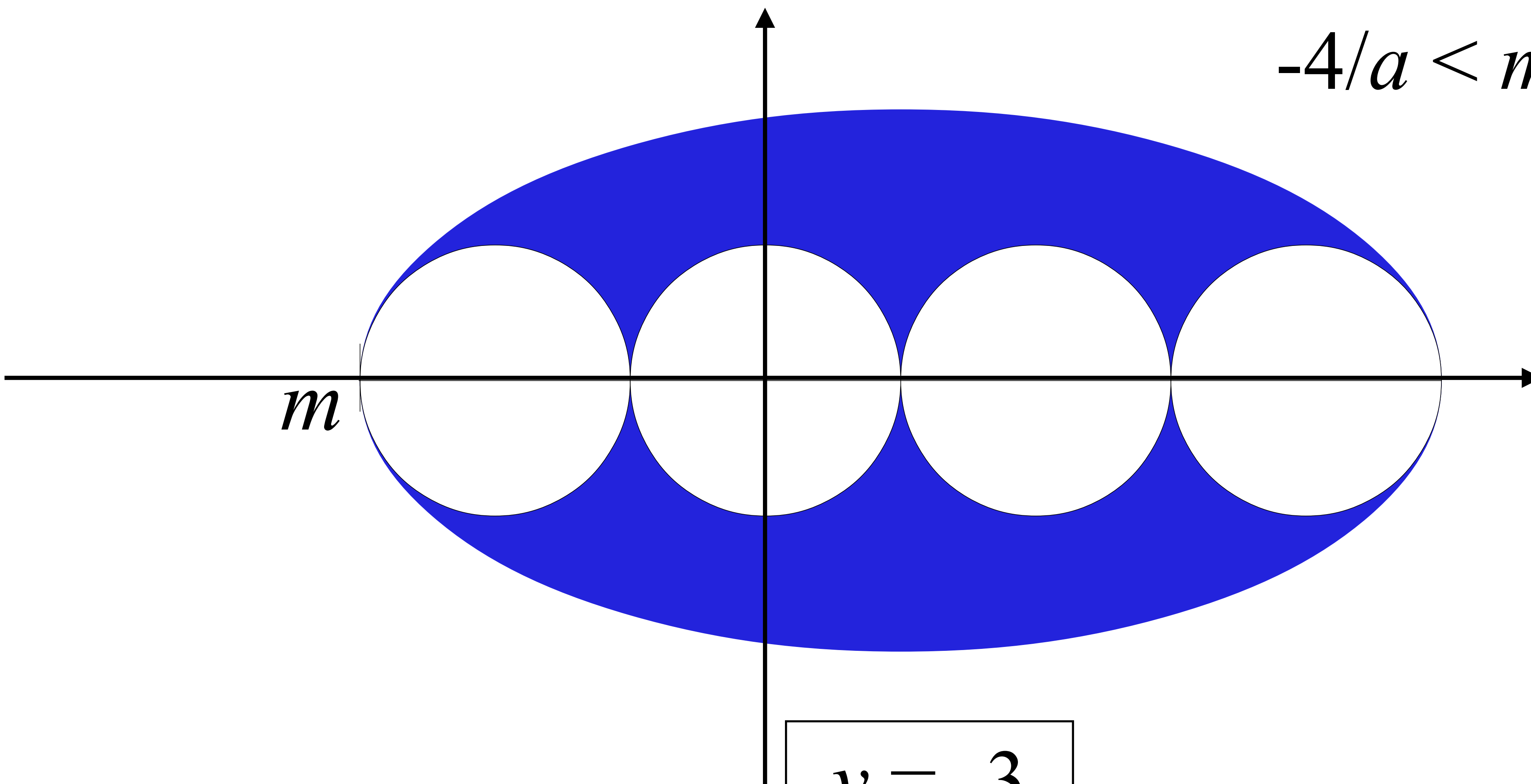


Domain-wall fermion : gapless mode emerging at boundary between ν=0 and ν=1 SPTs, where 't Hooft anomaly cancels.

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ sum of chiral charges of species with $m < 0$

$-4/a < m < -2/a$

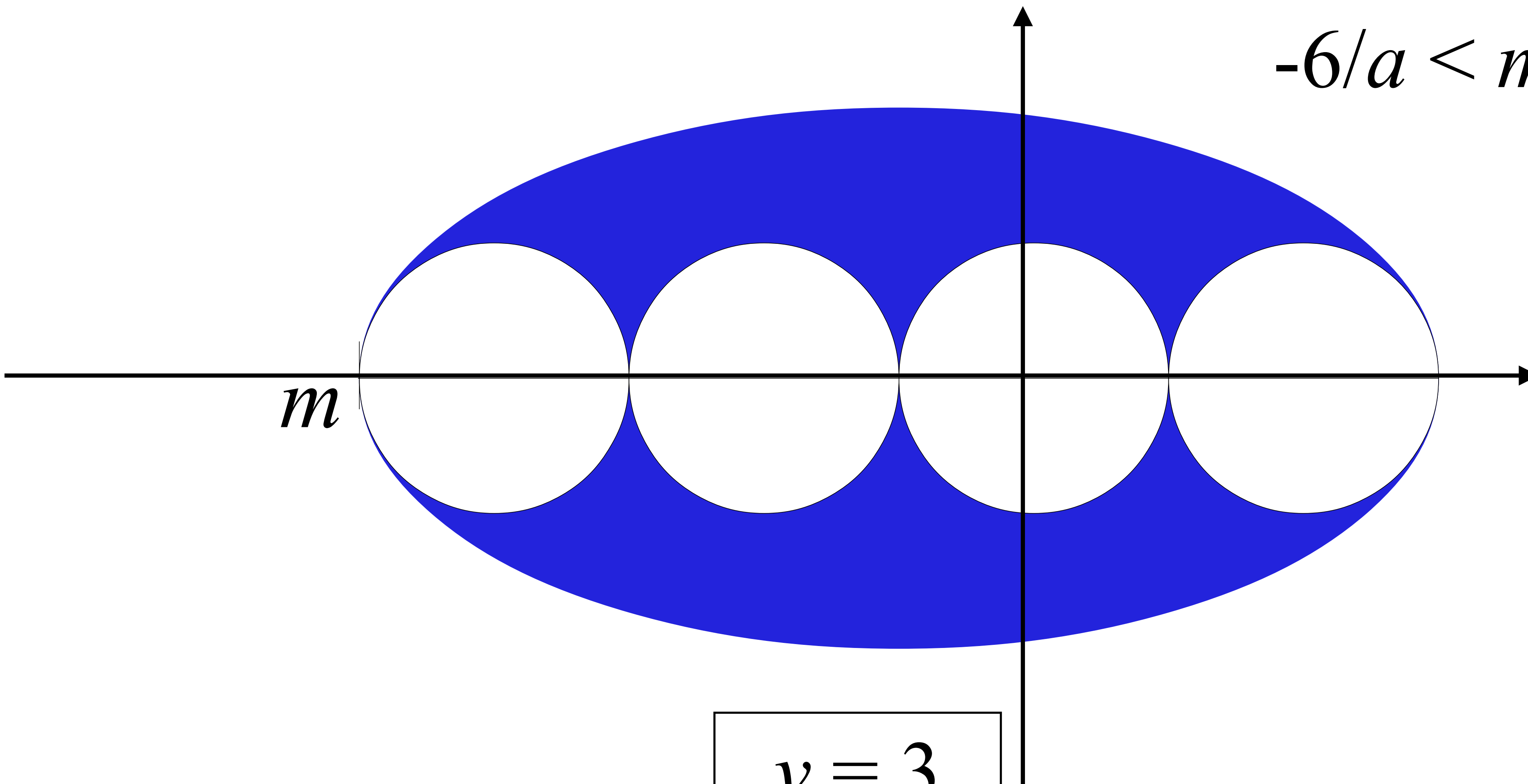


$\nu = -3$

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ sum of chiral charges of species with $m < 0$

