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JÜLICH FORSCHUNGSZENTRUM

Dispersive analysis of the nucleon electromagnetic form factors

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by ERC, EXOTIC

erc European Research Council Established by the European Commission

CONTENTS

Theoretical framework

Review: Lin, Hammer, UGM, Eur. Phys. J. **A 57** (2021) 255

BASIC DEFINITIONS

- Nucleon matrix elements of the em vector current J_μ^I

$$\langle N(p')|J_\mu^I|N(p)\rangle = \bar{u}(p')\left[F_1^I(t)\,\gamma_\mu + i\,\frac{F_2^I(t)}{2m}\,\sigma_{\mu\nu}q^\nu\right]u(p)$$

⋆ isospin $I = S, V$ (isoScalar, isoVector) $[= (p \pm n)/2]$

⋆ four-momentum transfer $t \equiv q^2 = (p' - p)^2 \equiv -Q^2$

⋆ F_1 = Dirac form factor, F_2 = Pauli form factor

⋆ Normalizations: $F_1^V(0) = F_1^S(0) = 1/2$, $F_2^{S,V}(0) = (\kappa_p \pm \kappa_n)/2$

⋆ Sachs form factors: $G_E = F_1 + \frac{t}{4m^2}F_2\,,\;\; G_M = F_1 + F_2$

⋆ Nucleon radii: $F(t) = F(0)\,\left[1 + t\langle r^2\rangle/6 + \ldots\right]$ [except for the neutron charge ff]

WHY DISPERSION RELATIONS for the NUCLEON FFs ?

- Model-independent approach $\rightarrow$ important non-perturbative tool to analyze data
- Dispersion relations are based on fundamental principles: unitarity & analyticity
- Connect data from small to large momentum transfer
  as well as time- and space-like data
- Allow for a **simultaneous analysis** of all four em form factors
- Spectral functions encode perturbative and non-perturbative physics
  e.g. vector meson couplings, multi-meson continua, pion cloud, . . .
- Spectral functions also encode information on the strangeness vector current
  $\rightarrow$ sea-quark dynamics, strange matrix elements
- Allow to extract nucleon electric and magnetic radii
- Can be matched to chiral perturbation theory [and not the other way around!]

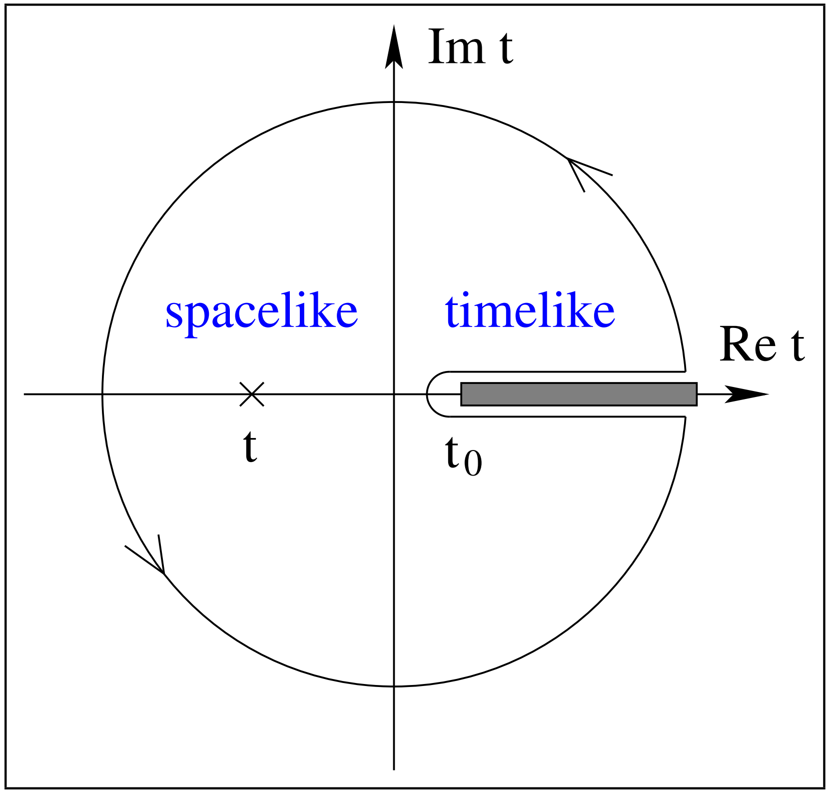
DISPERSION RELATIONS

Federbush, Goldberger, Treiman, Drell, Zachariasen, Frazer, Fulco, Höhler, . . .

- The form factors have cuts in the interval $[t_n, \infty[\ (n = 0, 1, 2, \ldots)$ and also poles

$\Rightarrow$ Dispersion relations for $F_i(t)$ $(i = 1, 2)$:

$$F_i(t) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \, \frac{\mathrm{Im}\, F_i(t')}{t' - t}$$



- no subtractions
  [only proven in perturbation theory]
- suppression of higher mass states
- central objects: spectral functions

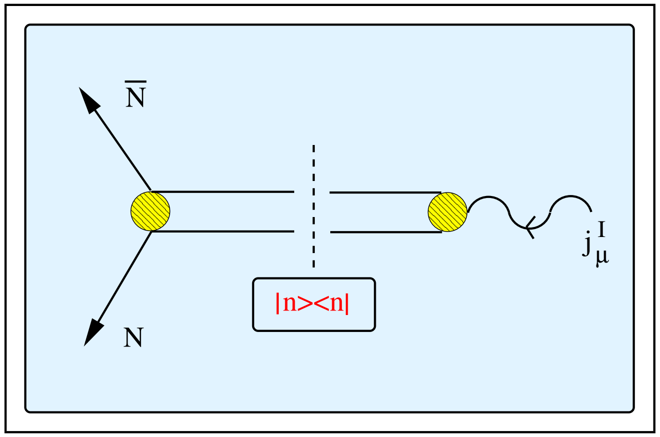
Im $F_i(t)$

– cuts $\hat{=}$ multi-meson continua
– poles $\hat{=}$ vector mesons

SPECTRAL FUNCTIONS – GENERALITIES

- Spectral decomposition: Chew, Karplus, Gasiorowicz, Zachariasen (1958)

$$\mathrm{Im}\,\langle \bar{N}(p')N(p)|J_\mu^I|0\rangle \sim \sum_n \langle \bar{N}(p')N(p)|n\rangle\langle n|J_\mu^I|0\rangle \Rightarrow \mathrm{Im}\,F$$



⋆ on-shell intermediate states

⋆ generates imaginary part

⋆ accessible physical states

- *Isoscalar* intermediate states: $3\pi, 5\pi, \ldots, K\bar{K}, K\bar{K}\pi, \pi\rho, \ldots +$ poles $\to t_0 = 9M_\pi^2$

- *Isovector* intermediate states: $2\pi, 4\pi, \ldots +$ poles $\to t_0 = 4M_\pi^2$

- Note that some poles are *generated* from the appropriate continua

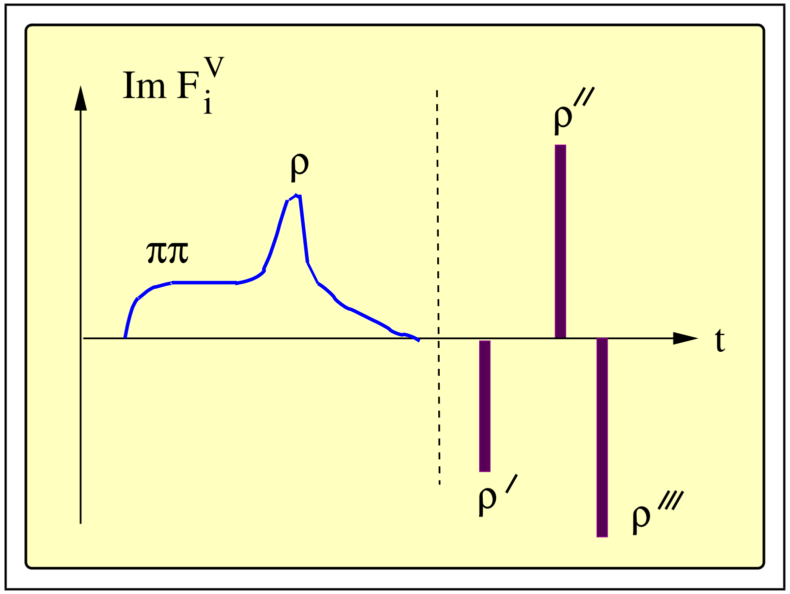
ISOVECTOR SPECTRAL FUNCTIONS

Frazer, Fulco, Höhler, Pietarinen, . . .

- exact 2π continuum is known from threshold $t_0 = 4M_\pi^2$ to $t \simeq 40\, M_\pi^2$

$$\text{Im}\, G^V_{E,M}(t) = \frac{q_t^3}{\sqrt{t}} F_\pi(t)^\star f^1_\pm(t)$$

  - $\star$ $F_\pi(t)$ = pion vector form factor
  - $\star$ $f^1_\pm(t)$ = P-wave pion-nucleon partial waves in the t-channel



- Spectral functions inherit singularity on the second Riemann sheet in $\pi N \to \pi N$

$$t_c = 4M_\pi^2 - M_\pi^4/m^2 \simeq 3.98\, M_\pi^2$$

$\to$ strong shoulder $\to$ isovector radii

- This singularity can also be analyzed in CHPT

Bernard, Kaiser, UGM, Nucl. Phys. A **611** (1996) 429

- For a recent determination of the 2π continuum, see HKRHM, EPJA **52** (2016) 331
- Higher mass states represented by poles (not necessarily physical masses)

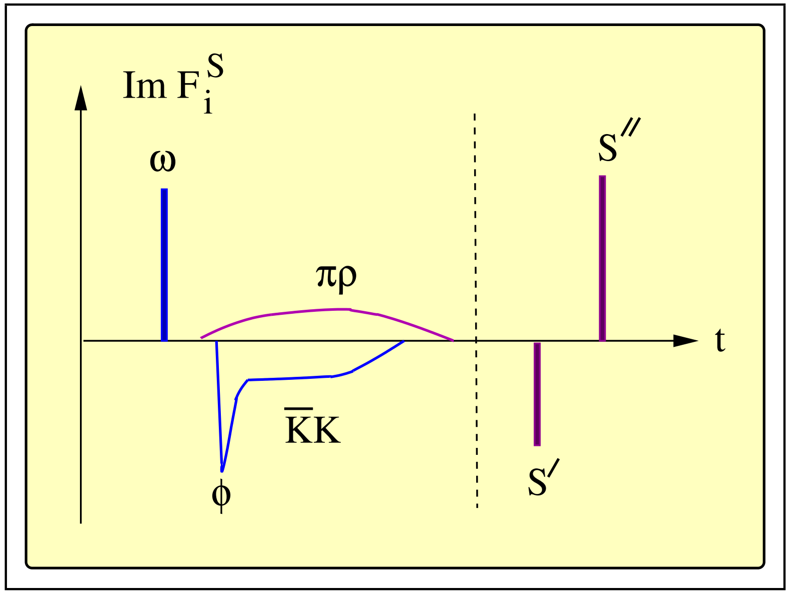
ISOSCALAR SPECTRAL FUNCTIONS

- $K\bar{K}$ continuum can be extracted from analytically cont. KN scattering amplitudes

  $\rightarrow$ analytic continuation must be stabilized

  $\rightarrow$ generates most of the ϕ contribution

  Hammer, Ramsey-Musolf, Phys. Rev. C **60** (1999) 045204, 045205



- Further strength in the ϕ-region generated by correlated $\pi\rho$ exchange

  $\rightarrow$ strong cancellations ($K\bar{K}, K^*K, \pi\rho$)

  $\rightarrow$ takes away sizeable strength from the ϕ

  UGM, Mull, Speth, van Orden, Phys. Lett. B **408** (1997) 381

- Spectral functions exhibit anomalous threshold (analyzed in 2-loop CHPT)

$$t_c = M_\pi^2 \left(\sqrt{4 - M_\pi^2/m^2} + \sqrt{1 - M_\pi^2/m^2}\right)^2 \simeq 8.9\, M_\pi^2$$

