Nucleon radial density moments

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Paper 2: In preparation

Key Words: *Form Factor; Probability Density Function; Radial Density Moments; Observational Bias; Proton Charge Radius*.

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Outline

- A **new approach** to extract the **moments of a probability density function with integral forms of its Fourier transform**: focus on the nucleon charge density
- Accessing **new observables** with the new method
- The conceptual implication of this new method on **the experimental determination** of the spatial moments of the charge density
- A reanalysis of some **proton electric form factor data** with **application of the new method**

Determining moments of the charge density

- **Ideally: the charge density $\rho_E(\mathbf{r})$ is known** from the fourier transform of $G_E(k^2)$

$$(r^\lambda, \rho_E) = \int \mathrm{d}^3\mathbf{r}\, r^\lambda \rho_E(\mathbf{r})$$

  - **Experimentally complicated**

- (r^{2j}, ρ_E) **from the derivative of electric Form Factor (FF)** $G_E(k^2)$

$$\langle r^{2j} \rangle \equiv (r^{2j}, \rho_E) = (-1)^j \frac{(2j+1)!}{j!} \frac{\mathrm{d}^j G_E(k^2)}{\mathrm{d}(k^2)^j} \Big|_{k^2=0}$$

  - Relies on **zero-momentum extrapolation** of the k^2-dependency
    - **Sensitive to the functional form** used in the extrapolation
    - **Sensitive to the interpolation** boundaries of the FF
  - Only **even** moments of positive order

Determining moments of the charge density

- Moments beyond the **second order** are of interest :

  Complementary information on the charge distribution inside the nucleon.

- **Negative orders** are relevant for the study of the high-momentum dependence of the form factor:

  Essential to understand **short range effects** near the nucleon's center

- **High positive order** moments probe the low-momentum behavior of the form factor:

  Scan of the **density** close to the nucleon's surface.

Goal: determination of spatial moments of densities at any order directly in momentum space

Mathematical formulation of the new method

- **Momentum density** $\tilde{f}(\mathbf{k}) \equiv \tilde{f}(k) = \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})$
- **Spatial density** $f(\mathbf{r}) \equiv f(r) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k})$

$$(r^\lambda, f) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3\mathbf{k}\, \tilde{f}(\mathbf{k}) \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} r^\lambda$$

Finite term: (r^λ, f) — Divergent term: $\int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} r^\lambda$

- **The divergent term needs to be regularized as by definition the moment r^λ is finite**

$$g_\lambda(\mathbf{k}) \equiv g_\lambda(k) = \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} r^\lambda$$

- **Two methods to regularize:**
  - The principle value regularization: integral method 1 (IM1)
  - The exponential regularization: integral method 2 (IM2)

The principle value regularization IM1

- **The integral** $g_\lambda(\mathbf{k}) \equiv g_\lambda(k) = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} r^\lambda$ satisfies the relation $g_\lambda(t\mathbf{k}) = \frac{1}{t^{\lambda+3}} g_\lambda(\mathbf{k})$
  - The integral is to be considered as a distribution and counter terms ($\tilde{f}_{2j}$) **need to be subtracted to insure convergence**

$$(r^\lambda, f) = \mathcal{N}_\lambda \int_0^\infty \mathrm{d}k \left\{ \frac{\tilde{f}(k)}{k^{\lambda+1}} \right\}$$ with $$\mathcal{N}_\lambda = \frac{2^{\lambda+2}}{\sqrt{\pi}} \frac{\Gamma(\frac{\lambda+3}{2})}{\Gamma(-\frac{\lambda}{2})}$$ with the regularized moment $$(\underbrace{r^{m-\eta}}_{\lambda}, f) = \mathcal{N}_{m-\eta} \int_0^\infty \mathrm{d}k \frac{\tilde{f}(k) - \sum_{j=0}^{n} \tilde{f}_{2j} k^{2j}}{k^{m-\eta+1}}$$

$$\tilde{f}_{2j} = \frac{1}{j!} \left. \frac{\mathrm{d}^j \tilde{f}(k)}{\mathrm{d}(k^2)^j} \right|_{k=0}$$

