WAYNE STATE UNIVERSITY

Lessons from Nucleon EM Form Factors

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Richard J. Hill, GP PRD **82**, 113005 (2010)

Bhubanjyoti Bhattacharya, Richard J. Hill, GP PRD **84**, 073006 (2011)

Zachary Epstein, GP, Joydeep Roy PRD **90**, 074027 (2014)

The proton electric radius problem

[Richard J. Hill, GP PRD **82** 113005 (2010)]

Form Factors

- Matrix element of EM current between nucleon states give rise to two form factors ($q = p_f - p_i$)

$$\langle N(p_f)|\sum_q e_q\,\bar{q}\gamma^\mu q|N(p_i)\rangle = \bar{u}(p_f)\left[\gamma^\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}}{2m}F_2(q^2)q^\nu\right]u(p_i)$$

- Sachs electric and magnetic form factors

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2) \qquad G_M(q^2) = F_1(q^2) + F_2(q^2)$$

$$G_E^p(0) = 1 \qquad G_M^p(0) = \mu_p \approx 2.793$$

- The slope of G_E^p

$$\langle r^2\rangle_E^p = 6\left.\frac{dG_E^p}{dq^2}\right|_{q^2=0}$$

determines the charge radius $r_E^p \equiv \sqrt{\langle r^2\rangle_E^p}$

Charge radius from atomic physics

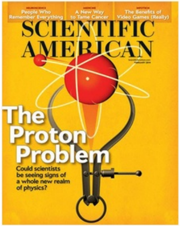
- Lamb shift in muonic hydrogen [Pohl et al. Nature **466**, 213 (2010)]
  $r_E^p = 0.84184(67)$ fm
  more recently $r_E^p = 0.84087(39)$ fm [Antognini et al. Science **339**, 417 (2013)]

- CODATA value [Mohr et al. RMP **80**, 633 (2008)]
  $r_E^p = 0.87680(690)$ fm
  more recently $r_E^p = 0.87750(510)$ fm [Mohr et al. RMP **84**, 1527 (2012)]
  extracted mainly from (electronic) hydrogen

- (more than) **5σ discrepancy!**

How to resolve the puzzle?

- Almost 5 years after first measurement puzzle is still not resolved



(Cover story of February 2014 Scientific American)

- Is it new physics?

- Is it a problem with the theoretical prediction?
  [Richard. J. Hill, GP PRL **107** 160402 (2011), and in progress]

- We can also extract it from electron-proton scattering data
  What does the PDG say?

What does the PDG say?

K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010)

p CHARGE RADIUS

This is the rms charge radius, $\sqrt{\langle r^2 \rangle}$.

VALUE (fm)	*DOCUMENT ID*		*TECN*	*COMMENT*
0.8768±0.0069	MOHR	08	RVUE	2006 CODATA value
• • • We do not use the following data for averages, fits, limits, etc. • • •				
0.897 ±0.018	BLUNDEN	05		SICK 03 + 2γ correction
0.8750±0.0068	MOHR	05	RVUE	2002 CODATA value
0.895 ±0.010 ±0.013	SICK	03		$e p \to e p$ reanalysis
0.830 ±0.040 ±0.040	[24] ESCHRICH	01		$e p \to e p$
0.883 ±0.014	MELNIKOV	00		1S Lamb Shift in H
0.880 ±0.015	ROSENFELDR.	00		$e p$ + Coul. corrections
0.847 ±0.008	MERGELL	96		$e p$ + disp. relations

Citation: K. Nakamura *et al.* (Particle Data Group), JPG **37**, 075021 (2010) (URL: http://pdg.lbl.gov)

0.877 ±0.024	WONG	94	reanalysis of Mainz $e p$ data
0.865 ±0.020	MCCORD	91	$e p \to e p$
0.862 ±0.012	SIMON	80	$e p \to e p$
0.880 ±0.030	BORKOWSKI	74	$e p \to e p$
0.810 ±0.020	AKIMOV	72	$e p \to e p$
0.800 ±0.025	FREREJACQ...	66	$e p \to e p$ (CH_2 tgt.)
0.805 ±0.011	HAND	63	$e p \to e p$

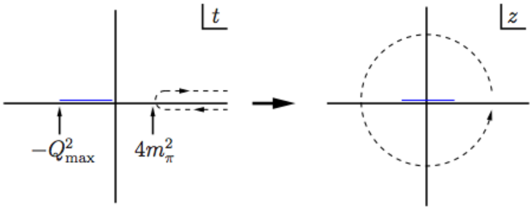
[24] ESCHRICH 01 actually gives $\langle r^2 \rangle = (0.69 \pm 0.06 \pm 0.06)$ fm^2.