$-6/a < m < -4/a$



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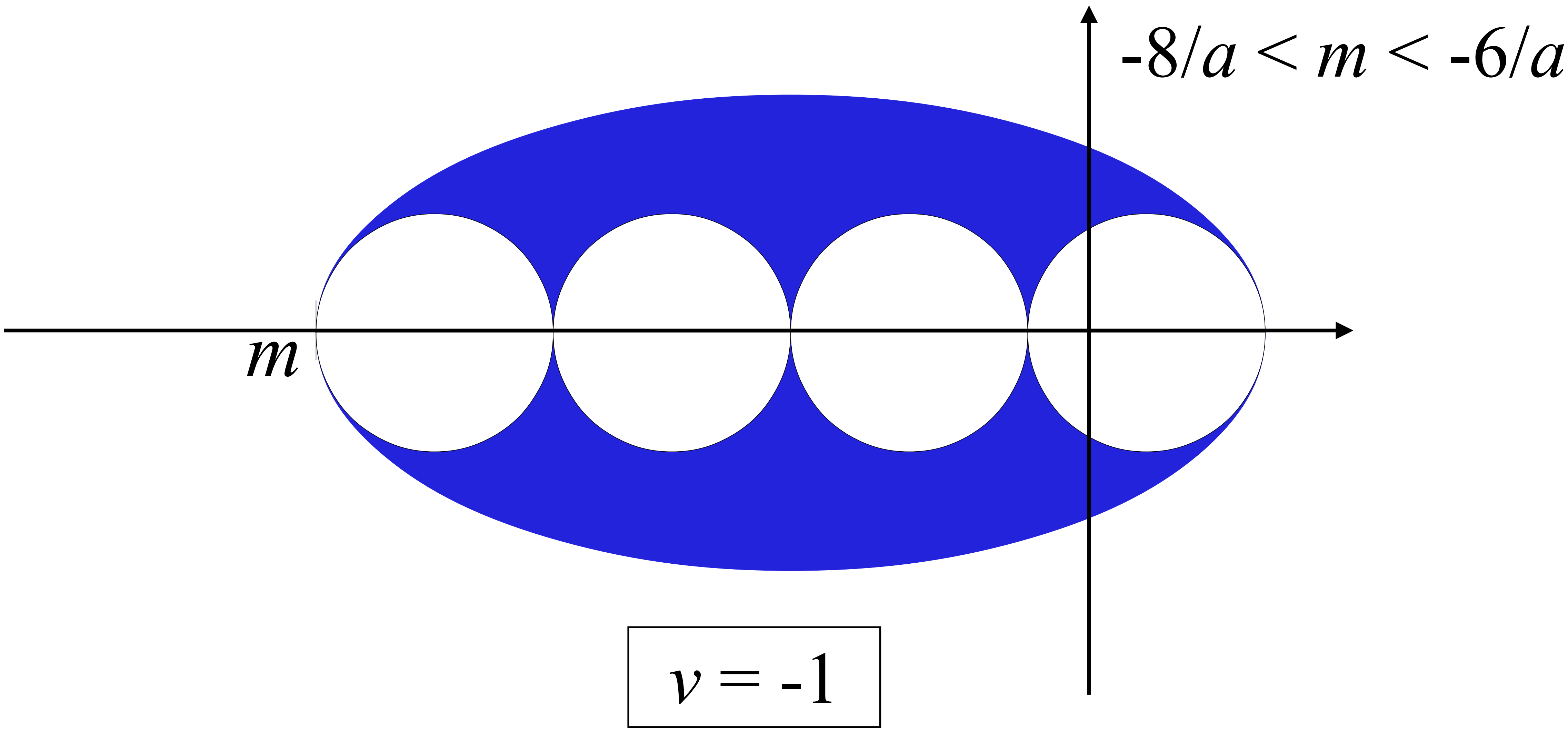
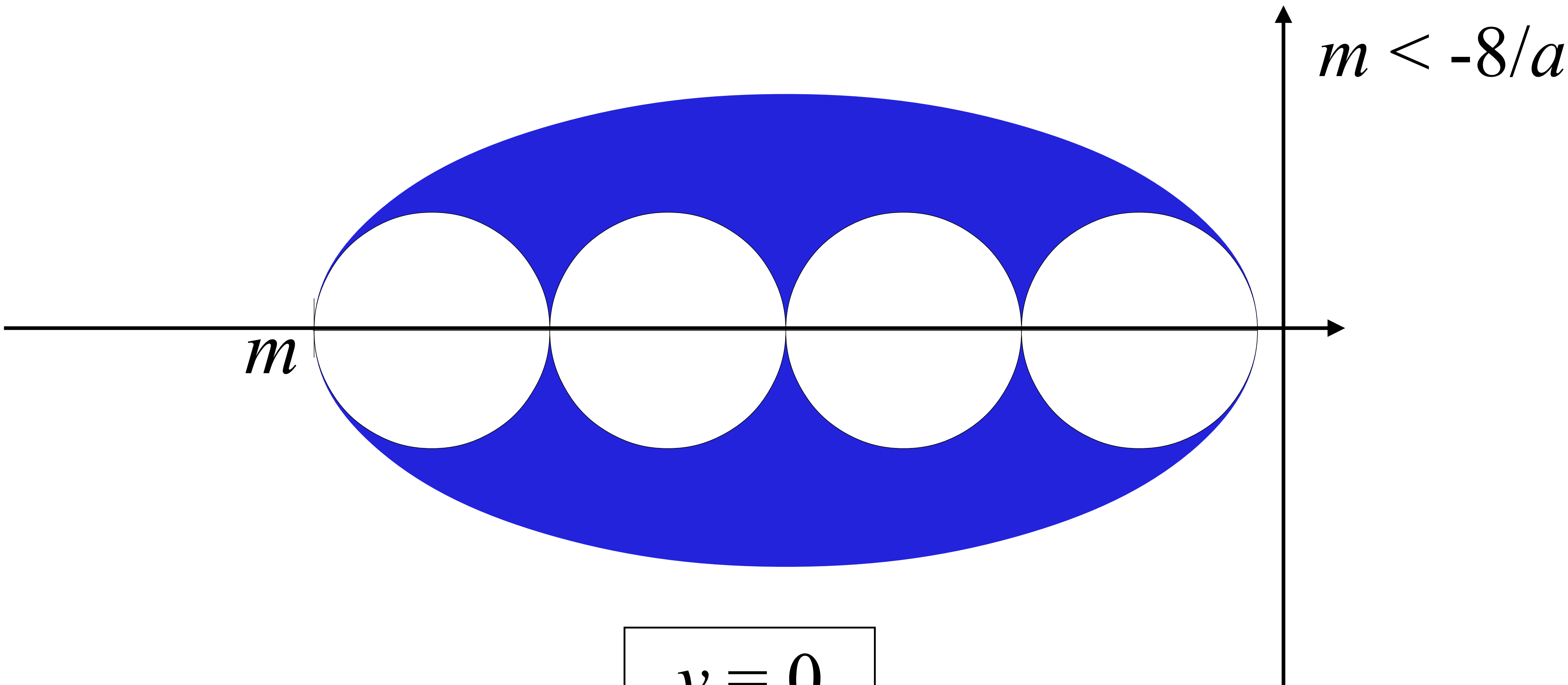
Wilson fermion as U(1) SPT phases

Topological # of SPT ~ sum of chiral charges of species with $m < 0$



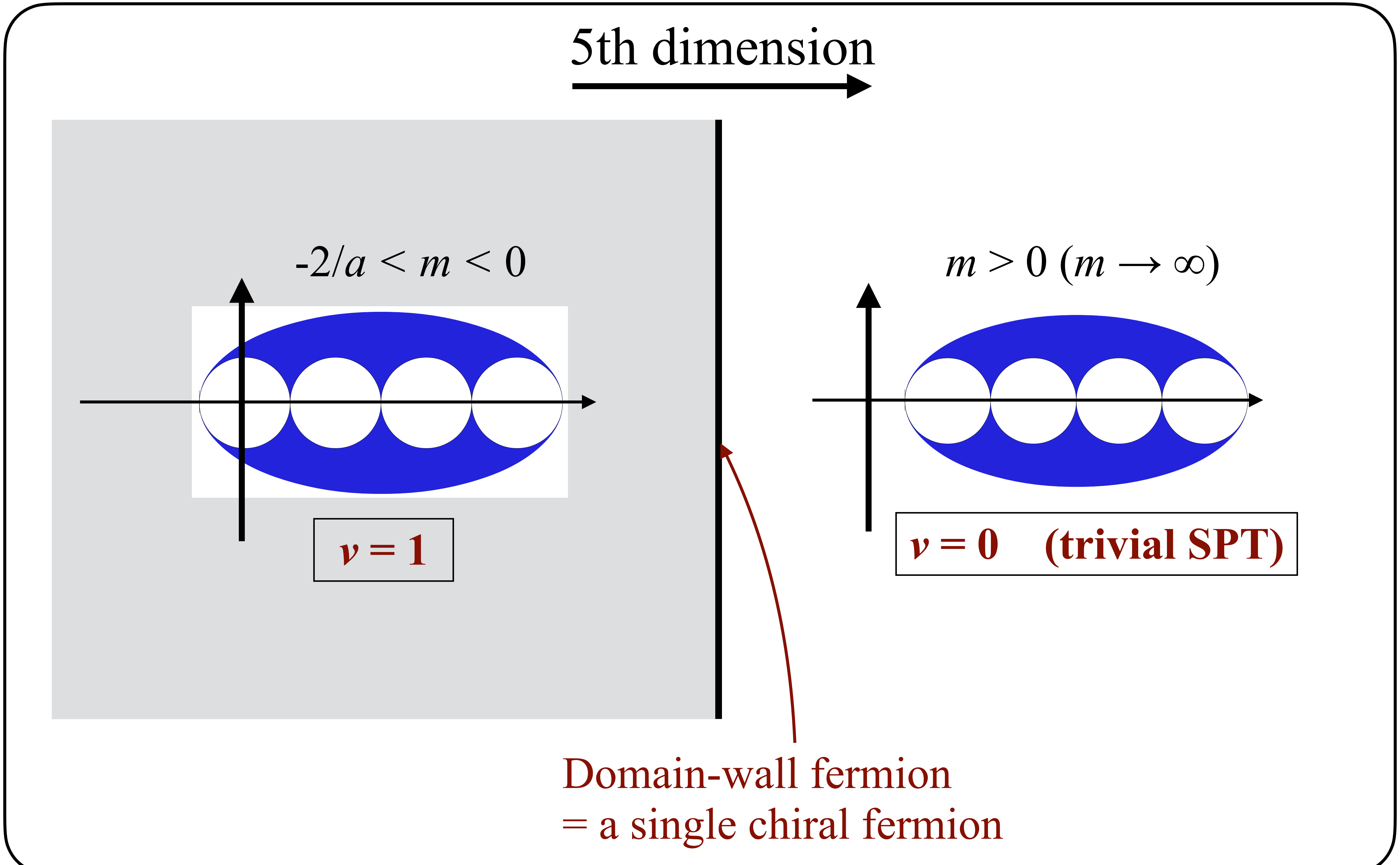
Wilson fermion as U(1) SPT phases

Topological # of SPT ~ sum of chiral charges of species with $m < 0$



$\nu = 0$

Domain-wall fermion as boundary gapless mode



2. Lattice fermions & 't Hooft anomaly

Tanizaki, TM (19)
TM, Yumoto (20)

What is 't Hooft anomaly ?