- **The divergence appearing in the normalization term is canceled by the divergence in the integral**

$$(r^m, f) = \lim_{\eta \to 0^+} (r^{m-\eta}, f) \qquad m \text{ even}$$

$$(r^m, f) = (r^{m-\eta}, f)|_{\eta=0} \qquad m \text{ odd}.$$

Condition: $\lambda \succ -3$

The exponential regularization IM2

- The integral $g_\lambda(\mathbf{k}) \equiv g_\lambda(k) = \int\limits_{\mathbb{R}^3} d^3\mathbf{r}\, e^{i\,\mathbf{k}\cdot\mathbf{r}} r^\lambda$ **can be taken as the weak limit of the convergent integral**

$$g_\lambda(\mathbf{k}) = \lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}^3} d^3\mathbf{r}\, r^\lambda e^{-\epsilon r}\, e^{i\,\mathbf{k}\cdot\mathbf{r}} = \lim_{\epsilon\to 0^+} \mathcal{I}_\lambda(k,\epsilon) \quad \text{with} \quad \mathcal{I}_\lambda(k,\epsilon) = \frac{4\pi\, \Gamma(\lambda+2)\, \sin\left[(\lambda+2)\mathrm{Arctan}\,(k/\epsilon)\right]}{k(k^2+\epsilon^2)^{\frac{\lambda}{2}+1}}$$

- **The moment r^λ can then be written as**

$$(r^\lambda, f) = \frac{2}{\pi}\, \Gamma(\lambda+2) \lim_{\epsilon\to 0^+} \int\limits_0^\infty dk\, \tilde{f}(k)\, \frac{k \sin\left[(\lambda+2)\mathrm{Arctan}\,(k/\epsilon)\right]}{(k^2+\epsilon^2)^{\lambda/2+1}}$$

- **For integer values of λ**

$$(r^m, f) = \frac{2}{\pi}\, (m+1)! \lim_{\epsilon\to 0^+} \epsilon^{m+2} \int\limits_0^\infty dk\, \tilde{f}(k)\, \frac{k}{(k^2+\epsilon^2)^{m+2}}\, \Phi_m(k/\epsilon)$$

with $$\Phi_m(k/\epsilon) = \sum_{j=0}^{m+2} \sin\left(\frac{j\pi}{2}\right) \frac{(m+2)!}{j!(m+2-j)!} \left(\frac{k}{\epsilon}\right)^j$$

Condition: $\lambda > -3$

Saturation behavior of the moments

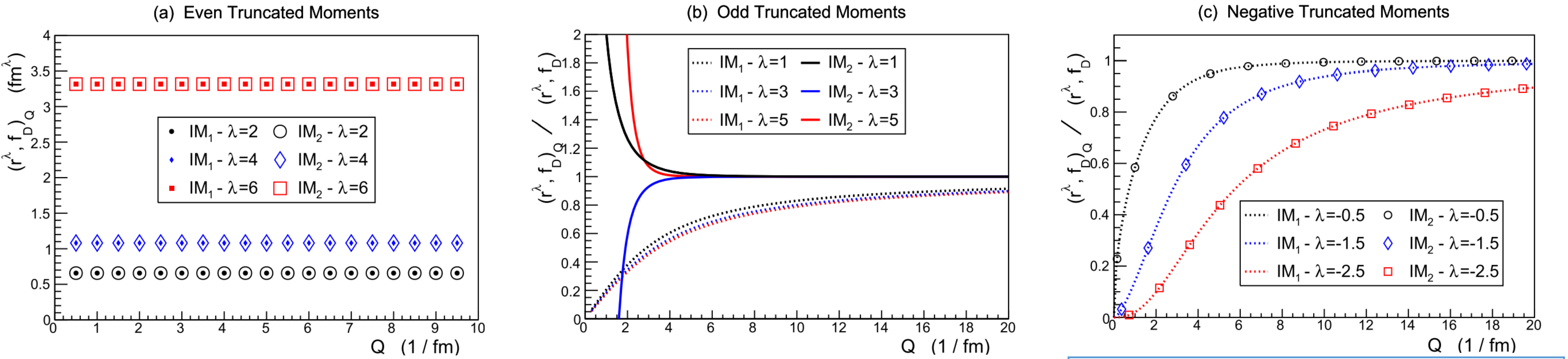
- **Moments evaluation requires an experimentally defined asymptotic limit; however:**
  - Momentum dependence of the integrands denominator **scales at large momentum**
  - Integrals are most likely to saturate at a momentum value well below infinity.
  - Cut-off Q replaces the infinite integral boundary : **truncated moments**.