$\rightarrow$ effectively masked

Bernard, Kaiser, M, Nucl. Phys. A **611** (1996) 429

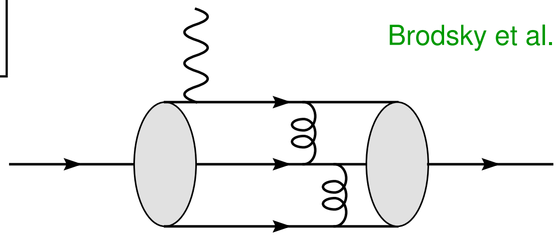
- Higher mass states represented by poles (with a finite width)

CONSTRAINTS ON THE SPECTRAL FUNCTIONS

- Normalizations: electric charges, magnetic moments
- Radii *not* imposed [except for the neutron charge radius] Filin et al., Phys. Rev. C **103** (2021) 024313
- Superconvergence relations ≅ leading pQCD behaviour

$$F_1(t) \sim 1/t^2 \, , F_2(t) \sim 1/t^3 \;\; (\text{helicity} - \text{flip})$$

Brodsky et al.



$$\Rightarrow \int_{t_0}^{\infty} \text{Im}\, F_1(t)\, dt = 0 \, , \quad \int_{t_0}^{\infty} \text{Im}\, F_2(t)\, dt = \int_{t_0}^{\infty} \text{Im}\, F_2(t)\, t\, dt = 0$$

- Various ways of implementing the asymptotic QCD behaviour

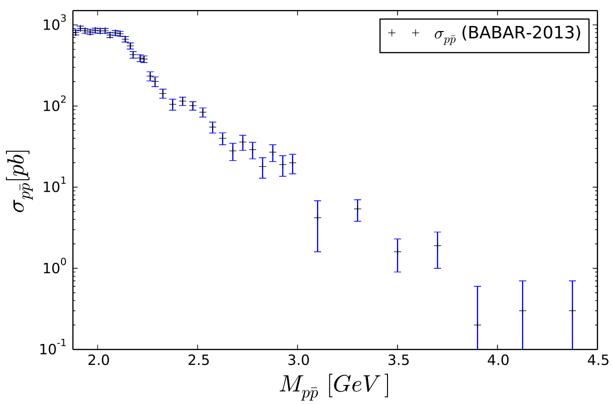
⇒ severely restricts the number of fit parameters

FORM FACTORS IN THE TIME–LIKE REGION

- Xsection for $e^+e^- \leftrightarrow \bar{p}p, \bar{n}n$ in the one-photon approximation

$$\sigma(s) = \frac{4\pi\alpha_{\rm EM}^2\beta}{3s}\, C(s) \left[|G_M(s)|^2 + \frac{2m_N^2}{s}\,|G_E(s)|^2\right], \quad \beta = \frac{k_p}{k_e}$$

$$= \frac{4\pi\alpha_{\rm EM}^2\beta}{3s}\, C(s) \left(1 + \frac{2m_N^2}{s}\right) \; |G_{\rm eff}|^2$$



- $G_{E,M}(s)$ are complex for $s \geq 4m_N^2$

- Threshold constraint: $\quad G_E(4m_N^2) = G_M(4m_N^2)$

- Gamov-Sommerfeld factor (only for the proton):

$$C = \frac{y}{1-e^{-y}}, \quad y = \frac{\pi\alpha_{\rm EM} m_p}{k_p}, \quad \sqrt{s} = 2\sqrt{m_p^2 + k_p^2}$$

- Data from $e^+e^- \to \bar{N}N$ & $\bar{N}N \to e^+e^-$: strong threshold enhancement & oscillations

SUMMARY: SPECTRAL & FIT FUNCTIONS

- Representation of the pole contributions: vector mesons
  [NB: can be extended for finite width]

$$\text{Im}\, F_i^V(t) = \sum_v \pi a_i^v\, \delta(t - M_v^2)\,, \quad a_i^v = \frac{M_v^2}{f_V} g_{vNN} \;\Rightarrow\; F_i(t) = \sum_v \frac{a_i^v}{M_v^2 - t}$$

- *Isovector* spectral functions:

$$\text{Im}\, F_i^V(t) = \text{Im}\, F_i^{(2\pi)}(t) + \sum_{v=v_1,v_2,\ldots} \pi\, a_i^v \delta(t - M_v^2) + \sum_{V=V_1,\ldots} \text{Im}\, F_i^{V_i} \quad (i = 1, 2)$$

- *Isoscalar* spectral functions:

$$\text{Im} F_i^S(t) = \text{Im} F_i^{(K\bar{K})}(t) + \text{Im} F_i^{(\pi\rho)}(t) + \sum_{v=\omega,\phi,s_1,s_2,\ldots} \pi\, a_i^v \delta(t - M_v^2) + \sum_{S=S_1,\ldots} \text{Im}\, F_i^{S_i}$$

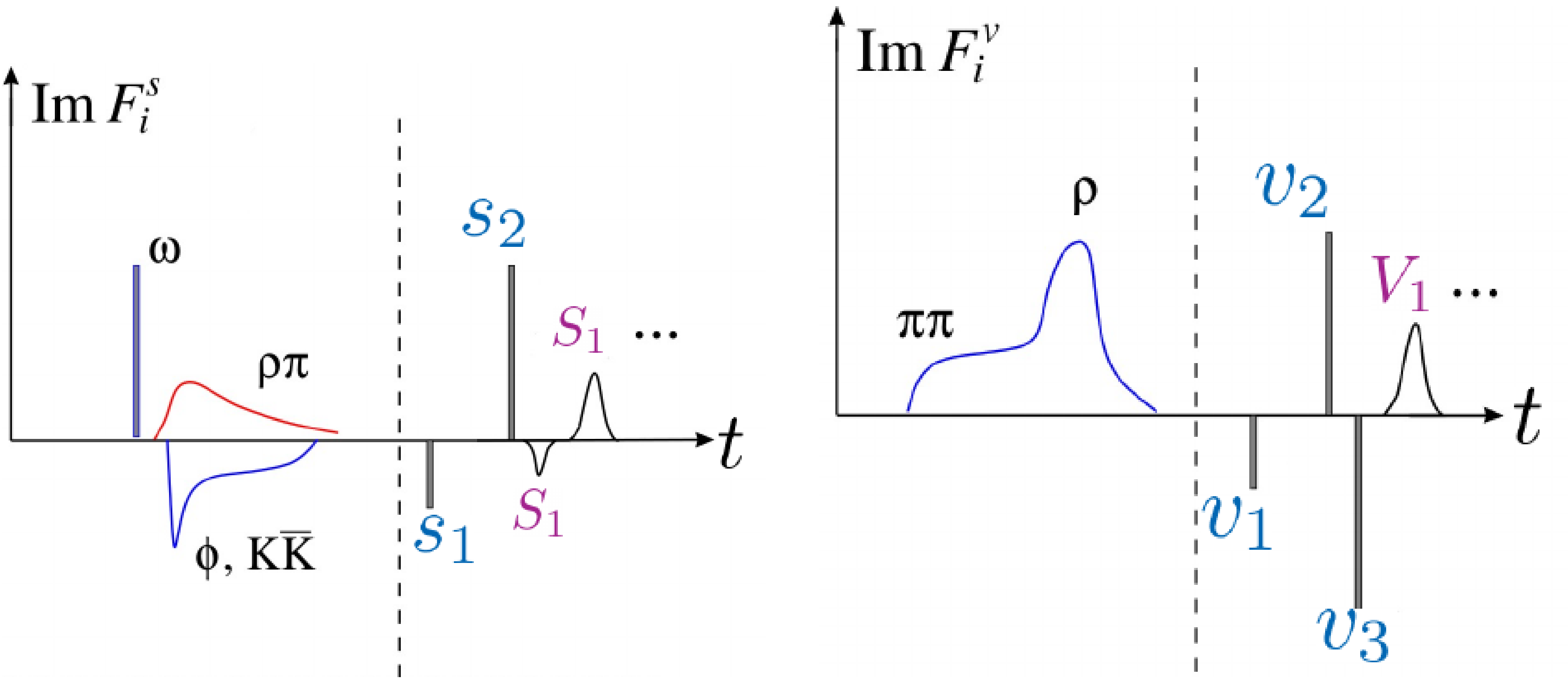
- Parameters: 2 for the ω, ϕ, 3 (4) for each other V-mesons minus # of constraints

- Additional broad poles w/ 3 parameters above the $\bar{N}N$ threshold [structures in the time-like ffs]

- Ill-posed problem $\rightarrow$ extra constraint: minimal # of poles to describe the data

SUMMARY: SPECTRAL FUNCTIONS

• Cartoons of the isoscalar/isovector spectral functions:



Fit procedure & theoretical uncertainties

Review: Lin, Hammer, UGM, Eur. Phys. J. **A 57** (2021) 255

DATA BASIS

- Data basis:

Region	Observables	Coll.	$\|t\|(GeV^2)$	number
Spacelike$(t < 0)$	$d\sigma/d\Omega$	MAMI	0.00384–0.977	1422
		PRad	0.000215–0.058	71
	$\mu_p G_E^p/G_M^p$	JLab	1.18–8.49	16
	G_E^n	world	0.14–1.47	25
	G_M^n	world	0.071–10.0	23
Timelike$(t > 0)$	$\|G_{\rm eff}^p\|$	world	3.52–20.25	153
	$\|G_{\rm eff}^n\|$	world	3.53–9.49	27
	$\|G_E^p/G_M^p\|$	BaBar	3.52–9.0	6
	$d\sigma/d\Omega$	BESIII	3.52–3.80	10

- Number of data/fit parameters:

$$\#_{\rm data} = 1753\,, \quad \#_{\rm fitpara.} = \underbrace{4}_{\omega+\phi} + 3\underbrace{(N_s + N_v)}_{best\ fit:\ 3+5} + 4\underbrace{(N_S + N_V)}_{best\ fit:\ 3+3} - 11 + \underbrace{31 + 2}_{\rm norm.}$$

FIT PROCEDURE

- Fit strategy (weigthed average): $\chi^2_{\rm total} = \sum_{i \text{ in data basis}} \frac{\chi^2_i}{\#\text{ data points}}$

- Two definitions of χ^2 [(un)correlated errors]:

$$\chi^2_1 = \sum_i \sum_k \frac{(n_k C_i - C(Q_i^2, \theta_i, \vec{p}))^2}{(\sigma_i + \nu_i)^2}$$

$$\chi^2_2 = \sum_{i,j} \sum_k (n_k C_i - C(Q_i^2, \theta_i, \vec{p}))[V^{-1}]_{ij}(n_k C_j - C(Q_j^2, \theta_j, \vec{p}))$$

$$V_{ij} = \sigma_i \sigma_j \delta_{ij} + \nu_i \nu_j$$

  - C_i = cross section data
  - $C(Q_j^2, \theta_j, \vec{p})$ = XS for a given FF parametrization
  - $\vec{p}$ = parameter values for a given FF parametrization
  - n_k = normalization constants for various data sets k
  - V_{ij} = covariance matrix w/ stat. (σ_i) & syst. (ν_i) errors

TYPES of UNCERTAINTIES

- Systematic and statistical, only recently fully incorporated

- Systematics: Vary the number of poles
  around the best solution
  so that $\delta\chi^2_{\rm total} \leq 1\%$

  Höhler et al. (1976)

Conf.
Total chi^2
Reduced chi^2

Change 1% of total chi^2 of best fit: 2875~2933

33 3063.09 2.01652					
43 2978.98 1.96502	34 3054.05 2.01455				
53 2944.36 1.94604	44 2958.66 1.95549	35 3022.62 1.99777			
54 2925.07 1.93713	63 2930.96 1.94104	45 2965.49 1.96390	36 3030.21 2.00676		
64 2904.04 1.92703	55 2906.76 1.92884	73 2954.51 1.96052	37 3058.50 2.02953	46 3062.15 2.03195	
74 2897.13 1.92629	56 2900.41 1.92846	65 2922.20 1.94295	83 2967.10 1.97281	47 3008.90 2.00060	38 3032.76 2.01646
66 2891.57 1.92643	57 2909.94 1.93867	84 2910.37 1.93895	75 2911.52 1.93972	48 2987.44 1.99030	
76 2892.43 1.93086	58 2897.60 1.93431	85 2899.90 1.93585	67 2901.95 1.93722		
86 2885.39 1.93002	77 2890.74 1.93360	68 2891.36 1.93402			
87 2886.00 1.93431	78 2891.97 1.93832				
88 2886.83 1.93877					