- D-dim QFT with global symmetry G
- Introduce non-dynamical background G-gauge field A
- Partition function is sometimes ambiguous under G-gauge transf.

$$Z[A + d\theta] = Z[A] \cdot \exp(i\mathcal{A}[\theta, A])$$

➡ G has 't Hooft anomaly

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G has 't Hooft anomaly

ex.) N_f-flavor massless QCD with $SU(N_\mathrm{f})_\mathrm{L} \times SU(N_\mathrm{f})_\mathrm{R}$

$$Z[A_L + D_L\theta_L, A_R + D_R\theta_R] = Z[A_L, A_R] \exp\left[\frac{iN_c}{24\pi^2}\int \mathrm{tr}\left\{\theta_R d\left(A_R dA_R + \frac{A_R^3}{2}\right) - \theta_L d\left(A_L dA_L + \frac{A_L^3}{2}\right)\right\}\right]$$

$SU(N_\mathrm{f})_\mathrm{A}$ has 't Hooft anomaly

’t Hooft anomaly matching

- Let the gauge field A weakly coupled to spectator fermion ψ
- Set the anomaly canceled → A can be dynamical
- In RG flow, the anomaly from ψ is intact. G is also unbroken.

➡ ’t Hooft anomaly is RG-invariant ! $\mathcal{A}_{\mathrm{UV}}[\theta, A] = \mathcal{A}_{\mathrm{IR}}[\theta, A]$

➡ it gives constraints on IR strongly-coupled physics

't Hooft anomaly matching

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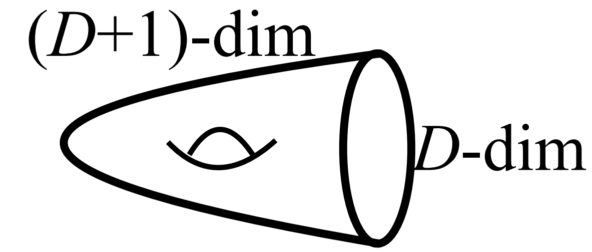
➡ it gives constraints on IR strongly-coupled physics

◆ It is nothing but what occurs in Standard Model

Standard Model = QCD w/ gauged flavor sym. & weakly coupled leptons

- Gauge anomaly cancelled in both UV and IR
- The anomaly from lepton sector is unchanged in RG
- The anomaly in QCD sector is RG-invariant (even though SSB)

't Hooft anomaly matching



- Regard the system as boundary of $(D+1)$d SPT $\quad Z[A]\cdot\exp(iS_{D+1}[A])$
- There is cancellation of 't Hooft anomaly $\quad \delta_\theta S_{D+1}[A] = -\mathcal{A}[\theta, A]$
- In RG flow the anomaly of bulk SPT system is intact

't Hooft anomaly is RG-invariant ! $\quad \mathcal{A}_{\mathrm{UV}}[\theta, A] = \mathcal{A}_{\mathrm{IR}}[\theta, A]$
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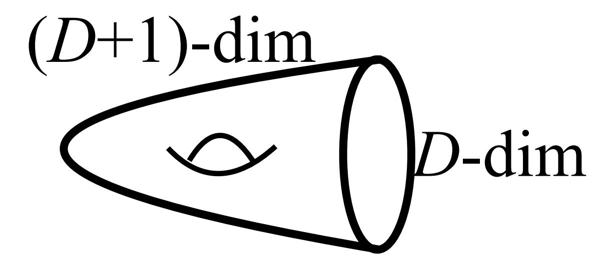
't Hooft anomaly matching



- Regard the system as boundary of $(D+1)$d SPT $Z[A]\cdot\exp(iS_{D+1}[A])$
- There is cancellation of 't Hooft anomaly $\delta_\theta S_{D+1}[A] = -\mathcal{A}[\theta, A]$
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't Hooft anomaly is RG-invariant ! $\mathcal{A}_{\mathrm{UV}}[\theta, A] = \mathcal{A}_{\mathrm{IR}}[\theta, A]$
it gives constraints on IR strongly-coupled physics

ex.) SU(N_{f})$_{\mathrm{A}}$ 't Hooft anomaly in N_{f}-flavor massless QCD

- At UV it has 't Hooft anomaly, thus it also does at IR
- Trivially gapped phase (confined phase without SSB) is forbidden
- It indicates **spontaneous chiral symmetry breaking**

Existence of 't Hooft anomaly means absence of trivially gapped phase

Mixed 't Hooft anomaly

- Consider theory with global symmetries G_1 and G_2
- Gauge one of them by background G_1-gauge field A_1
- It means the symmetry G_2 can be broken

⇒ G_1 and G_2 have Mixed 't Hooft anomaly

Mixed 't Hooft anomaly

- Consider theory with global symmetries G_1 and G_2
- Gauge one of them by background G_1-gauge field A_1
- It means the symmetry G_2 can be broken

$\Rightarrow$ G_1 and G_2 have Mixed 't Hooft anomaly

ex.) 3d free massless Dirac fermion with U(1) & T

$$\mathcal{Z}[A] = |\mathcal{Z}[A]| \exp(\mathrm{i}\eta[A]/2)$$ gauged U(1) partition function

$\Rightarrow$ $$\mathcal{Z}[\mathsf{T} \cdot A] = \mathcal{Z}[A] \exp\left(-\frac{\mathrm{i}}{4\pi} \int A \mathrm{d} A\right)$$ under T transformation

U(1) & T has mixed 't Hooft anomaly : 3d boundary of 4d U(1) × T SPT

Recent progress in ’t Hooft anomaly

- Generalization to systems without fermions
- Generalization to higher-form symmetries
- Generalization to compactified theories

• SU(N) YM with θ=π	Gaiotto, Kapustin, Komargodski, Seiberg (17)
• Bifundamental gauge theories with θ=π	Tanizaki, Kikuchi (17)
• CP^{N-1} models on R^2 and $R\times S^1$	Komargodski, Sharon, Thorngren, Zhou (17) Tanizaki, TM, Sakai (17)
• RW-symmetric QCD and QCD(adj.)	Shimizu, Yonekura (17)
• QCD with θ=π and Dashen phase	Gaiotto, Komargodski, Seiberg (17)
• N-flavor QCD on $R^3\times S^1$	Tanizaki, TM, Sakai(17) Tanizaki, Kikuchi, TM, Sakai (17)
• Extension of Lieb-Schultz-Mattis theorem	Cho, Hsieh, Ryu (17) Kobayashi, Shiozaki, Kikuchi, Ryu (18)
• SU(N) spin system & Flag sigma model	Yao, Hsieh, Oshikawa(18) Tanizaki, Sulejmanpasic (18) Hongo, TM, Tanizaki (18)
• Charge-q Schwinger model	Anber, Poppitz (18) Armoni, Sugimoto (18) TM, Tanizaki, Unsal (19)
• **Lattice Wilson fermion & Aoki phase**	TM, Tanizaki (19)

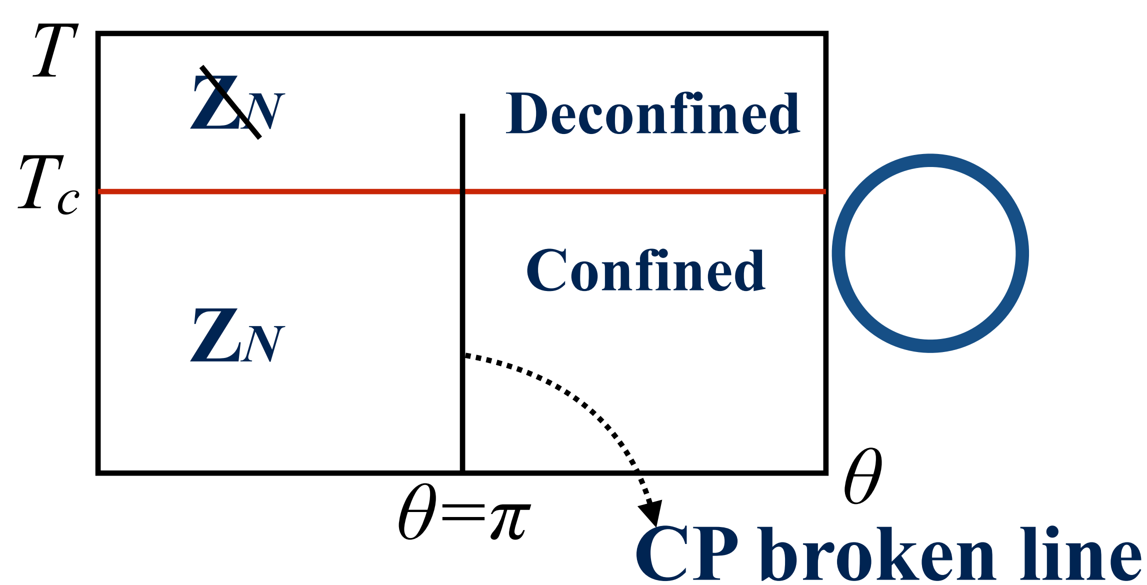
SU(N) Yang-Mills theory with $\theta=\pi$ on $R_3 \times S_1$