Consequences:

- The integral method advocates the necessity of **fitting all available data** for the determination of the functional form
- The model needs only to be **integrable over the domain where data exists**. The convergence of the integrand is insured by a cut-off.
- Can **access moments with orders** $\lambda > -3$, integer and non − integer moments
  - Allows to experimentally determine all moments, even if the inverse Fourier transform of the FF doesn't exist
- The extraction of even moments are unchanged w.r.t. previous determinations
- **The extraction of odd moments represents a first experimental determination**

Validation of the integral method

Use the polynomial ratio parametrization $\tilde{f}_K(\mathbf{k}) \equiv \tilde{f}_K(k) = \dfrac{1 + a_1 k^2}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}$ J.J. Kelly, Phys. Rev. C 70 (2004) 068202.



Q-independence is reproduced by each prescription: the **IM recovers the same quantities as the derivative method**. The practical constraint is to obtain an **appropriate description of the data over a large k^2-domain**.

Different saturation behaviours of the two regularization methods: **IM1 asks for large Q-values, IM2 rapidly saturates about 6 fm^{-1}**
Sensitive to the high-momentum behaviour of the FF.

Same behavior for both methods IM1 and IM2
Convergence is not guaranteed within the domain covered by experimental data
Negative moments are of particular importance: behavior of the charge density near the nucleon's center

Application to real data

Very preliminary

- **Select data extracted from electron scattering experiments**
  - via a Rosenbluth separation
  - for kinematical conditions where the magnetic form factor contribution to the cross section is strongly suppressed
- **Use the functional form**

$$\tilde{f}_K(\mathbf{k}) \equiv \tilde{f}_K(k) = \frac{1 + a_1 k^2}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}$$

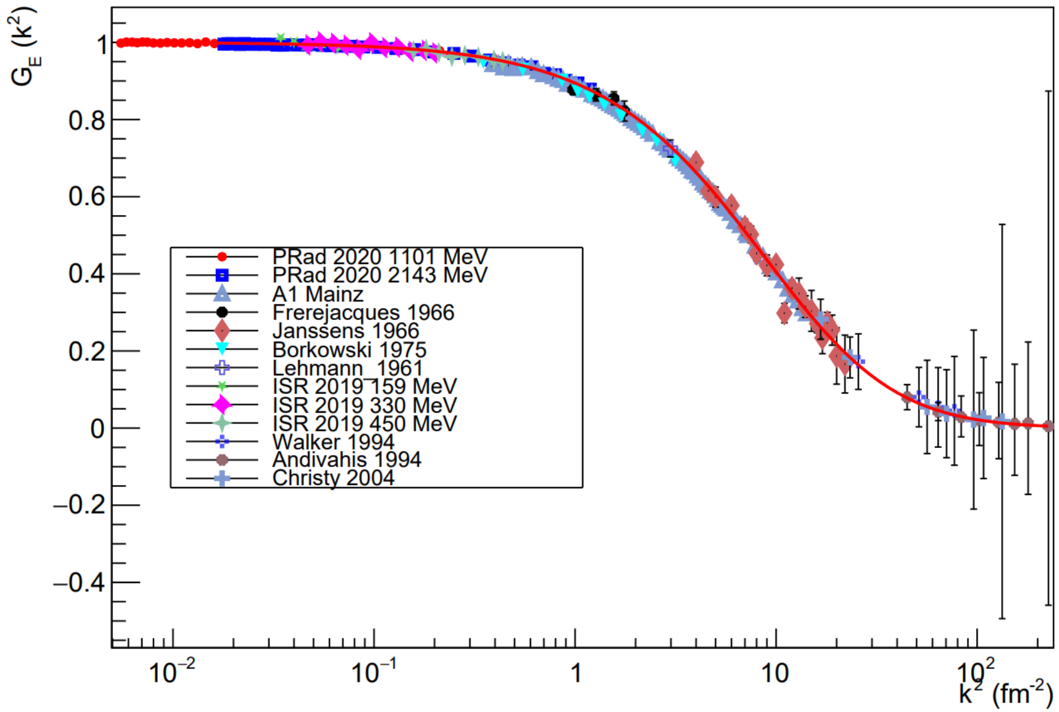
- **Fit simultaneously the different datasets**
  - The **same functional behavior** is assumed for each dataset
  - **A separate normalization** parameter is considered for each dataset