- Statistical errors, use
  - bootstrap method — Efron, Tibshirani (1986)
  - Bayesian analysis — Bayes (1763)

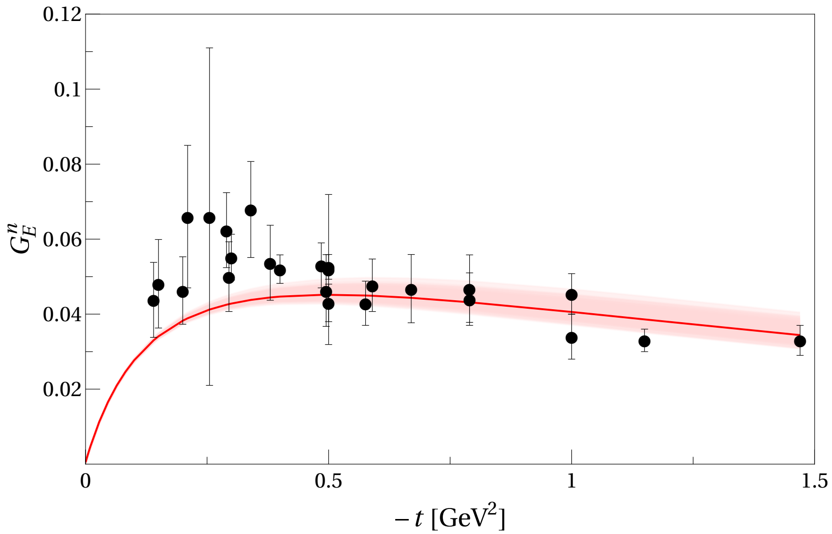
  $\Rightarrow$ both give the same, bootstrap simpler to implement

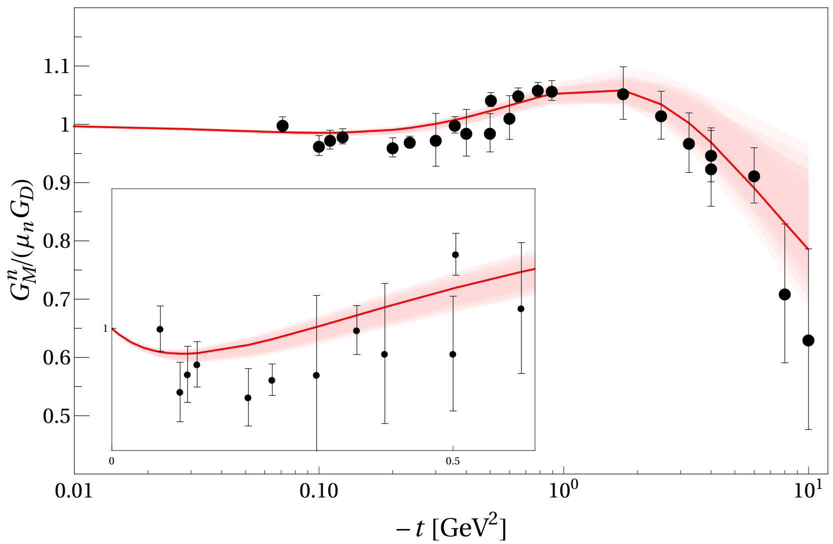
Results

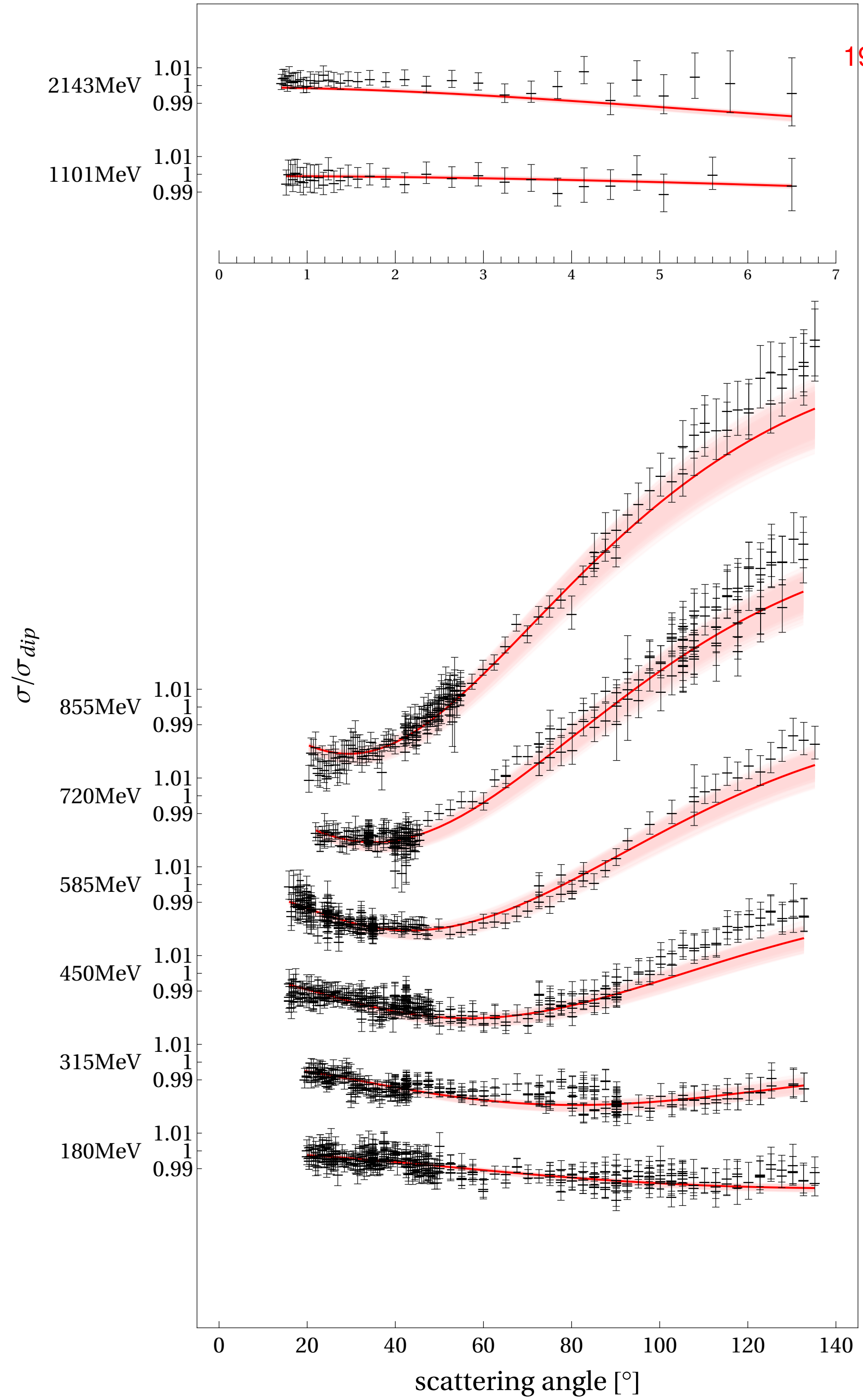
Lin, Hammer, UGM, Phys. Rev. Lett. **128** (2022) 5, 052002

SPACE-LIKE RESULTS I

- ep scattering data and neutron ffs:
  [error bands from bootstrap, χ^2/dof=1.223]

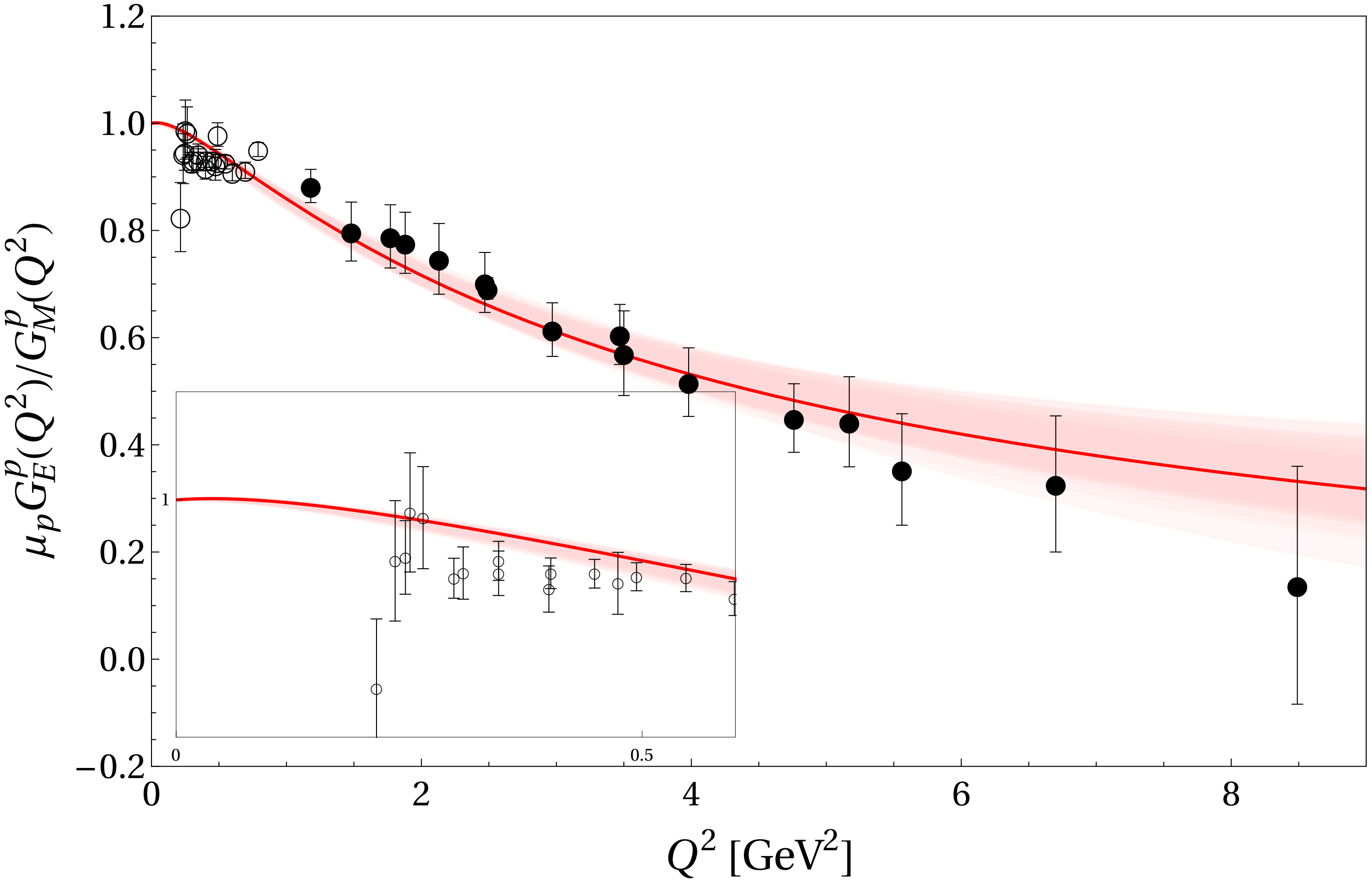






SPACE-LIKE RESULTS II

- Proton form factor ratio [Jlab data]:

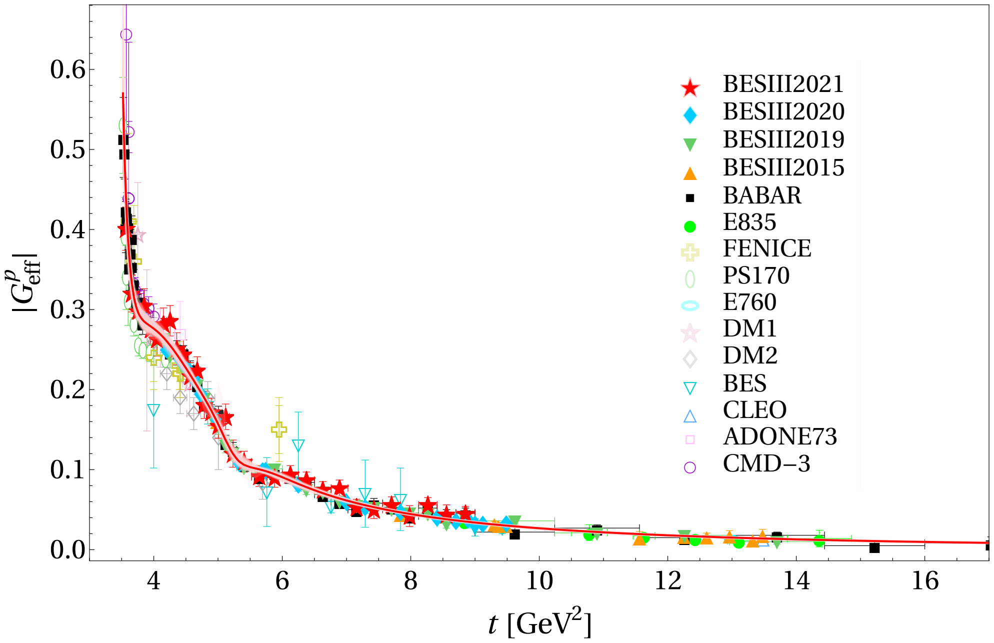


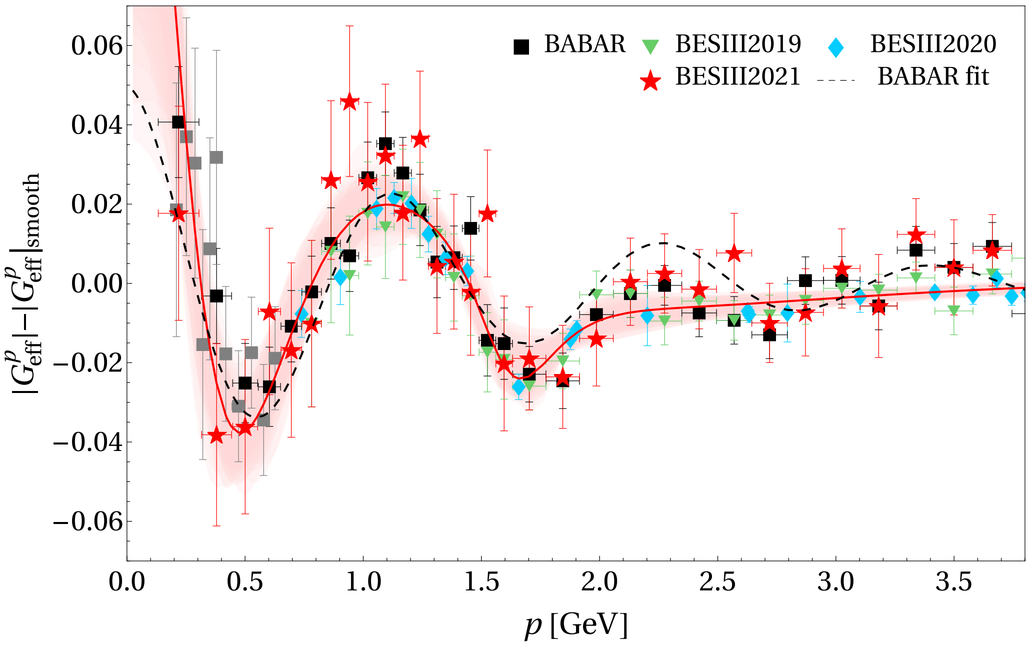
— empty symbols: not fitted

— zero crossing disfavored

TIME-LIKE RESULTS I

- Proton effective form factor:





open symbols not fitted

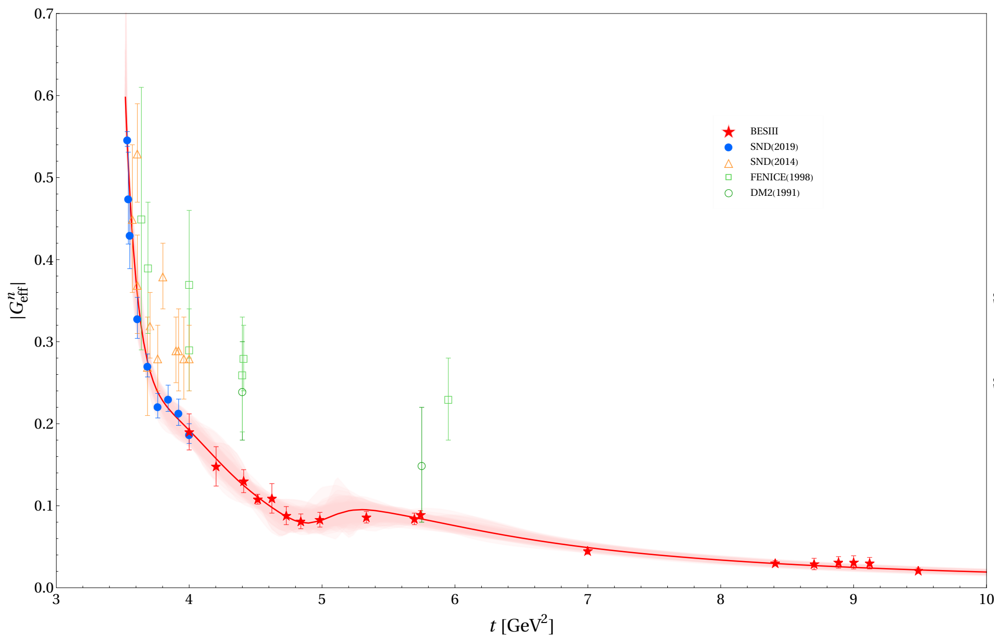
$$|G^p_{\rm eff}|_{\rm smooth} = \frac{7.7}{(1+t/14.8)(1-t/0.71)^2}$$

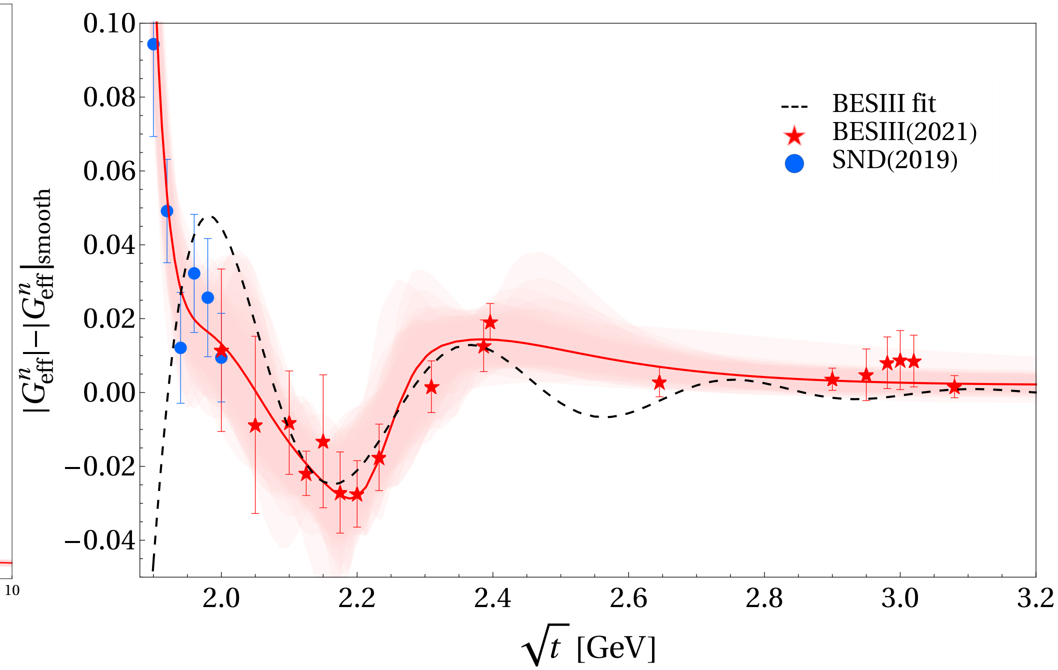
BaBar fit formula:

$$F_p = A^{\rm osc}\exp(-B^{\rm osc}p) \times \cos(C^{\rm osc}p + D^{\rm osc})$$

TIME-LIKE RESULTS II

- Neutron effective form factor:





open symbols not fitted

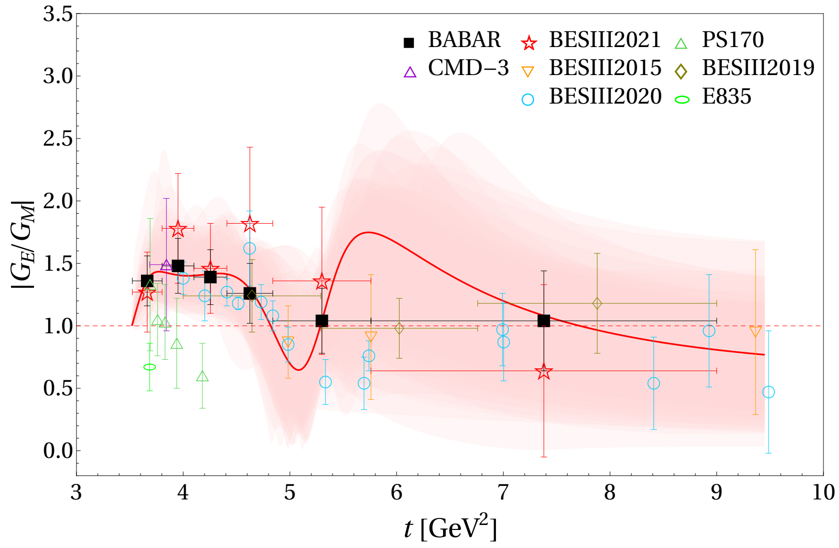
$$|G^n_{\rm eff}|_{\rm smooth} = \frac{4.87}{(1+t/14.8)(1-t/0.71)^2}$$

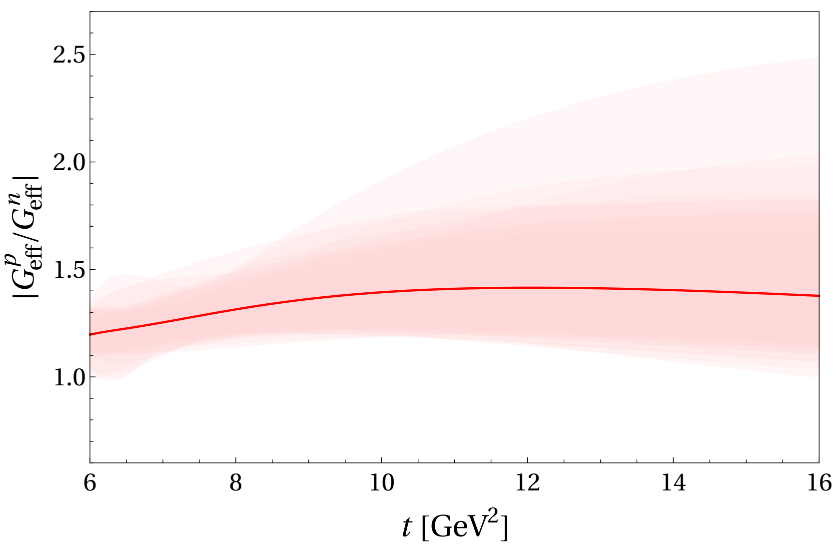
BESIII fit formula:

$$F_p = A^{\rm osc}\exp(-B^{\rm osc}p) \times \cos(C^{\rm osc}p + D^{\rm osc})$$

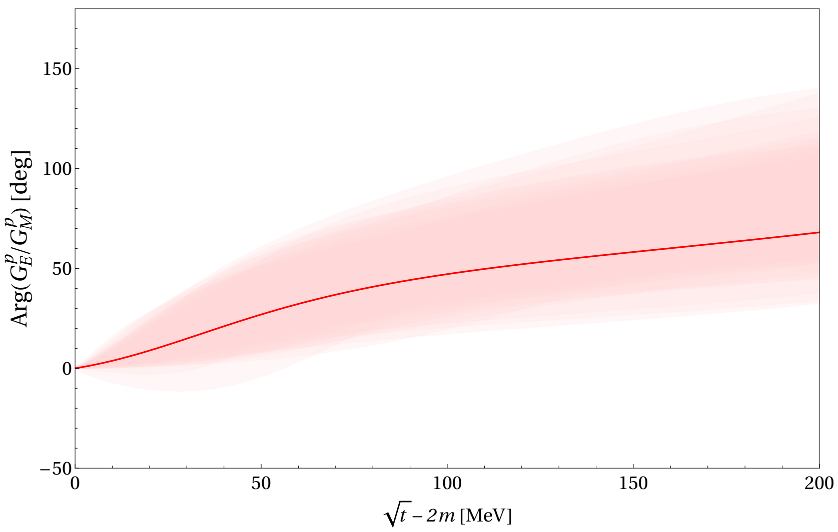
TIME-LIKE RESULTS III

- More fits and predictions [pQCD & phases]





open symbols not fitted



The proton radius

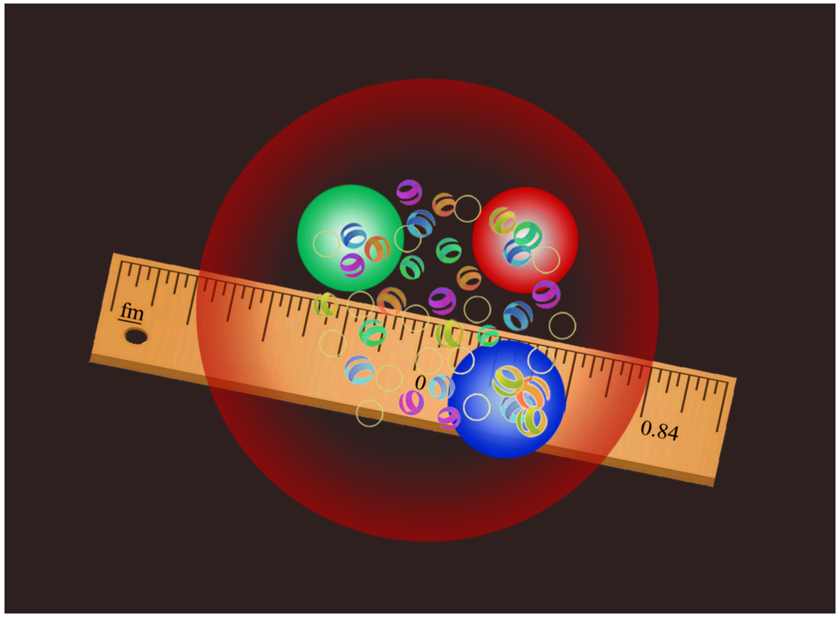


Fig. courtesy Yong-Hui Lin

PROTON CHARGE RADIUS

- Definition: $r_p^2 \equiv -6\, G'_E(0)$

- Measurements:

  – Leptonic hydrogen Lamb shift [in principle 2 numbers: r_p & R_∞]

$$\Delta E_{LS} = \Delta E_1 + \Delta E_2 C(r_p^2) + \mathcal{O}(m_{\rm red}\alpha_{\rm EM}^2)$$

$$C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(m_{\rm red}\alpha_{\rm EM}^2)$$

  – Lepton-proton scattering (Rosenbluth sep.)

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\rm Mott}}{d\Omega}\frac{1}{1+\tau}\left(G_E^2 + \frac{\tau}{\varepsilon}G_M^2\right)(1+\delta_{\rm rad.}) + \mathcal{O}(m_{\rm red}\alpha_{\rm EM}^2)$$

- The neglected sibling, the proton magnetic radius:

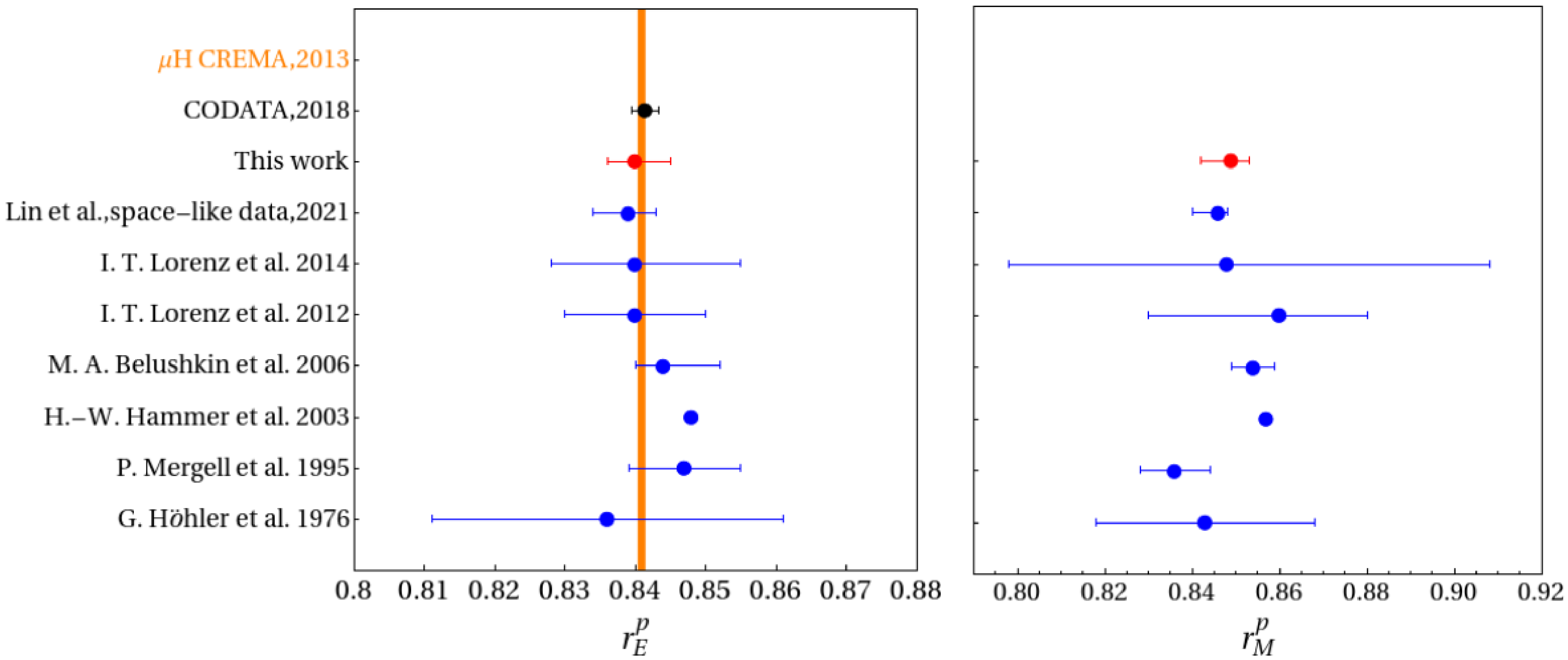
$$(r_p^M)^2 \equiv -(6/\mu_p)\, G'_M(0)$$

PROTON CHARGE & MAGNETIC RADIUS

- Our determination incl. statistical and systematic errors:

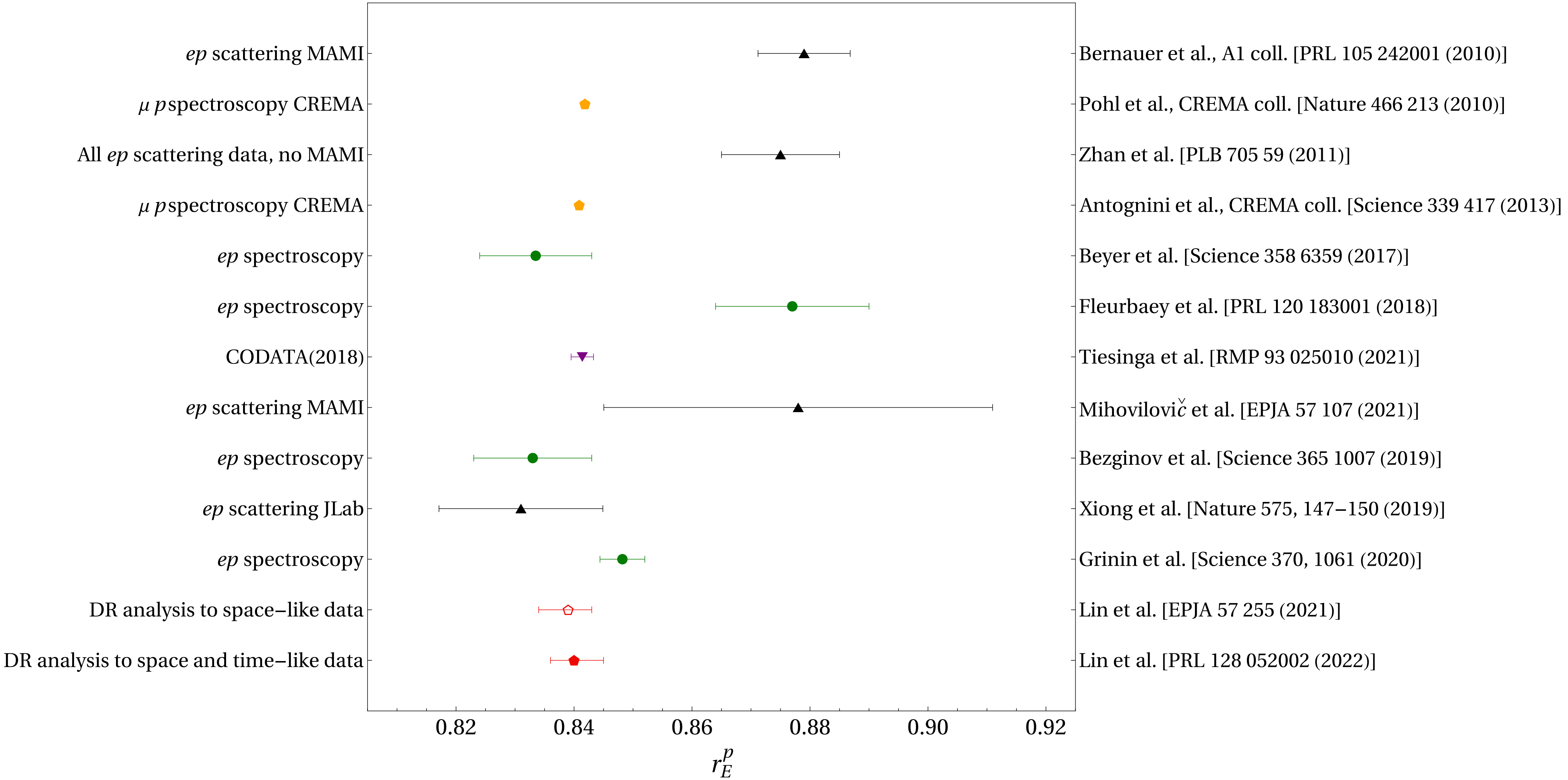
$$r_E^p = 0.840^{+0.003+0.002}_{-0.002-0.002}\ \mathrm{fm}\ ,\quad r_M^p = 0.849^{+0.003+0.001}_{-0.003-0.004}\ \mathrm{fm}$$

- Comparison to earlier DR determinations (and some data)



PROTON CHARGE RADIUS

- Comparison to recent measurements:

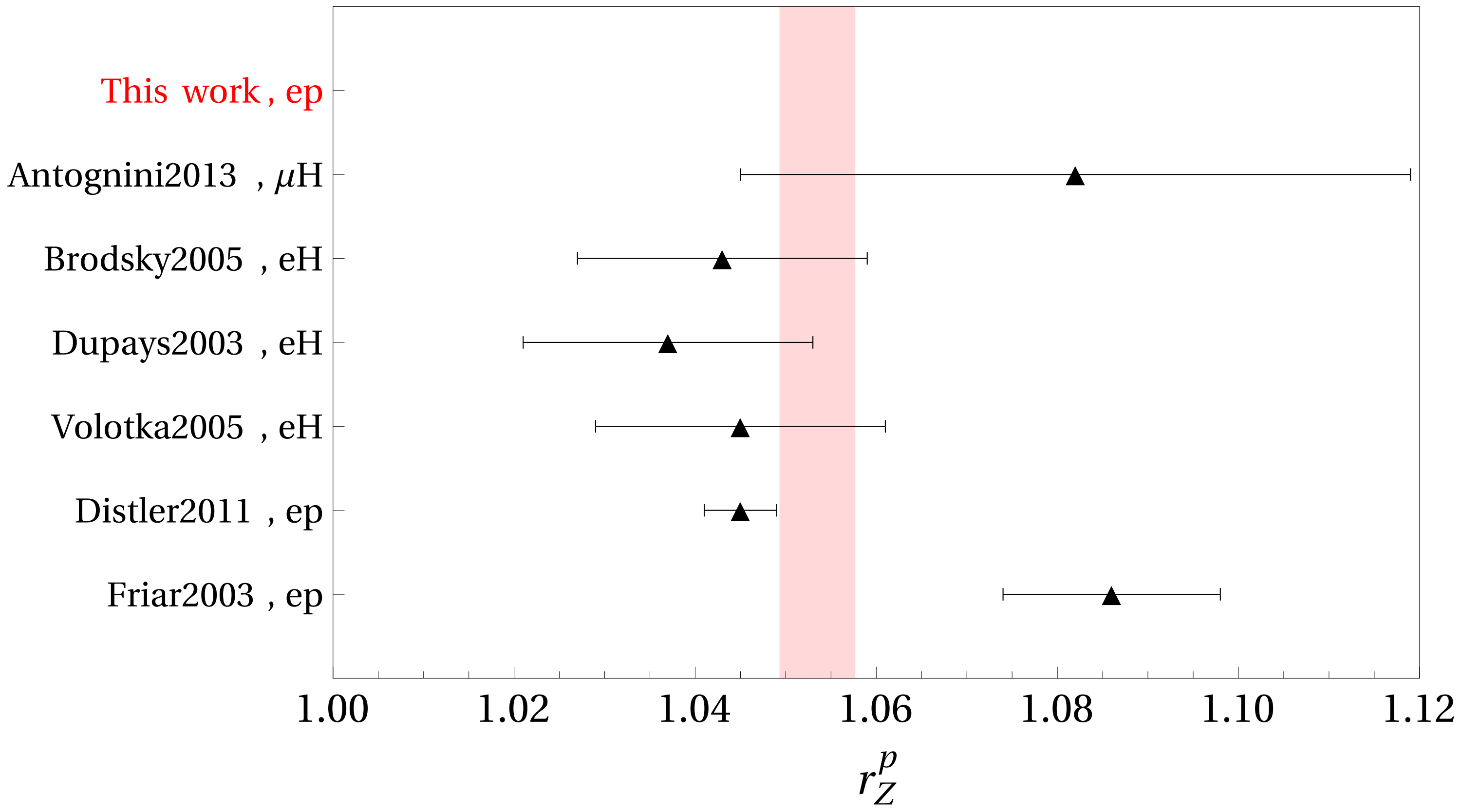


ZEMACH RADIUS & THIRD MOMENT

- Our determination incl. statistical and systematic errors:

$$r_Z = \frac{2}{\pi}\int_{-\infty}^{0}\frac{dt}{t\sqrt{-t}}\left(\frac{G_E(t)G_M(t)}{1+\kappa_p} - 1\right) = 1.054^{+0.003+0.000}_{-0.002-0.001}\ \mathrm{fm}$$

$$\langle r^3\rangle_{(2)} = \frac{24}{\pi}\int_{-\infty}^{0}\frac{dt}{t^2\sqrt{-t}}\left(G_E^2(t) - 1 - \frac{t}{3}\langle r^2\rangle_p\right) = 2.310^{+0.022+0.014}_{-0.018-0.015}\ \mathrm{fm}^3$$



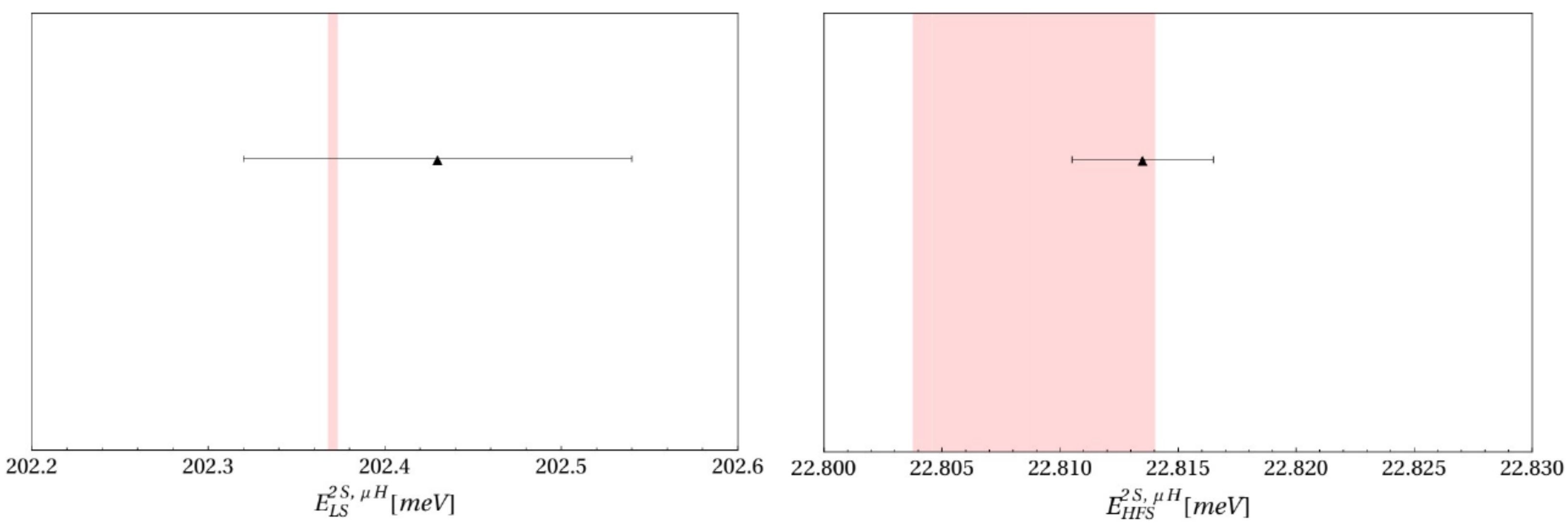
COMPARISON TO SPECTROSCOPY OBSERVABLES

- Relevant formulas:

$$\Delta E_{LS} = 206.0336(15) - 5.2275(10)\langle r_p^2\rangle + 0.0347\langle r^3\rangle_{(2)}$$

$$\Delta E_{HFS} = 22.9843(30) - 0.1621(10)\, r_Z$$

Antognini et al., Ann. Phys. **331** (2013) 127



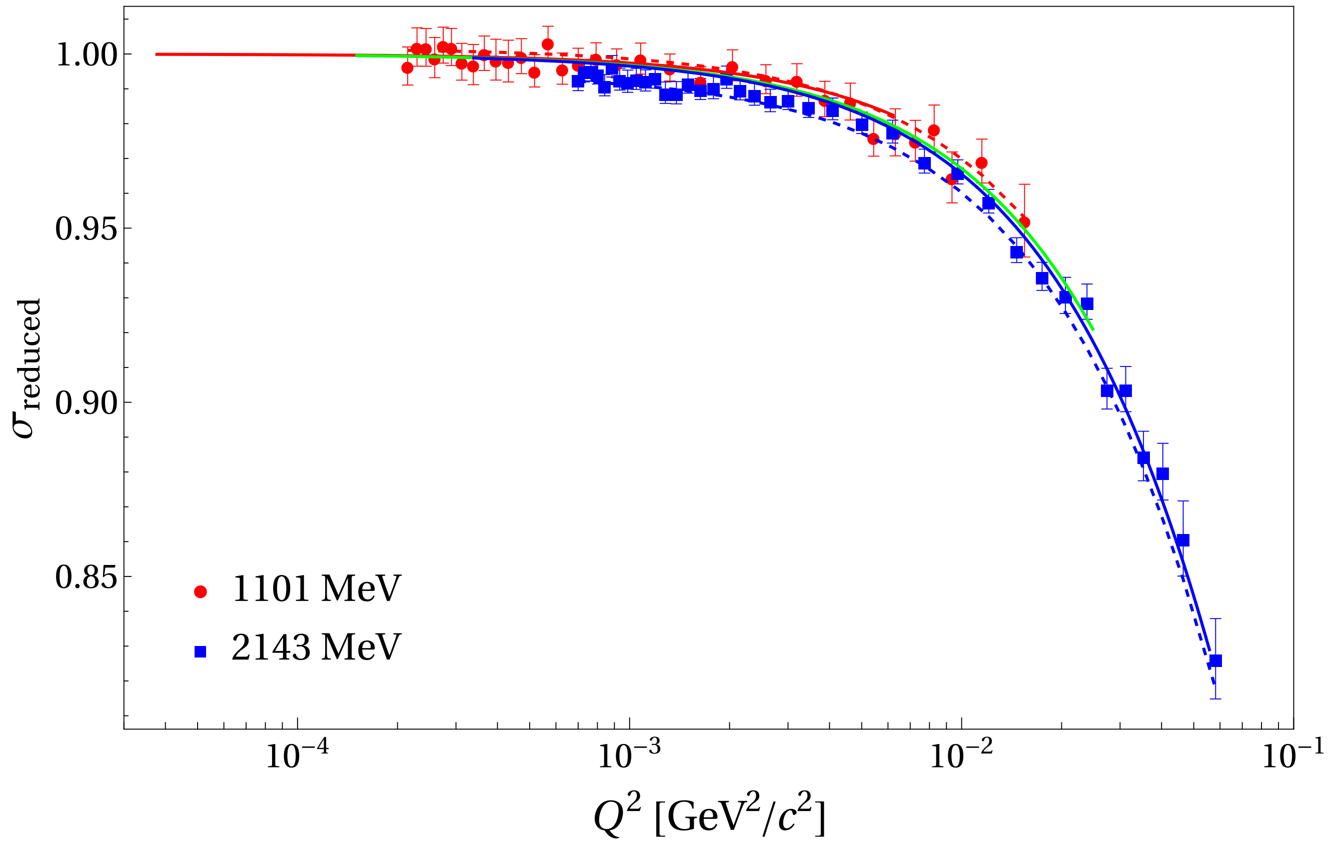
black triangles = our results

red bands from Antognini et al., Science **339** (2013) 417

PREDICTIONS for PRAD-II

Lin, Hammer, UGM, Phys. Lett. B 827 (2022) 136981 [2111.09619 [hep-ph]]

- Predictions for the upcoming PRad-II and e^+p scattering



$\hookrightarrow$ Predictions for $E_\gamma = 0.7, 1.4, 2.1$ GeV

NEVTRON RADII

- The charge squared neutron $(r_E^n)^2$ radius was mostly input in DR analyses, but not the magnetic one