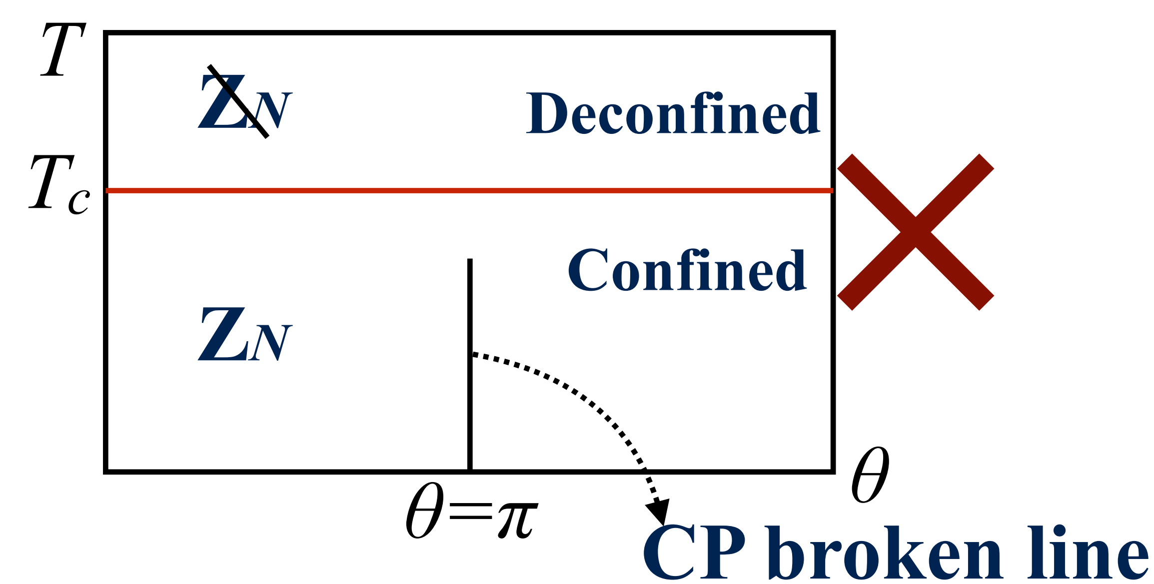
Gaiotto, Kapustin, Komargodski, Seiberg (17)

CP symmetry & Z_N 1-form symmetry (Center symmetry)

By introducing background gauge field for Z_N 1-form symmetry, one finds CP is broken : mixed 't Hooft anomaly

Mixed 't Hooft anomaly indicates SSB of CP or Z_N 1-form symmetry even at finite-temperature : trivially gapped phase forbidden!





Use of 't Hooft anomaly matching

't Hooft anomaly of G at UV

't Hooft anomaly of G at IR
=
Trivially gapped phase is prohibited

SSB of symmetry G
(gapped)

CFT
(gapless)

topological phase
(gapless)

Use of 2D 't Hooft anomaly matching

't Hooft anomaly of G at UV

't Hooft anomaly of G at IR
=
Trivially gapped phase is prohibited

SSB of discrete sym.
(gapped)

CFT
(gapless)

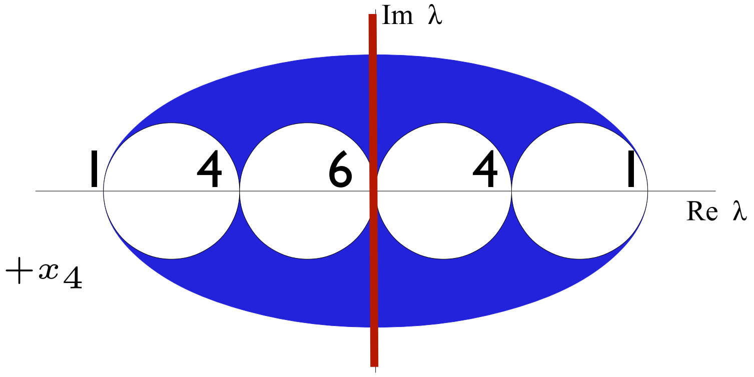
Is ’t Hooft anomaly matching applicable to lattice field theory?

Central-branch Wilson

Creutz, Kimura, TM (11)
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

◆ Wilson without onsite terms $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2}\sum_{x,\mu}\bar{\psi}_x[\gamma_\mu(U_{x,\mu}\psi_{x+\mu} - U_{x,-\mu}\psi_{x-\mu}) - (U_{x,\mu}\psi_{x+\mu} + U_{x,-\mu}\psi_{x-\mu})]$$



⇨ Extra U(1) for 6 flavors!

$$\psi_x \to e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}\psi_x,\ \ \bar{\psi}_x \to \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

- Flavor-chiral symmetry

$$\Gamma_X^{(+)} \in \left\{\mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\,,\ (-1)^{n_\mu}i\gamma_\mu\gamma_5\,,\ (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

$$\Gamma_X^{(-)} \in \left\{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\ \gamma_5\,,\ (-1)^{n_\mu}\gamma_\mu\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\ (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

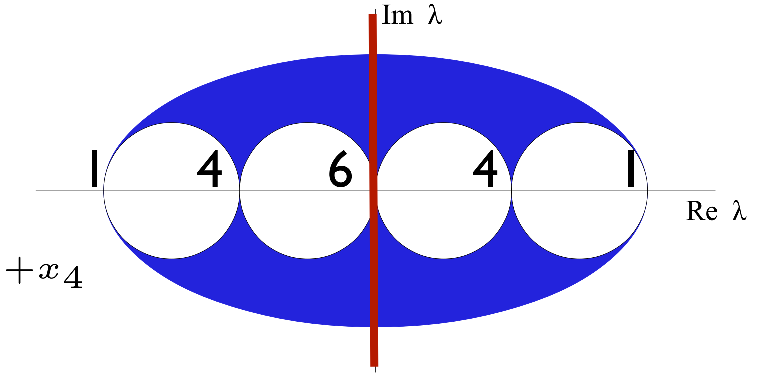
$$\psi_n \to \psi'_n = \exp\left[\mathrm{i}\sum_X\left(\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]\psi_n\,,\quad \bar{\psi}_n \to \bar{\psi}'_n = \bar{\psi}_n\exp\left[\mathrm{i}\sum_X\left(-\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]$$

Central-branch Wilson

Creutz, Kimura, TM (11)
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

◆ Wilson without onsite terms $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2} \sum_{x,\mu} \bar{\psi}_x [\gamma_\mu (U_{x,\mu}\psi_{x+\mu} - U_{x,-\mu}\psi_{x-\mu}) - (U_{x,\mu}\psi_{x+\mu} + U_{x,-\mu}\psi_{x-\mu})]$$



Extra U(1) for 6 flavors!

$$\psi_x \to e^{i\theta(-1)^{x_1+x_2+x_3+x_4}} \psi_x, \;\; \bar{\psi}_x \to \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

- Flavor-chiral symmetry

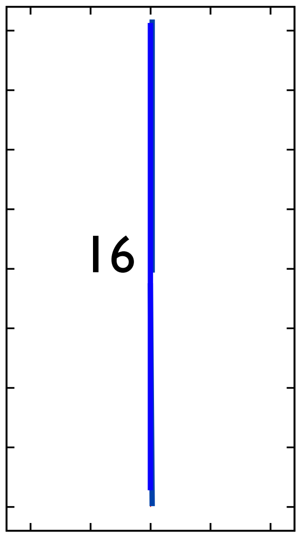
$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4 \,, \qquad\qquad \right\}$$

$$\Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\ldots+n_4} \mathbf{1}_4 \,, \qquad\qquad \right\}$$

$$\psi_n \to \psi_n' = \exp\left[\mathrm{i} \sum_X \left(\theta_X^{(+)} \Gamma_X^{(+)} + \theta_X^{(-)} \Gamma_X^{(-)} \right) \right] \psi_n \,, \quad \bar{\psi}_n \to \bar{\psi}_n' = \bar{\psi}_n \exp\left[\mathrm{i} \sum_X \left(-\theta_X^{(+)} \Gamma_X^{(+)} + \theta_X^{(-)} \Gamma_X^{(-)} \right) \right]$$

Flavor-chiral symmetries of Wilson fermion

Naive

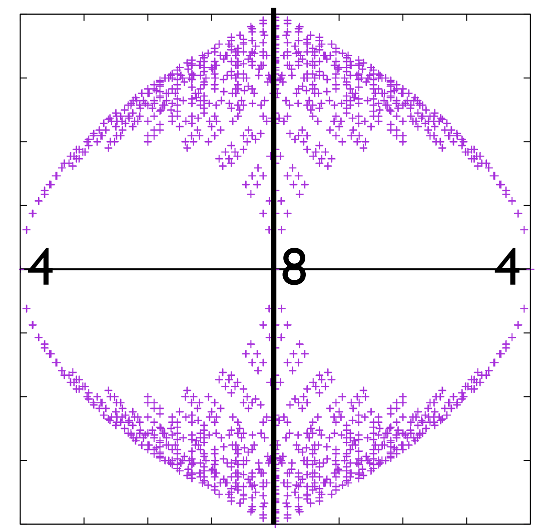


$C_{12} + C_{34}$

$C_\mu = (T_{+\mu} + T_{-\mu})/2$

$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$

CB Wilson'



U(4)×U(4) → U(2)×U(2)

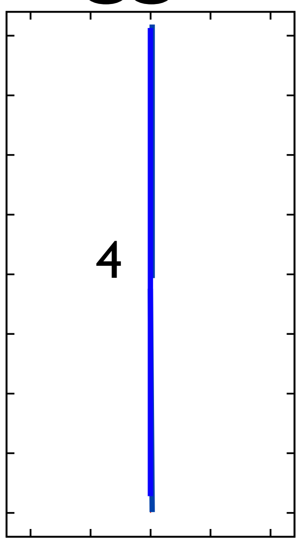
$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\qquad (-1)^{n_{1,2}}\frac{i\,[\gamma_1\,,\gamma_2]}{2}\,,\ (-1)^{n_{3,4}}\frac{i\,[\gamma_3\,,\gamma_4]}{2} \right\}$$

$$\Gamma_X^{(-)} \in \left\{ (-1)^{\check{n}_{1,3}}\frac{i\,[\gamma_1\,,\gamma_3]}{2}\,,\ (-1)^{\check{n}_{2,4}}\frac{i\,[\gamma_2\,,\gamma_4]}{2}\,,\ (-1)^{\check{n}_{1,4}}\frac{i\,[\gamma_1\,,\gamma_4]}{2}\,,\ (-1)^{\check{n}_{2,3}}\frac{i\,[\gamma_2\,,\gamma_3]}{2} \right\}$$