Author (Year)	k^2 range (fm^{-2})	nb. of data points
Xiong (2019)	0.005-14.9	71
Bernauer (2014)	0.39-14.15	77
Lehmann (1961)	1.05-2.98	1
Frerejacque (1966)	0.975-1.76	4
Janssens (1966)	4-30	20
Borkowski (1975)	0.35-3.15	10
Walker (1994)	25.65-77	4
Andivahis (1994)	44.85-226	8
Christy (2004)	16.65-133	7
Mihovilovic (2019)	0.0256 - 0.436	25

Application to real data: fit results

Very preliminary

- **Acceptable fit quality** with a reduced χ^2 of $\sim$1.9
- The evaluated **normalization parameters** reflect **the imprecision** on the determination of the absolute normalization for each experiment



Experiment	Normalization values	± (stat.)	± (sys.)
Frerejacques	0.907	0.0091	$5.700\ \times10^{-05}$
Janssens	1.005	0.0112	$4.500\ \times10^{-05}$
Borkowski	0.981	0.0025	$4.000\ \times10^{-05}$
Bernauer	0.990	0.0008	$3.200\ \times10^{-05}$
Xiong 1.1 GeV	1.000	0.0002	$3.600\ \times10^{-04}$
Xiong 2.1 GeV	0.984	0.0002	$2.400\ \times10^{-04}$
Walker	1.083	0.3798	0.0320
Andivahis	0.986	0.3009	0.3399
Christy	1.052	0.1612	3.2853×10^{-04}
Lehmann	0.993	0.0301	1.0600×10^{-04}
Mihovilovic 195 MeV	1.001	1.1322×10^{-03}	2.5755×10^{-03}
Mihovilovic 330 MeV	0.999	9.6429×10^{-04}	1.6312×10^{-03}
Mihovilovic 495 MeV	1.001	6.4563×10^{-04}	1.8105×10^{-03}

a_1 $(10^{-1}\ \text{fm}^2)$	b_1 $(10^{-1}\ \text{fm}^2)$	b_2 $(10^{-1}\ \text{fm}^4)$	b_3 $(10^{-3}\ \text{fm}^6)$
8.8005	9.9560	1.0300	2.9455
$(\delta a_1)_{Sta.}$ $(10^{-1}\ \text{fm}^2)$	$(\delta b_1)_{Sta.}$ $(10^{-1}\ \text{fm}^2)$	$(\delta b_2)_{Sta.}$ $(10^{-1}\ \text{fm}^4)$	$(\delta b_3)_{Sta.}$ $(10^{-3}\ \text{fm}^6)$
0.0054	0.0114	0.0601	0.0687
$(\delta a_1)_{Sys.}$ $(10^{-1}\ \text{fm}^2)$	$(\delta b_1)_{Sys.}$ $(10^{-1}\ \text{fm}^2)$	$(\delta b_2)_{Sys.}$ $(10^{-1}\ \text{fm}^4)$	$(\delta b_3)_{Sys.}$ $(10^{-3}\ \text{fm}^6)$
0.0003	0.0004	0.0001	0.0002

Application to real data: evaluation of moments

Very preliminary

- **Moments are evaluated for different values of λ**
  - With a Q^2 cutoff at 2 GeV^2
  - With $Q^2 \to \infty$
- **Both evaluations are compatible for positive valued moment orders**
  - The Q^2 cutoff provides a faithful representation of the physical moments
- **Negative moments show discrepancy when a cutoff is taken into account**
  - Motivation for measurements of the electric FF at large k^2
  - Would provide better understanding of the charge density near the nucleon's center

λ	$\langle r^\lambda \rangle_Q$ (fm^λ)	$\langle r^\lambda \rangle$ (fm^λ)
-2	7.671	8.699
-1	2.050	2.088
0	1.000	1.000
1	0.718	0.718
2	0.693	0.693
3	0.865	0.865
4	1.444	1.444
5	3.641	3.641
6	15.25	15.25
7	95.95	95.95

Truncated moments evaluated for the cutoff $Q^2 \to 52\mathrm{fm}^{-2}$

Exact moments evaluated in the limit $Q^2 \to \infty$

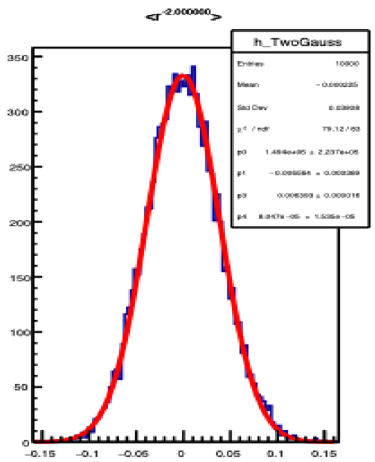
Application to real data: evaluation of errors