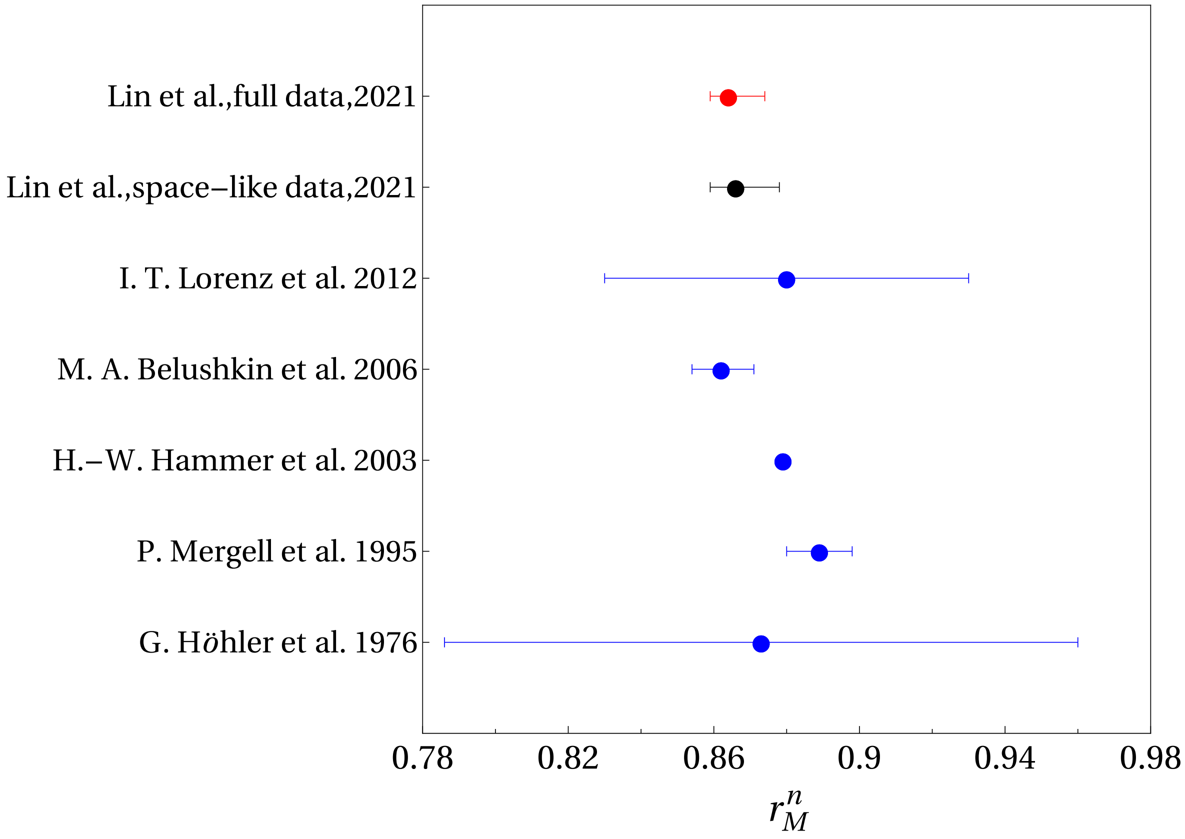
$$r_M^n = 0.864^{+0.004+0.006}_{-0.004-0.001}\ \text{fm}$$

$\hookrightarrow$ rather stable over time

$\hookrightarrow$ but larger variation

$\hookrightarrow$ always the largest em radius!

$\hookrightarrow$ lattice QCD gives rather comparable isovector radii (p& n)



SUMMARY & OUTLOOK

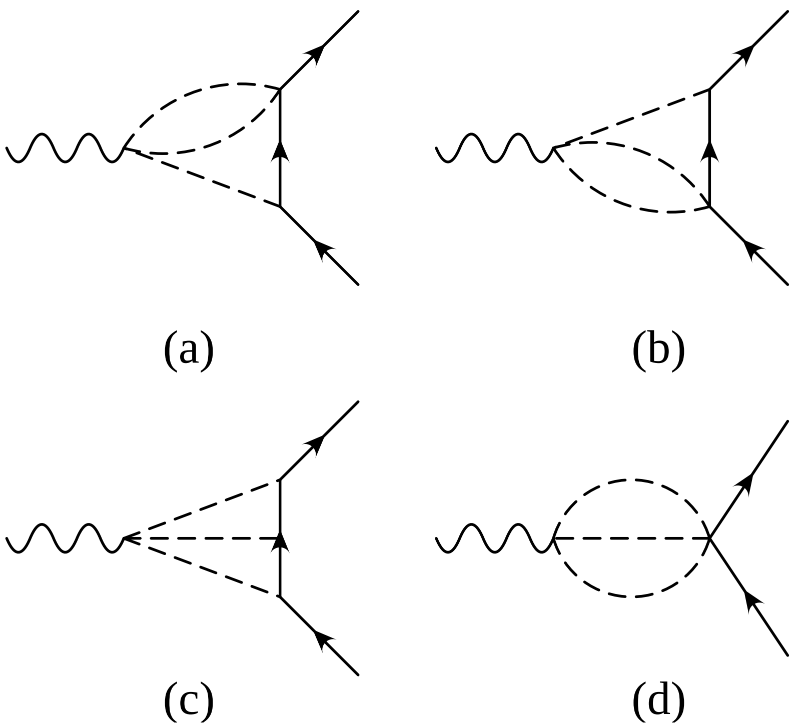
- Dispersion theory is the best tool to analyze the nucleon em FFs
- Description of all data, time- and space-like
- Always a small proton charge radius (0.84 fm), magnetic one bigger (0.85 fm)
- Most recent experiments tend to the small radius
- From a puzzle to precision — Hammer, UGM, Sci.Bull. **65** (2020) 257
- Predictions for PRad-II & positron-proton scattering (also MESA/Mainz)
- Magnetic radius of the neutron is the largest one
- Theory challenge: better understanding of the oscillations in $|G_{\rm eff}^{p,n}|$
- Experimental challenges
  - $\hookrightarrow$ proton form factor ratio at $Q^2 \simeq 10\,{\rm GeV}^2$
  - $\hookrightarrow$ more resolved form factor measurements in the time-like region
  - $\hookrightarrow$ μp scattering - testing lepton flavor universality (MUSE, AMBER)

Spares

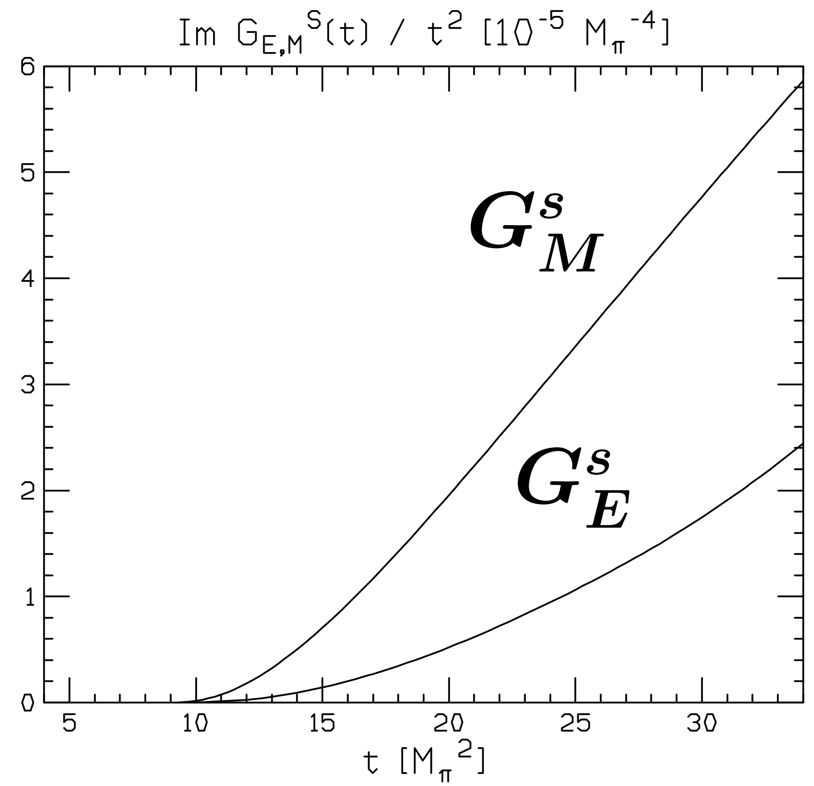
ISOSCALAR SPECTRAL FUNCTION

Bernard, Kaiser, UGM, Nucl. Phys. A **611** (1996) 429 [hep-ph/9607428]

• Two-loop CHPT calculation



• Electric/magnetic spectral fcts



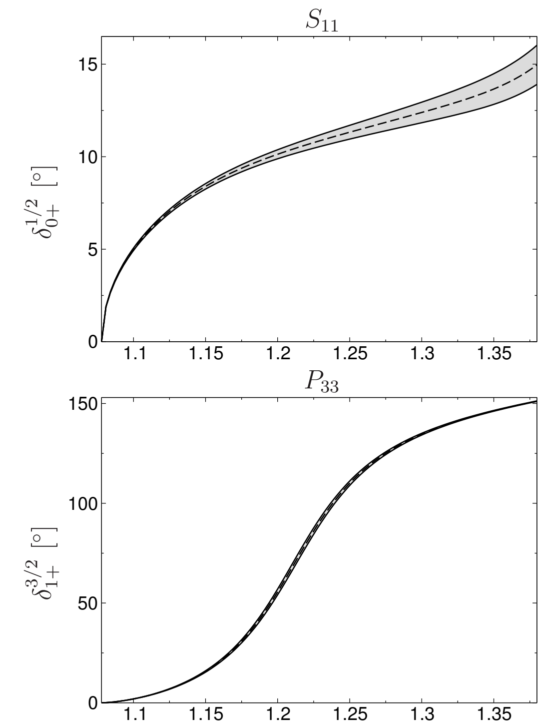
⋆ **no** shoulder on the left wing

⋆ **clean** omega-pol dominance

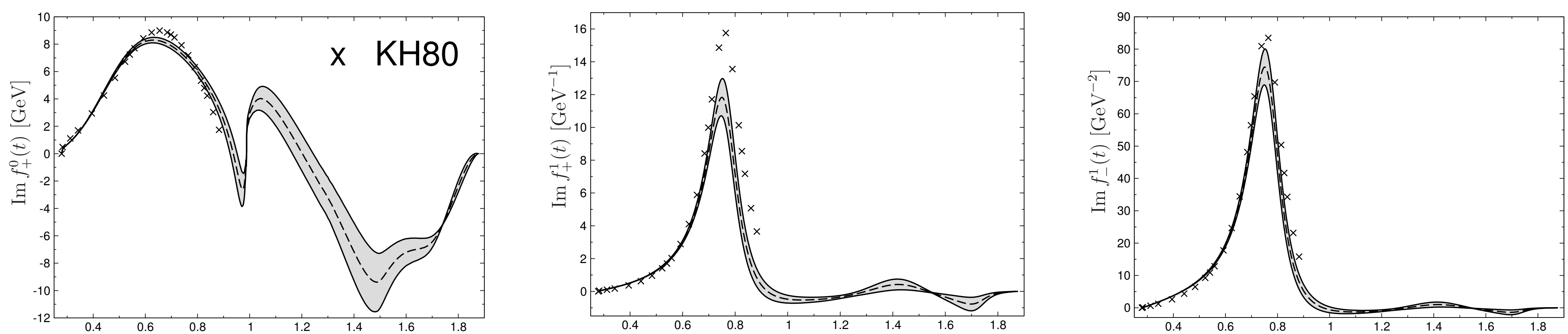
Once more on the isovector spectral functions

Hoferichter, Kubis, Ruiz de Elvira, Hammer, UGM, Eur. Phys. J. A **52** (2016)331 [arXiv:1609.06722 [nucl-th]]