TM, Yumoto (20)

Flavor-chiral symmetries of Wilson fermion

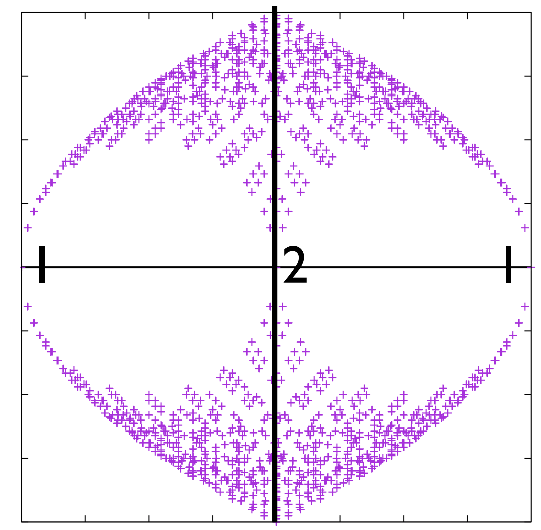
Staggered



$$i(\eta_{12}C_{12} + \eta_{34}C_{34})$$

$$(\eta_{\mu\nu})_{xy} = \epsilon_{\mu\nu}\eta_\mu\eta_\nu\delta_{x,y},\ (\epsilon_{\mu\nu})_{xy} = (-1)^{x_\mu + x_\nu}\delta_{x,y},$$

CB staggered-Wilson



Adams (09)
Hoebling(10)
de Forcrand et.al. (10)

$$\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \times \{U^\epsilon(1)\}_{m=0} \quad \Rightarrow \quad \{C_T,\ C_T',\ \Xi_\mu',\ R_{12},\ R_{34},\ R_{24}R_{31}\}$$

$$C_T' :\ \chi_x \to \bar{\chi}_x^T,\ \bar{\chi}_x \to \chi_x^T,\ U_{x,\mu} \to U_{x,\mu}^*$$

Sharpe (12)
TM, Kimura, Nakano, Ohnishi (12)

2D Central-branch Wilson

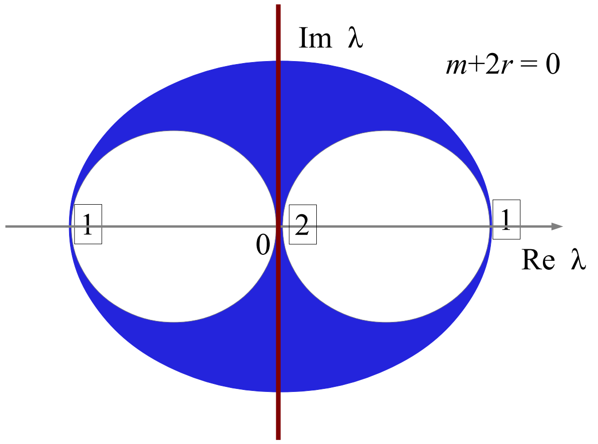
Tanizaki, TM (19)

◆ Wilson without onsite terms $m + 2r = 0$

$$S_{\mathrm{CB}} = \sum_{n,\mu} \left(\bar{\psi}_n \gamma_\mu D_\mu \psi_n - r \bar{\psi}_n C_\mu \psi_n \right)$$

Extra U(1) for 2 flavors!

$$U(1)_{\overline{V}} : \psi_n \mapsto \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta} \psi_n, \ \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}$$



Dirac eigenvalue distribution

- Flavor-chiral symmetry for 2D naive fermion

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_2 \,, \ (-1)^{n_1+n_2} \gamma_3 \,, \ (-1)^{\check{n}_\mu} \gamma_\mu \right\}$$

$$\Gamma_X^{(-)} \in \left\{ (-1)^{n_1+n_2} \mathbf{1}_2 \,, \ \gamma_3 \,, \ (-1)^{n_\mu} \gamma_\mu \right\}$$

2D Central-branch Wilson

Tanizaki, TM (19)

◆ Wilson without onsite terms $m + 2r = 0$

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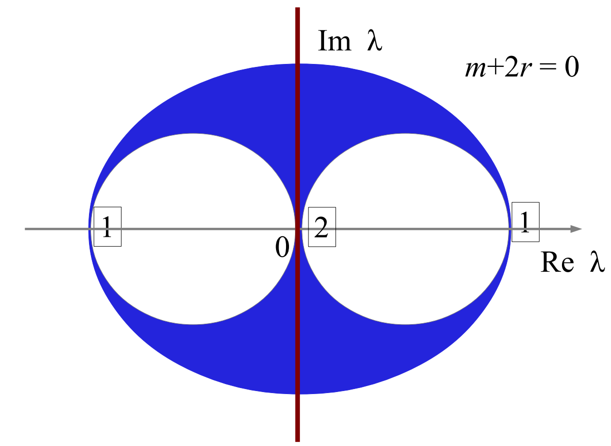


Dirac eigenvalue distribution

Extra U(1) for 2 flavors!

$$U(1)_{\overline{V}} : \psi_n \mapsto \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta} \psi_n,\ \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}$$

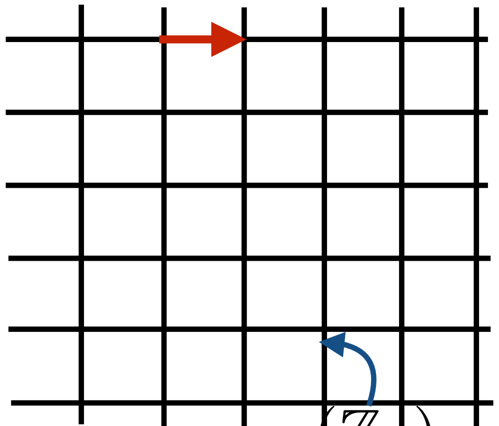
- Flavor-chiral symmetry for central-branch Wilson fermion

$$\Gamma_X^{(+)} \in \{ \mathbf{1}_2 \qquad \qquad \}$$

$$\Gamma_X^{(-)} \in \{ (-1)^{n_1+n_2} \mathbf{1}_2 \qquad \qquad \}$$

Symmetries of CB Wilson

Tanizaki, TM (19)



◆ Relevant symmetries

$$U(1)_V : \psi_n \mapsto \mathrm{e}^{\mathrm{i}\alpha}\psi_n, \quad \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{-\mathrm{i}\alpha}$$

$$U(1)_{\overline{V}} : \psi_n \mapsto \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}\psi_n, \ \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}$$

$$(\mathbb{Z}_2)_{\mathrm{lat.trans.}} : \quad \psi(x,y) \mapsto \psi(x+1,y) \quad \psi(x,y) \mapsto \psi(x,y+1)$$

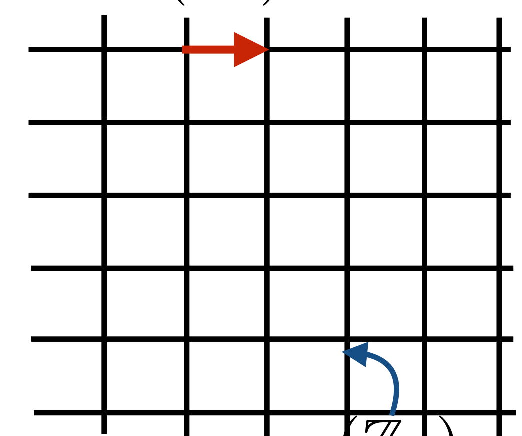
$$(\mathbb{Z}_2)_\chi : \quad \psi(x,y) \mapsto \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\gamma_3}\psi(y,-x), \ \overline{\psi}(x,y) \mapsto \overline{\psi}(y,-x)\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\gamma_3}$$

$$\boxed{G_{\mathrm{CB\ fermion}} = \frac{U(1)_V \times [U(1)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\mathrm{lat.\ trans.}}]}{(\mathbb{Z}_2)_F} \times (\mathbb{Z}_2)_\chi}$$

’t Hooft anomaly matching tells CB-Wilson Schwinger model is gapless or has SSB of Z_2

Symmetries of CB Wilson

Tanizaki, TM (19)



◆ Relevant symmetries

$$U(1)_V : \psi_n \mapsto \mathrm{e}^{\mathrm{i}\alpha}\psi_n, \quad \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{-\mathrm{i}\alpha}$$

$$U(1)_{\overline{V}} : \psi_n \mapsto \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}\psi_n, \ \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}$$

$$(\mathbb{Z}_2)_{\mathrm{lat.trans.}} : \quad \psi(x,y) \mapsto \psi(x+1,y) \quad \psi(x,y) \mapsto \psi(x,y+1)$$

$$(\mathbb{Z}_2)_\chi : \quad \psi(x,y) \mapsto \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\gamma_3}\psi(y,-x), \ \overline{\psi}(x,y) \mapsto \overline{\psi}(y,-x)\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\gamma_3}$$

$$G_{\mathrm{CB\ fermion}} = \frac{U(1)_V \times [U(1)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\mathrm{lat.\ trans.}}]}{(\mathbb{Z}_2)_F} \times \boxed{(\mathbb{Z}_2)_\chi}$$

$$O(2) \subset \frac{SU(2)_{\mathrm{flavor}}}{(\mathbb{Z}_2)_F} \qquad \subset U(1)_A$$