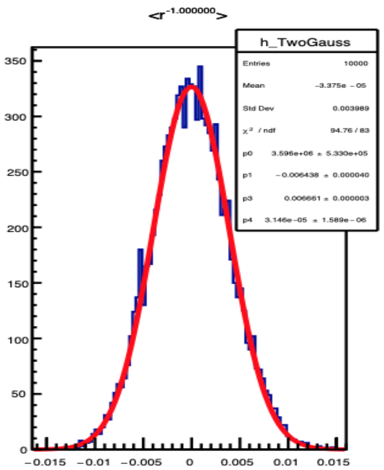
- **Statistical error** coming from **experimental data**
- Sources of **systematic errors**:
    1. Originating from the systematic error that is reported by each considered **experiment on the EFF**
    2. Bias that could be generated on the fit parameters from **the fitting model itself**
    3. Error coming from the **choice of the fitting model.**
- **Errors are propagated to the evaluated moments using MC methods**
    - Take into account correlations between parameters to all orders

Procedure:

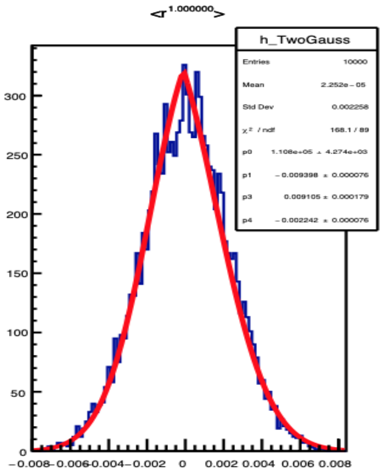
- Make replicas (10000) of data following the assumption of each error source
- Each replica is fitted with the chosen fitting model
- The moments are estimated from each replica
- A dedicated study of the bias and variance of the replicas is performed from which the different error sources are obtained

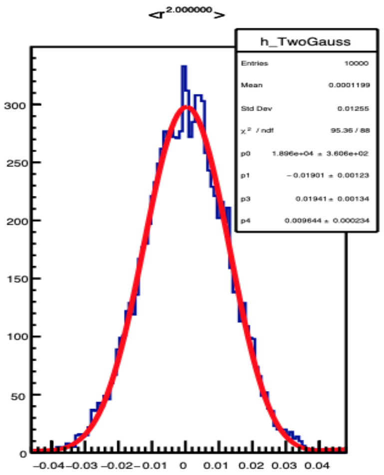
Very preliminary

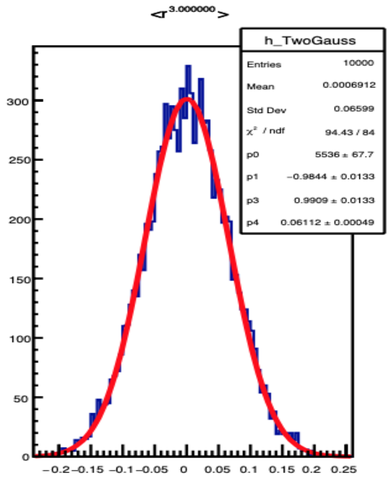


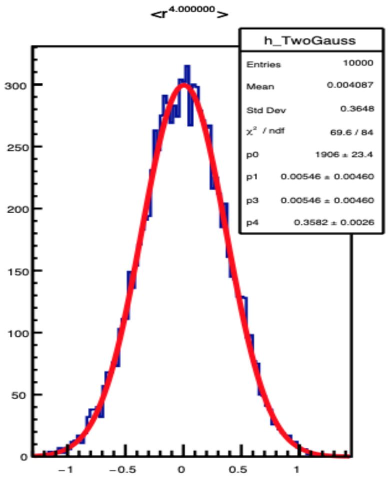


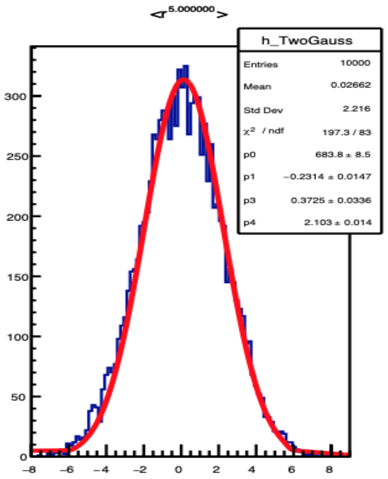
10000 replicas
Plot: (fitted – expected) value for each moment