ROY-STEINER EQUATION ANALYSIS



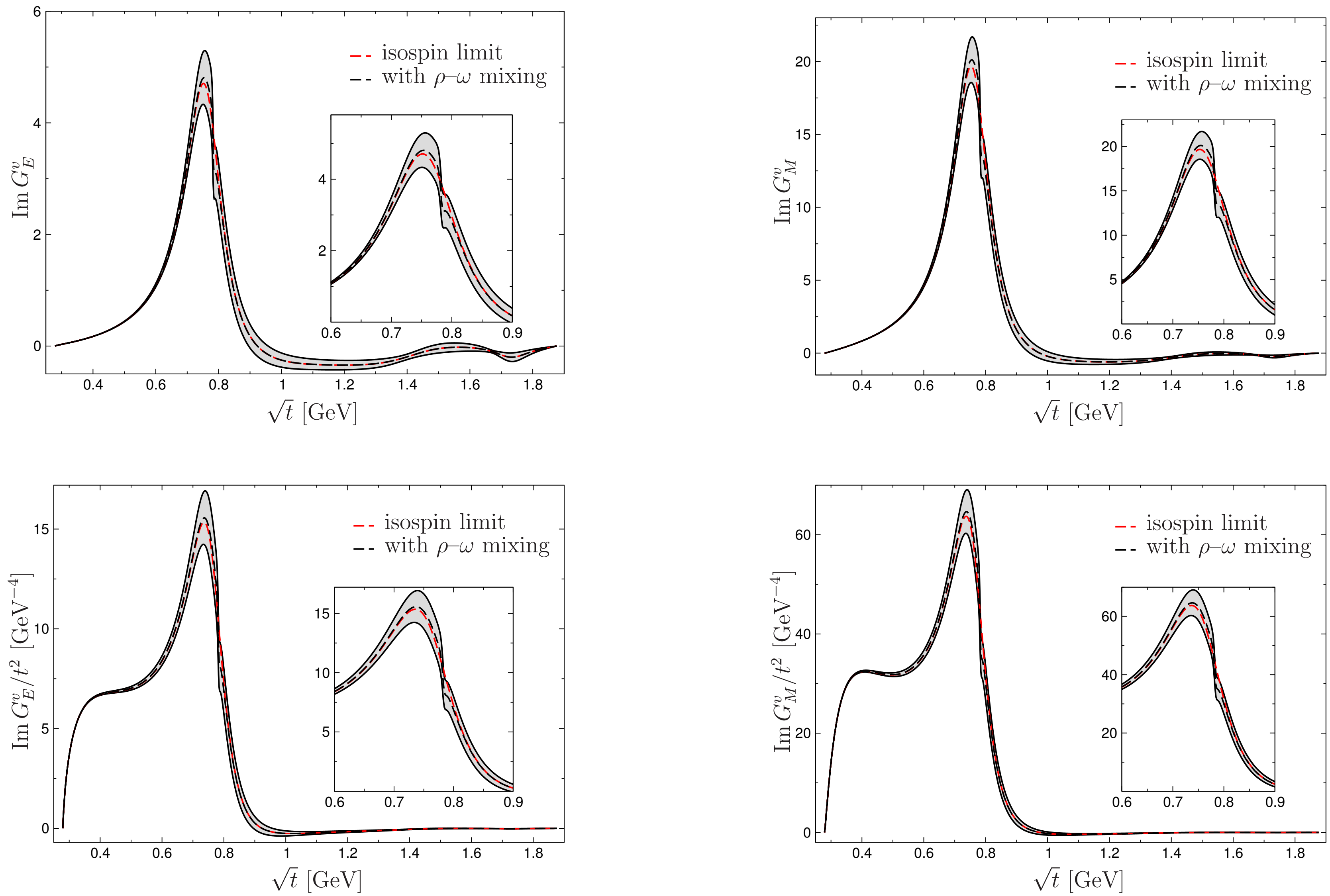
- improve the isovector spectral functions by
  - ↪ updated πN amplitudes from Roy-Steiner equations
  - ↪ include modern data (esp. pionic hydrogen & deuterium)
  - ↪ better treatment of isospin-violating effects
  - ↪ construct the pion FF from precise knowledge of $\delta_1^1(s)$
  - ↪ perform systematic error analysis



Hoferichter, Ruiz de Elvira, Kubis, UGM, Phys. Rev. Lett. **115** (2015) 092301; Phys. Rev. Lett. **115** (2015) 192301; Phys. Rept. **625** (2016) 1; J.Phys. G**45** (2018) 024001

NEW ISOVECTOR SPECTRAL FUNCTIONS

- Precise determinations of the isovector spectral functions

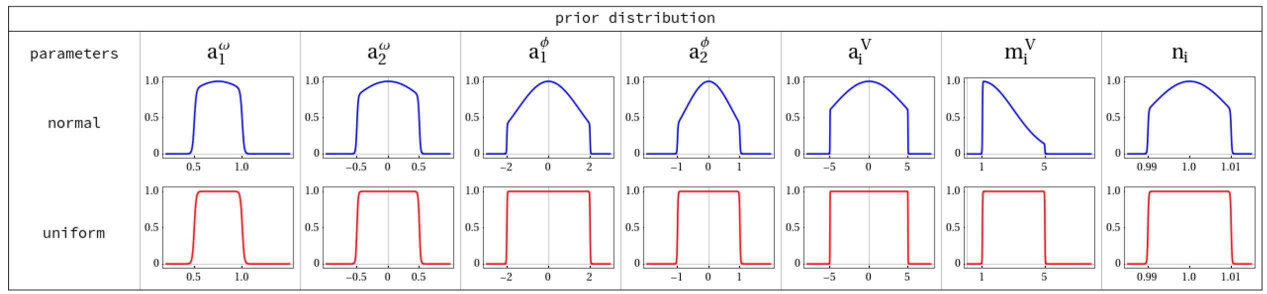


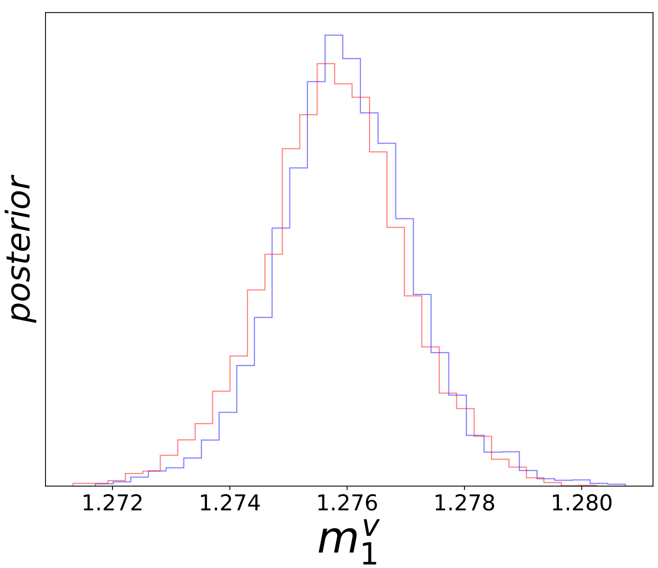
BAYESIAN ANALYSIS

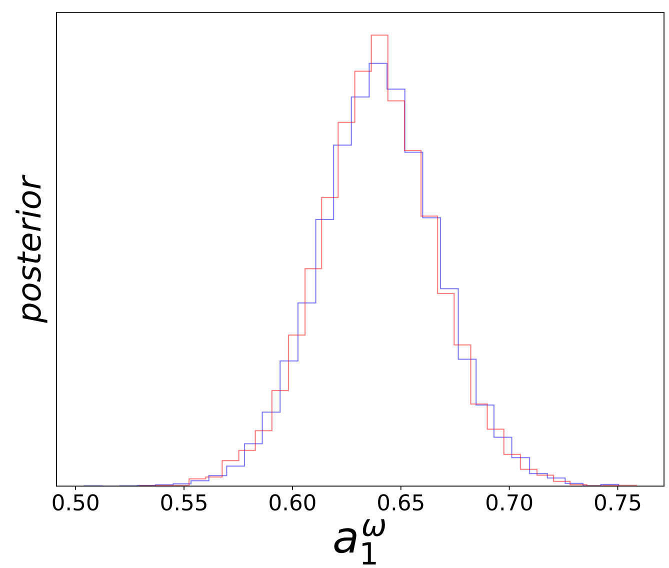
- Bayes theorem: $P(\text{parameters}|\text{data}) = \dfrac{P(\text{parameters})P(\text{data}|\text{parameters})}{P(\text{data})}$

  posterior $\sim$ prior $\times$ likelihood

- Bayesian analysis of the PRad data (71 data pts)



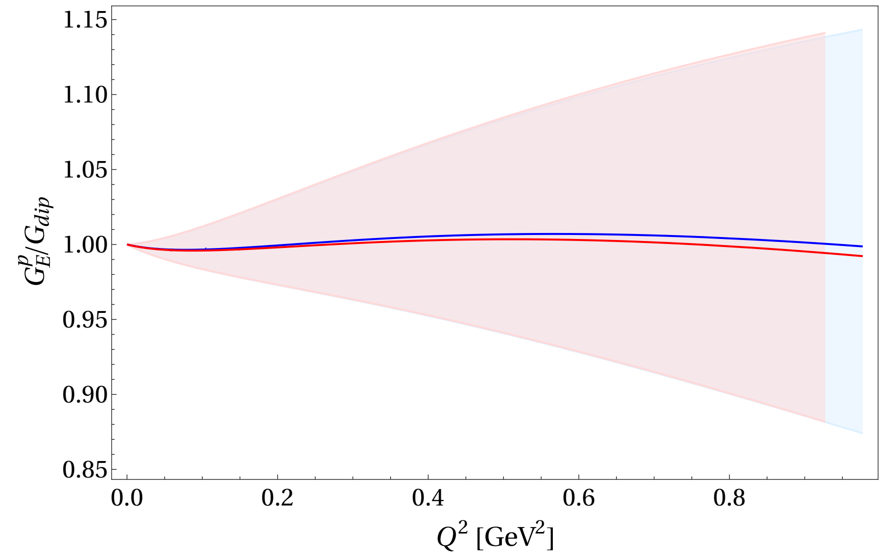


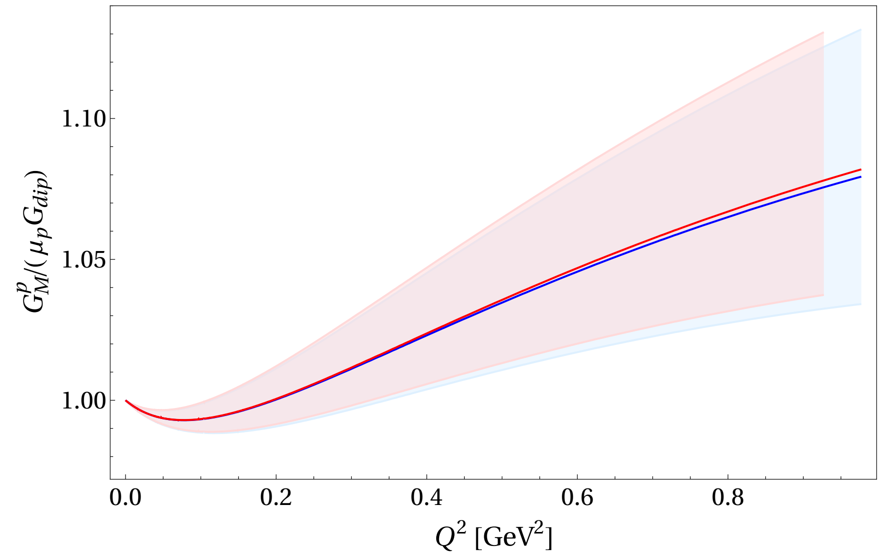


BAYESIAN versus BOOTSTRAP

- Bootstrap sampling in comparison to the Bayesian analysis [PRad data]

Method	r_E^p [fm]	r_M^p [fm]
Bayes normal	$\mathbf{0.828 \pm 0.011}$	$\mathbf{0.843 \pm 0.004}$
Bayes uniform	$\mathbf{0.828 \pm 0.011}$	$\mathbf{0.843 \pm 0.004}$
Bootstrap	$\mathbf{0.828 \pm 0.012}$	$\mathbf{0.843 \pm 0.005}$





$\hookrightarrow$ identical results, but bootstrap much faster

$\hookrightarrow$ will use the bootstrap method to determine the statistical error

PROTON CHARGE RADIUS - LEPTONIC LAMB SHIFT

- Energy levels in hydrogen:

$$E_{n\ell j} = R_\infty \left(-\frac{1}{n^2} + f_{n\ell j}\left(\alpha, \frac{m_e}{m_p}, \ldots\right) + \delta_{\ell 0} \frac{C_{\rm NS}}{n^3} r_p^2 \right)$$

$$f_{n\ell j}\left(\alpha, \frac{m_e}{m_p}, \ldots\right) = X_{20}\alpha^2 + X_{30}\alpha^3 + X_{31}\alpha^3 \ln\alpha + X_{40}\alpha^4 + \ldots$$

– n, ℓ, j - principal, orbital, total ang. momentum quantum numbers

– $f_{n\ell j}$ - relativistic corr's, vacuum effects, other QED corrections

– m_e/m_p enters through the coefficients X_{20}, X_{30}, ... (recoil)

– $C_{\rm NS}$ calculable leading order correction due to the finite r_p

– higher order charge distributions are included in $f_{n\ell j}$

$\Rightarrow$ must measure at least 2 transitions to pin down the two unknowns

$\Rightarrow$ this is done in recent measurements, but not before! [inconsistency]