Anomaly matching for QED with CB

- 2D QED with CB Wilson = Schwinger-like model

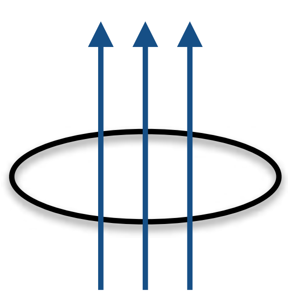
$$G = G_{\text{CB fermion}}/U(1)_V = \frac{U(1)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\text{lat. trans.}}}{(\mathbb{Z}_2)_F} \times (\mathbb{Z}_2)_\chi$$

Pay attention to discrete subgroup of vector-like symmetry

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \boxed{\frac{(\mathbb{Z}_4)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\text{lat. trans.}}}{(\mathbb{Z}_2)_F}} \subset \frac{U(1)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\text{lat. trans.}}}{(\mathbb{Z}_2)_F}$$

- Z_N background 1-form gauge field

→ nothing but Z_N twisted boundary conditions = Z_N flux



$$\mathrm{i}\tau_1 \in (\mathbb{Z}_4)_{\overline{V}} \qquad\qquad \tau_3 \in (\mathbb{Z}_2)_{\text{lat. trans.}}$$

Anomaly matching for QED with CB

- Gauging vector-like symmetry $\dfrac{(\mathbb{Z}_4)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\text{lat. trans.}}}{(\mathbb{Z}_2)_F}$

  Twisted boundary conditions with $i\tau_1 \in (\mathbb{Z}_4)_{\overline{V}}$ $\quad \tau_3 \in (\mathbb{Z}_2)_{\text{lat. trans.}}$

- $(\mathbb{Z}_2)_\chi$ transformation on the gauged action

  $(\mathbb{Z}_2)_\chi : \mathcal{Z}_{\text{twisted}} \mapsto -\mathcal{Z}_{\text{twisted}}$ $\quad$ Z_2 't Hooft anomaly

1. massless excitation
2. spontaneous symmetry breaking of $(\mathbb{Z}_4)_{\overline{V}}, \ (\mathbb{Z}_2)_{\text{lat.trans.}}, \ (\mathbb{Z}_2)_\chi$

- Several possibilities of low-energy dynamics……
- Is there a cond-mat system in the same universality class?
- Yes ! It is 1D XXZ Heisenberg spin chain system ! 

Anomaly matching for QED with CB

- XXZ spin chain

$$\hat{H} = -\sum_{\ell}(J_x \hat{X}_\ell \hat{X}_{\ell+1} + J_y \hat{Y}_\ell \hat{Y}_{\ell+1} + J_z \hat{Z}_\ell \hat{Z}_{\ell+1}) \qquad J \equiv J_x = J_y \neq J_z$$

Symmetries : $SO(2) \rtimes \mathbb{Z}_2 \times (\mathbb{Z}_2)_{\mathrm{lat.trans.}}$ same symmetry structure as CB-Wilson Schwinger model

- Known facts on the system

$|J_z/J| < 1$ gapless phase with spin-wave or spinon

Anomaly matching for QED with CB

◆ XXZ spin chain

$$\hat{H} = -\sum_{\ell}(J_x \hat{X}_\ell \hat{X}_{\ell+1} + J_y \hat{Y}_\ell \hat{Y}_{\ell+1} + J_z \hat{Z}_\ell \hat{Z}_{\ell+1}) \qquad J \equiv J_x = J_y \neq J_z$$

Symmetries : $\cancel{SO(2) \rtimes \mathbb{Z}_2} \times (\mathbb{Z}_2)_{\mathrm{lat.trans.}}$

same symmetry structure as CB-Wilson Schwinger model

◆ Known facts on the system

$|J_z/J| < 1$ gapless phase with spin-wave or spinon

$|J_z/J| > 1$ Z_2 SSB : ferromagnetic

Anomaly matching for QED with CB

- XXZ spin chain

$$\hat{H} = -\sum_{\ell} (J_x \hat{X}_\ell \hat{X}_{\ell+1} + J_y \hat{Y}_\ell \hat{Y}_{\ell+1} + J_z \hat{Z}_\ell \hat{Z}_{\ell+1}) \qquad J \equiv J_x = J_y \neq J_z$$

Symmetries : $SO(2) \rtimes \mathbb{Z}_2 \times \cancel{(\mathbb{Z}_2)_{\mathrm{lat.trans.}}}$

same symmetry structure
as CB-Wilson Schwinger model

- Known facts on the system

$|J_z/J| < 1$ gapless phase with spin-wave or spinon

$|J_z/J| > 1$ Z_2 SSB : anti-ferromagnetic

Anomaly matching for QED with CB

- XXZ spin chain

$$\hat{H} = -\sum_{\ell} (J_x \hat{X}_\ell \hat{X}_{\ell+1} + J_y \hat{Y}_\ell \hat{Y}_{\ell+1} + J_z \hat{Z}_\ell \hat{Z}_{\ell+1}) \qquad J \equiv J_x = J_y \neq J_z$$

Symmetries : $SO(2) \rtimes \mathbb{Z}_2 \times (\mathbb{Z}_2)_{\mathrm{lat.trans.}}$

same symmetry structure
as CB-Wilson Schwinger model

- Known facts on the system

$|J_z/J| < 1$ gapless phase with spin-wave or spinon

$|J_z/J| > 1$ Z_2 SSB : ferromagnetic or anti-ferromagnetic

⇨ consistent to our anomaly matching !

CB-Wilson Schwinger enables us to simulate Heisenberg spin chain

Anomaly matching for CB Gross-Neveu

◆ 2D Gross-Neveu with CB Wilson

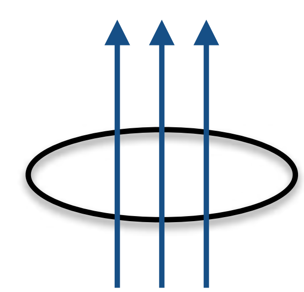
$$S = S_{\rm CB} + \frac{g^2}{2}\sum_{(x,y)}\left[\left(\overline{\psi}\psi(x,y)\right)^2 + \left(\overline{\psi}\mathrm{i}\gamma_3\psi(x,y)\right)^2\right]$$

⇒ symmetries : $\dfrac{(\mathbb{Z}_4)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\rm lat.trans.}}{(\mathbb{Z}_2)_F} \times (\mathbb{Z}_2)_\chi$

smaller but including all necessary for the anomaly

◆ Gauging vector-like symmetry with TBC

◆ $(\mathbb{Z}_2)_\chi$ transformation on the gauged action



⇒ $(\mathbb{Z}_2)_\chi : \mathcal{Z}_{\rm twisted} \mapsto -\mathcal{Z}_{\rm twisted}$ $\quad Z_2$ 't Hooft anomaly

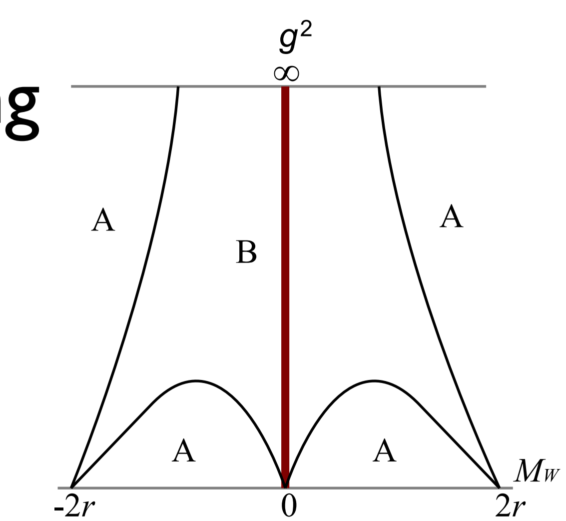
Anomaly matching for CB Gross-Neveu

$(\mathbb{Z}_2)_\chi : \mathcal{Z}_{\text{twisted}} \mapsto -\mathcal{Z}_{\text{twisted}}$ Z_2 't Hooft anomaly

1. massless excitation → unlikely for asymptotic-free model
2. two vacua by Z_2 spontaneous symmetry breaking among $(\mathbb{Z}_4)_{\overline{V}}$, $(\mathbb{Z}_2)_{\text{lat.trans.}}$, $(\mathbb{Z}_2)_\chi$

- Aoki phase conjecture $\langle \overline{\psi} \mathrm{i}\gamma_3 \psi \rangle$ yields following breaking

$$(\mathbb{Z}_4)_{\overline{V}} \xrightarrow{\text{SSB}} (\mathbb{Z}_2)_F$$



⇒ consistent to the anomaly matching condition !