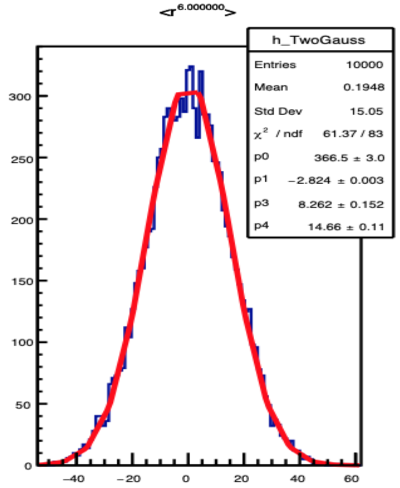


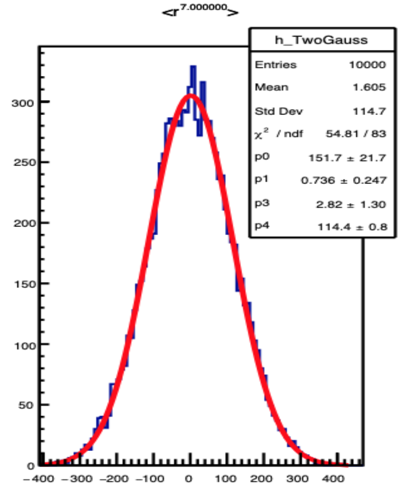












Very preliminary

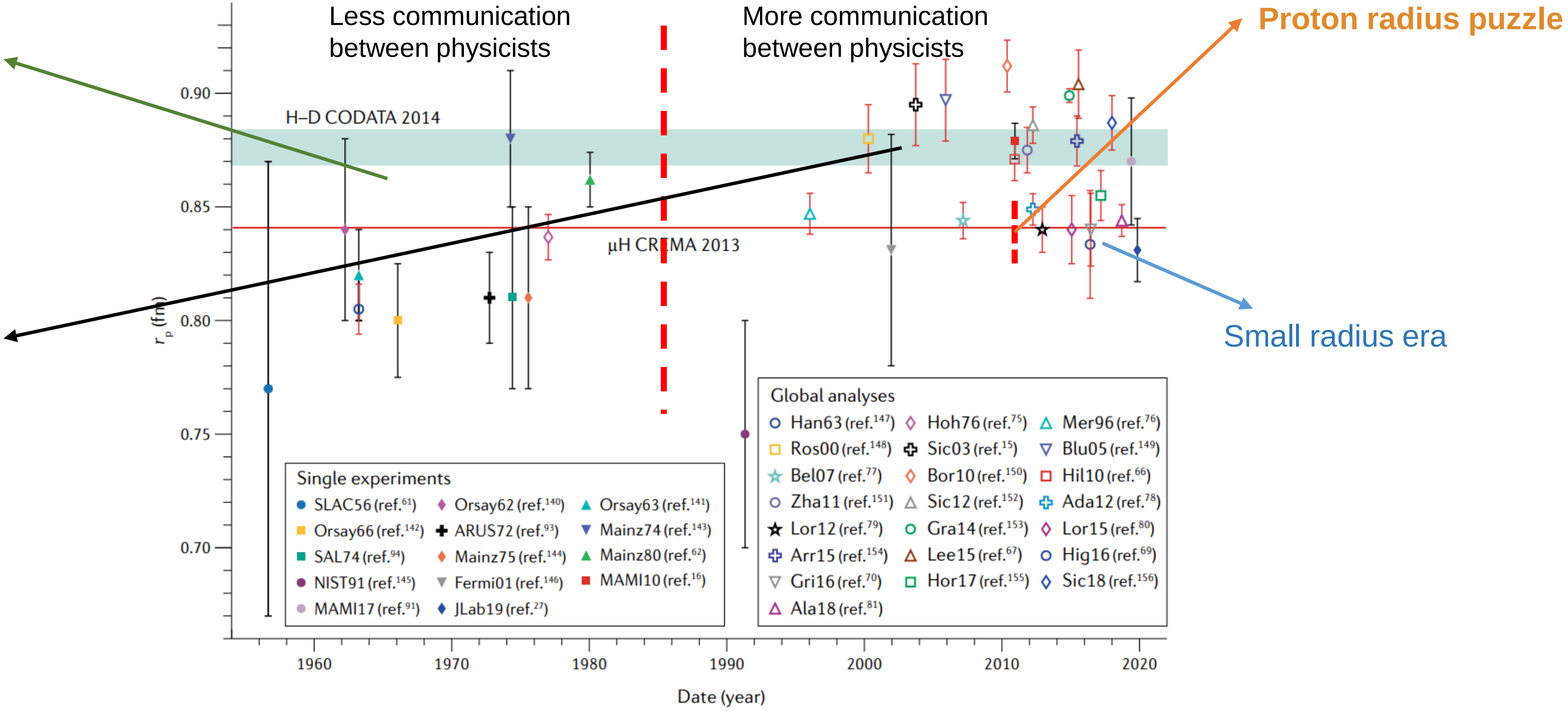
New experimental determination thanks to the IM