z expansion

- Analytic properties of $G_E^p(t)$ are known
  $G_E^p(t)$ is analytic outside a cut $t \in [4m_\pi^2, \infty]$
  [Federbush, Goldberger, Treiman, Phys. Rev. **112**, 642 (1958)]
  $e - p$ scattering data is in $t < 0$ region
- We can map the domain of analyticity onto the unit circle

$$z(t, t_{\rm cut}, t_0) = \frac{\sqrt{t_{\rm cut} - t} - \sqrt{t_{\rm cut} - t_0}}{\sqrt{t_{\rm cut} - t} + \sqrt{t_{\rm cut} - t_0}}$$

where $t_{\rm cut} = 4m_\pi^2$, $z(t_0, t_{\rm cut}, t_0) = 0$



- Expand G_E^p in a Taylor series in z: $G_E^p(q^2) = \sum_{k=0}^{\infty} a_k \, z(q^2)^k$

Comparison of series expansions

- Use data sets tabulated by Rosenfelder [arXiv:nucl-th/9912031] with $Q^2 < 0.04\,\mathrm{GeV}^2 \Rightarrow 0 \le z \le 0.1$ ($B \to \pi : |z| \lesssim 0.3$, $B \to D : |z| \lesssim 0.03$)

r_E^p in $10^{-18}m$

polynomial

continued fraction

z expansion (no bound)

z expansion ($|a_k| \le 10$)

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r_E^p in $10^{-18} m$

	$k_{\rm max} = 1$
polynomial	836^{+8}_{-9}
continued fraction	882^{+10}_{-10}
z expansion (no bound)	918^{+9}_{-9}
z expansion ($\|a_k\| \le 10$)	918^{+9}_{-9}

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r_E^p in $10^{-18} m$

	$k_{\max} = 1$	2
polynomial	836^{+8}_{-9}	867^{+23}_{-24}
continued fraction	882^{+10}_{-10}	869^{+26}_{-25}
z expansion (no bound)	918^{+9}_{-9}	868^{+28}_{-29}
z expansion ($\lvert a_k\rvert \le 10$)	918^{+9}_{-9}	868^{+28}_{-29}

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r_E^p in $10^{-18} m$

	$k_{\rm max} = 1$	2	3
polynomial	836^{+8}_{-9}	867^{+23}_{-24}	866^{+52}_{-56}
continued fraction	882^{+10}_{-10}	869^{+26}_{-25}	–
z expansion (no bound)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+64}_{-69}
z expansion ($\lvert a_k\rvert \le 10$)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+38}_{-59}

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r_E^p in $10^{-18} m$

	$k_{\max} = 1$	2	3	4
polynomial	836^{+8}_{-9}	867^{+23}_{-24}	866^{+52}_{-56}	959^{+85}_{-93}
continued fraction	882^{+10}_{-10}	869^{+26}_{-25}	–	–
z expansion (no bound)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+64}_{-69}	1022^{+102}_{-114}
z expansion ($\lvert a_k \rvert \leq 10$)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+38}_{-59}	880^{+39}_{-61}

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r_E^p in $10^{-18} m$

	$k_{\max} = 1$	2	3	4	5
polynomial	836^{+8}_{-9}	867^{+23}_{-24}	866^{+52}_{-56}	959^{+85}_{-93}	1122^{+122}_{-137}
continued fraction	882^{+10}_{-10}	869^{+26}_{-25}	–	–	–
z expansion (no bound)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+64}_{-69}	1022^{+102}_{-114}	1193^{+152}_{-174}
z expansion ($\|a_k\| \le 10$)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+38}_{-59}	880^{+39}_{-61}	880^{+39}_{-62}

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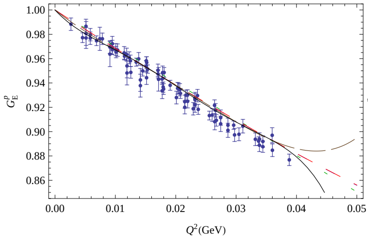
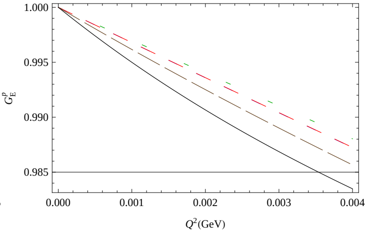
- Fit with two parameters agree well
- As we increase $k_{\rm max}$ the errors for the first three fits grow
- For the continued fraction fit for $k_{\rm max} > 3$ the slope is not positive
- To get a meaningful answer we must constrain a_k. How?