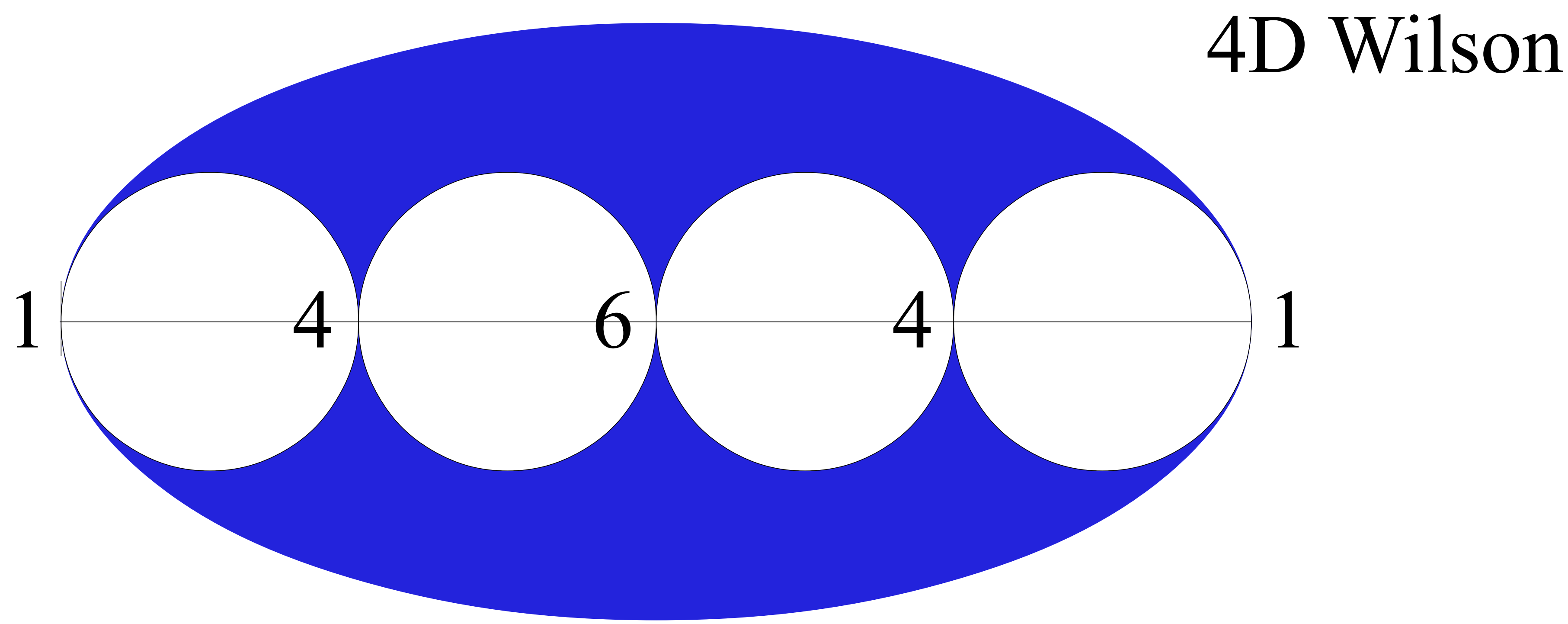
Mixed 't Hooft anomaly is matched by the existence of Aoki phase.

3. Species doubling & Betti numbers

Yumoto, TM (22)(23)

Reconsider Naive and Wilson

Yumoto, TM (22)



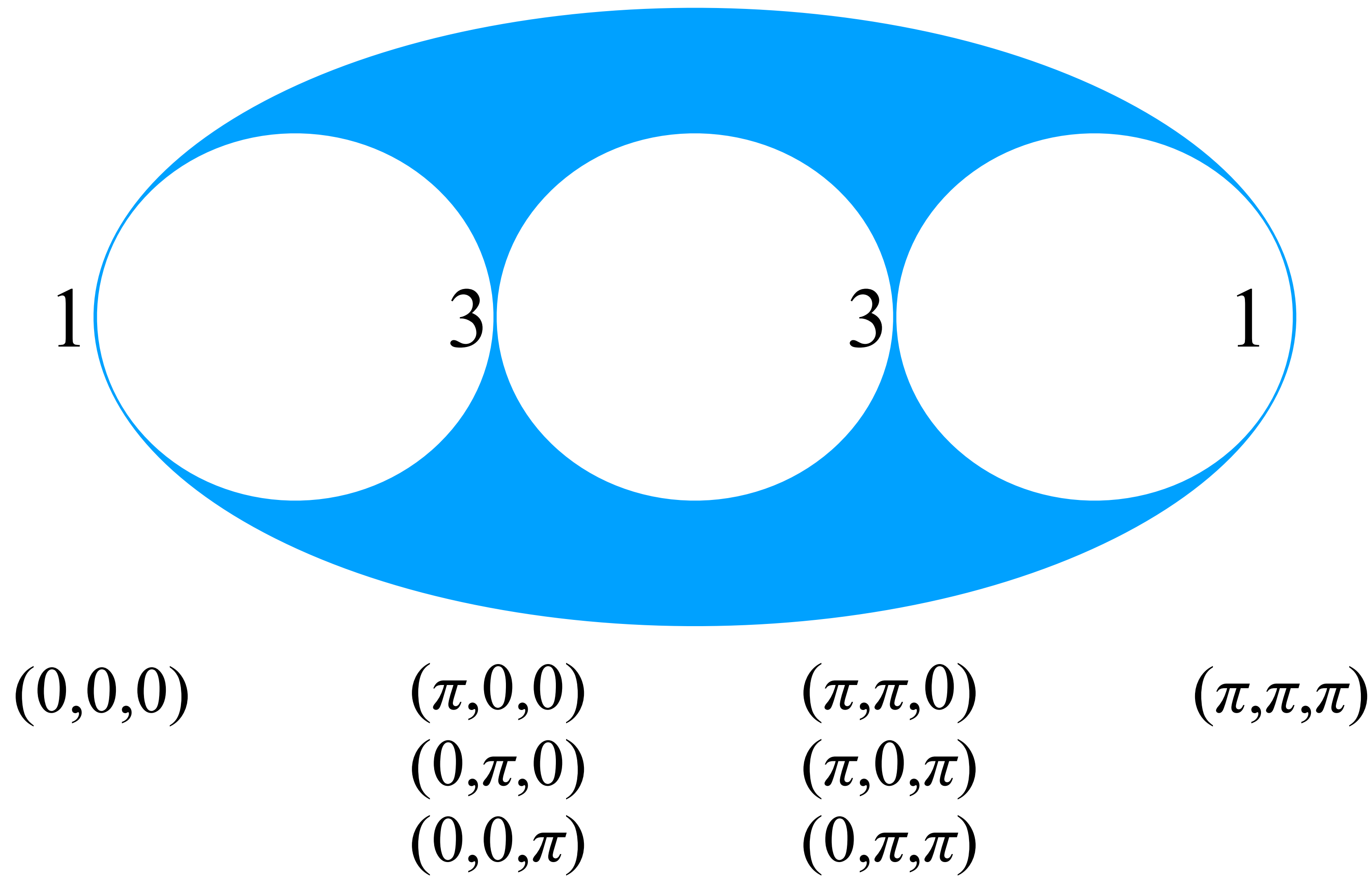
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$(0,\pi,0,0)$	$(\pi,0,\pi,0)$	$(\pi,\pi,0,\pi)$	
	$(0,0,\pi,0)$	$(\pi,0,0,\pi)$	$(\pi,0,\pi,\pi)$	
	$(0,0,0,\pi)$	$(0,\pi,0,\pi)$	$(0,\pi,\pi,\pi)$	
		$(0,0,\pi,\pi)$		
		$(0,\pi,\pi,0)$		

What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

3D Wilson

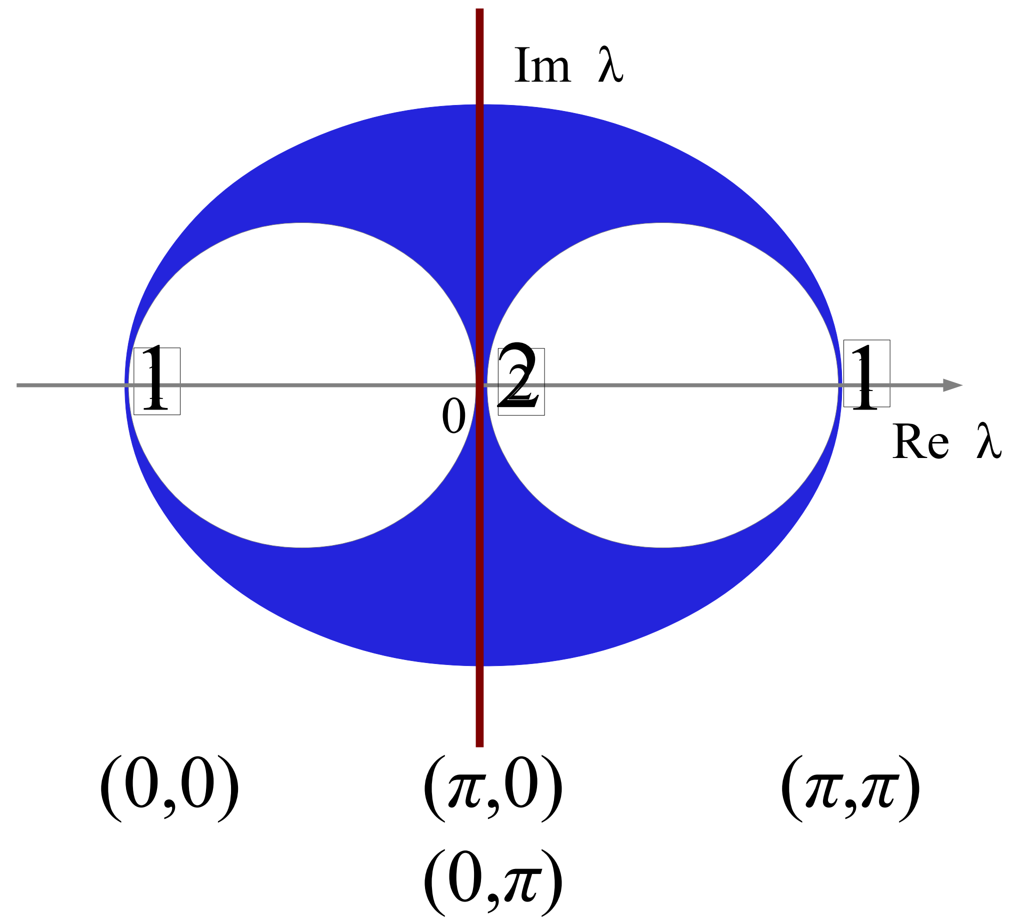


What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

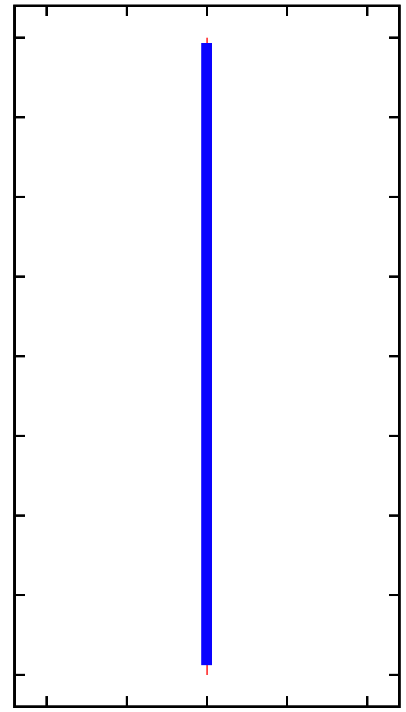
2D Wilson



What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)