As can be obtained from derivative forms of the FF

n	r^n (Q_C^2 = 52 fm^{-2})	± (stat.)	± (syst)	± (model)	± (choice of model)	r^n
-2	7.671	8.047×10^{-5}	0.0013	0.0029	2.306	8.699
-1	2.050	3.146×10^{-5}	0.0002	0.0004	0.172	2.088
0	1.00					1.000
1	0.718	0.0019	0.0002	0.0002	0.0134	0.718
2	0.693	0.0090	0.0012	0.0006	0.0276	0.693
3	0.865	0.0610	0.0066	0.0018	0.1035	0.865
4	1.444	0.3580	0.0285	0.0063	0.2062	1.444
5	3.641	2.1030	0.0131	0.0203	0.4018	3.641
6	15.253	14.660	1.1000	0.0634	0.7285	15.253
7	95.948	114.40	9.1080	0.2206	20.9622	95.948

Observational bias and the proton charge radius

Small radius era



Researchers are tempted -unconsciously- to find results that validate previous findings
Solution: **blind physical observables** while **optimizing the extraction strategy** so only fit quality and estimated precision guides the analysis

J-P. Karr, D. Marchand, E. Voutier Nature Reviews Physics **2**, 601–614 (2020)

Very preliminary

- **The obtained proton charge radius from our reanalysis is:**

$$r_p = 0.8326 \pm 0.0054^{\text{stat.}} \pm 0.0007^{\text{syst.}} \pm 0.0003^{\text{model}} \pm 0.0166^{\text{parametrization}}$$

- stat.: Reflects the statistical precision of the considered data sets
- syst.: Reflects the quality of the considered data sets
- model: Reflects the intrinsic bias generated by the fitting model
- parametrization: Reflects the bias induced from the choice of the extrapolation model

$r_p = 0.84184 \pm 0.00067^{\text{stat.+syst.}}$ **value from** R. Pohl et al. Nature 466 7303 (2010)

$r_p = 0.87900 \pm 0.00500^{\text{stat.}} \pm 0.00400^{\text{syst.}} \pm 0.00200^{\text{model}} \pm 0.00400^{\text{group}}$ **value from** J. C. Bernauer et al.Phys. Rev. Lett. 105 242001 (2010)

- Radius **compatible** with previous measurements
- The **systematic error on the choice of the parametrization** insures that the variation observed in the radius due to extrapolation is taken into account as a limitation of the measurement

Conclusions

- A **novel method** for the **determination of the moments** of the charge density via **integral forms of the electric FF**.
- The method **generalizes to any probability** density function
- The method **generalizes to any D-dimensional space**

  Important when considering the Dirac and Pauli form factors F1 and F2 for the estimation of a relativistic charge radius
- A conceptual implication on **the consideration of k^2 domain** for the fit
- The experimental extraction of **odd**, **even, fractional and negative** $(\lambda \succ -3)$ **moments** of the charge density
- Performed a **reanalysis of some experimental data** extracting the values of several moments of the charge density
- Extracted a **value for the radius taking all error sources** into consideration

Post-conclusion

- **Blinded analysis** is already a common practice in high energy particle physics
- We **need to take it as a standard procedure** in our community