4D Naive
on Torus

16

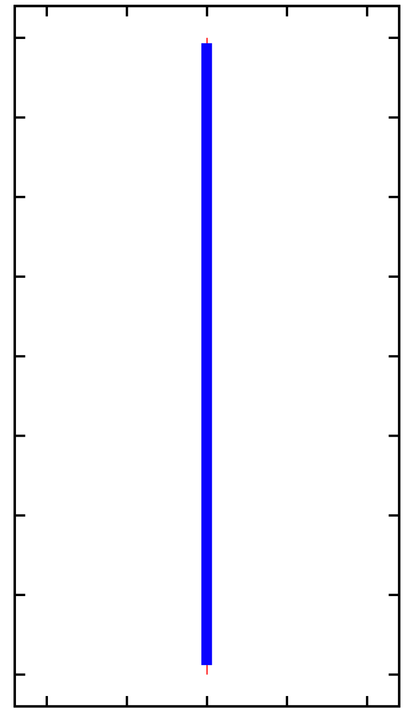
1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$\vdots$	$\vdots$	$\vdots$	

What is the meaning of the numbers?

The reason why $p=\pi$ becomes zero of Dirac operator is "periodicity"

Reconsider Naive and Wilson

Yumoto, TM (22)



4D Naive
on Torus

16

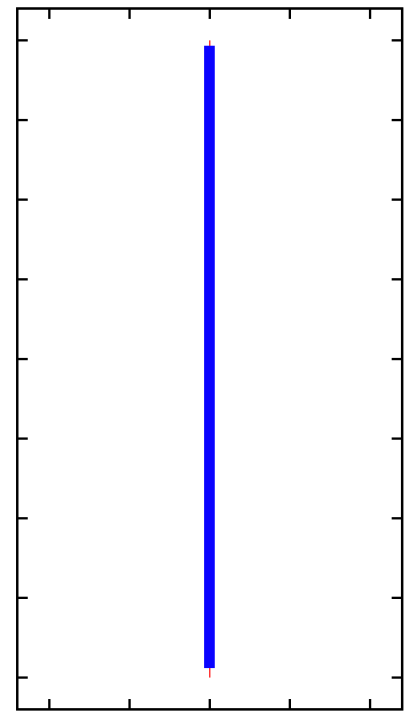
1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$\vdots$	$\vdots$	$\vdots$	

What is the meaning of the numbers?

It means these numbers are related to certain topological invariants

Reconsider Naive and Wilson

Yumoto, TM (22)



4D Naive
on Torus

16

1	4	6	4	1
(0,0,0,0)	(π,0,0,0)	(π,π,0,0)	(π,π,π,0)	(π,π,π,π)
	⋮	⋮	⋮	
# of 0d holes	# of 1d holes	# of 2d holes	# of 3d holes	# of 4d holes

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) \ (= \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- **4D torus**

$$\beta_0(M) = 1 \quad \beta_1(M) = 4 \quad \beta_2(M) = 6 \quad \beta_3(M) = 4 \quad \beta_4(M) = 1$$

$$\Downarrow$$

Sum of Betti numbers is 16 $\longrightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) \ (= \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- **3D torus**

$$\beta_0(M) = 1 \qquad \beta_1(M) = 3 \qquad \beta_2(M) = 3 \qquad \beta_3(M) = 1$$

$\Downarrow$

Sum of Betti numbers is 8 $\rightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) \; (= \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- **2D torus**

$$\beta_0(M) = 1 \qquad \beta_1(M) = 2 \qquad \beta_2(M) = 1$$

$\Downarrow$

Sum of Betti numbers is 4 $\rightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M)\ (= \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- **D-dim hyperball**

$$\beta_0(M) = 1 \quad \beta_1(M) = 0 \quad \beta_2(M) = 0 \quad \ldots..$$

Sum of Betti numbers is 1 $\rightarrow$ # of Dirac zero modes in free theory

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) \ (= \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- $\mathbf{T}^4 \times \mathbf{R}^1$

$\beta_0(M) = 1 \quad \beta_1(M) = 4 \quad \beta_2(M) = 6 \quad \beta_3(M) = 4 \quad \beta_4(M) = 1 \quad \beta_5(M) = 0$

$\Downarrow$

Sum of Betti numbers is 16 $\longrightarrow$ maximal # of species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M)\ (= \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- $\mathbf{T^2 \times R^2}$

$\beta_0(M) = 1 \quad \beta_1(M) = 2 \quad \beta_2(M) = 1 \quad \beta_3(M) = 0 \quad \beta_4(M) = 0$

Sum of Betti numbers is 4 $\rightarrow$ maximal # of species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M)\ (= \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1})$$

n-th Betti number is a rank of n-th homology group

- **2D Spheres**

$$\beta_0(M) = 1 \quad \beta_1(M) = 0 \quad \beta_2(M) = 1$$

$$\Downarrow$$

Kamata, Matsuura, TM, Ohta (16)
Yumoto, TM (21)

Sum of Betti numbers is 2 $\rightarrow$ # of Dirac zero modes in free theory

Topological invariants

Yumoto, TM (22)

	sum of $\beta_n(M)$	max # of free Dirac zeromodes
1D torus	1+1	2
2D torus	1+2+1	4
3D torus	1+3+3+1	8
4D torus	1+4+6+4+1	16
T^D	$(1+1)^D$	2^D
Hyperball	1+0+0+….	1
Sphere	1+0+0+…+1	2
$T^D \times R^d$	$2^D + 0$	2^D

Conjecture on fermion species

Yumoto, TM (22)

- **Conjecture**

A sum of Betti numbers of a continuum manifold
is equivalent to
a maximal number of exact Dirac zero modes
on the discretized version of the manifold.

It can be a theorem complementary to Nielsen-Ninomiya's no-go theory.

Sketch of proof

Yumoto, TM (23)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus and 1D ball by homology theory and Hodge theory.

↓

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4D Torus and Hyperball.

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*)$$

↓

Classify necessary conditions and complete proof.

Details of this conjecture will be discussed in Jun Yumoto's talk on 7th

Summary

- Wilson fermion is regarded asSPT, which has gapless mode at the boundary, or Domain-wall fermion.
- ’t Hooft anomaly matching is applicable to Lattice field theory. It may reveal phase structure specific to lattice.
- New conjecture on fermion doubling is proposed: Maximal # of exact Dirac zeromodes on discretized manifold is equal to sum of Betti numbers of the manifold.

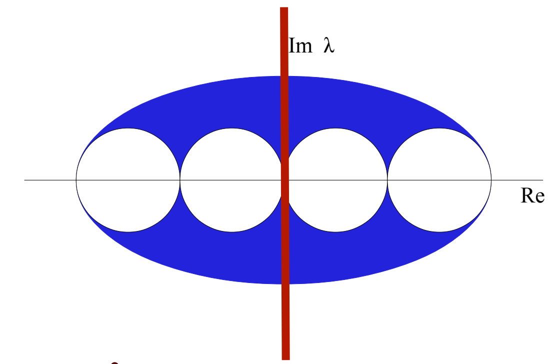
Semi-positivity of det(D)

Tanizaki, TM (19)

◆Use of extra U(1) symmetry $\mathsf{D}(-1)^{\sum_\mu n_\mu} = -(-1)^{\sum_\mu n_\mu}\mathsf{D}$

$$\mathsf{D}|R_\lambda\rangle = \lambda|R_\lambda\rangle \qquad \langle L_\lambda|\mathsf{D} = \lambda\langle L_\lambda|$$

$$\Rightarrow \quad \mathsf{D}(-)^{\sum_\mu n_\mu}|R_\lambda\rangle = -\lambda(-)^{\sum_\mu n_\mu}|R_\lambda\rangle, \qquad \langle L_\lambda|(-)^{\sum_\mu n_\mu}\mathsf{D} = -\lambda\langle L_\lambda|(-)^{\sum_\mu n_\mu}.$$



It shows that +λ and -λ make a pair.

◆Use of hermitian Dirac operator $H = \gamma_5\mathsf{D}$

$$H(-1)^{\sum_\mu n_\mu} = -(-1)^{\sum_\mu n_\mu}H \quad \Rightarrow \quad \{\pm\varepsilon_i\}_{i=1,\dots,N}$$

Pair of +ε and -ε

$$\Rightarrow \quad \det(\mathsf{D}) = \det(H) = \prod_{i=1}^{N}\varepsilon_i(-\varepsilon_i) = (-1)^N\prod_{i=1}^{N}\varepsilon_i^2 > 0$$

$N = N_1N_2N_3N_4$: even

This procedure shows semi-positivity $\det(\mathsf{D}) \geq 0$