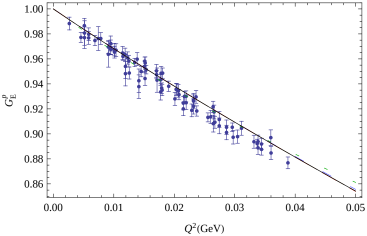
Comparison of Taylor and constrained z fits

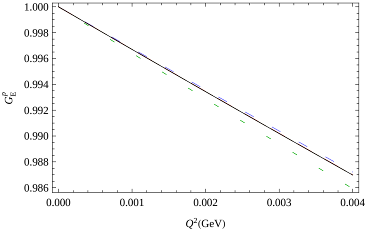
- Taylor fit





- Constrained z fit



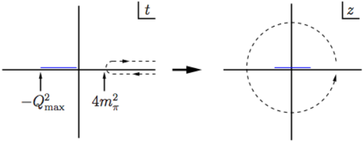


See also:

"Constrained curve fitting" : Lepage et al. Nucl.Phys.Proc.Suppl. 106 (2002) 12-20

Analytic structure and a_k

$$z(t, t_{\rm cut}, t_0) = \frac{\sqrt{t_{\rm cut} - t} - \sqrt{t_{\rm cut} - t_0}}{\sqrt{t_{\rm cut} - t} + \sqrt{t_{\rm cut} - t_0}}$$



- Analytic structure implies:
  Information about $\mathrm{Im}\, G_E^p(t + i0) \Rightarrow$ information about a_k
- $G(t) = \sum_{k=0}^{\infty} a_k\, z(t)^k$, z^k are orthogonal over $|z| = 1$

$$\begin{aligned} a_0 &= G(t_0) \\ a_k &= \frac{2}{\pi} \int_{t_{\rm cut}}^{\infty} \frac{dt}{t - t_0} \sqrt{\frac{t_{\rm cut} - t_0}{t - t_{\rm cut}}}\, \mathrm{Im}\, G(t) \sin[k\theta(t)]\,, \quad k \geq 1 \\ \sum_k a_k^2 &= \frac{1}{\pi} \int_{t_{\rm cut}}^{\infty} \frac{dt}{t - t_0} \sqrt{\frac{t_{\rm cut} - t_0}{t - t_{\rm cut}}} |G|^2 \end{aligned}$$

- How to constrain $\mathrm{Im}\, G(t)$?

Size of a_k: vector dominance ansatz

- The isovector and isoscalar form factors are

$$G_E^{(0)} = G_E^p + G_E^n\,, \quad G_E^{(1)} = G_E^p - G_E^n$$

- Assume vector dominance ansatz [Höhler NPB **95**, 210 (1975)]

$$F_i^{(I=0)} \sim \frac{\alpha_i m_\omega^2}{m_\omega^2 - t - i\Gamma_\omega m_\omega}\,, \quad F_i^{(I=1)} \sim \frac{\beta_i m_\rho^2}{m_\rho^2 - t - i\Gamma_\rho m_\rho}\,,$$

α_i and β_i are fixed by $F_i^I(0)$

- For $G(t) \sim 1/(t - m_V^2)$, $\mathrm{Im}\,G(t + i0) = -i\pi\delta(t - m_V^2)$
$\Rightarrow |a_k/a_0| \leq 2\sqrt{(t_{\rm cut} - t_0)/(m_v^2 - t_{\rm cut})}$
Taking $t_0 = 0$: $|a_k| < 1.3$ for $G_E^{(0)}$,$|a_k| < 0.78$ for $G_E^{(1)}$

- Conclusion: $|a_k| \leq 10$ is a very conservative estimate for this ansatz

Size of a_k: $\pi\pi$ continuum

- $\pi\,\pi$ is the lightest state that can contribute to $\mathrm{Im}\,G_E^{(1)}$

$$\mathrm{Im}\,G_E^{(1)}(t) = \frac{2}{m_N\sqrt{t}}\left(t/4 - m_\pi^2\right)^{\frac{3}{2}} F_\pi(t)^* f_+^1(t)$$

  $F_\pi(t)$ pion form factor, $f_+^1(t)$ is a partial amplitude for $\pi\pi \to N\bar{N}$ [Federbush et al. Phys. Rev. **112**, 642 (1958), Frazer et al. Phys. Rev. **117**, 1609 (1960), Belushkin et al. PRC **75**, 035202 (2007)]
- Since they share the same phase up to $t < 16m_\pi^2$, we can use $|F_\pi|$ (For determining bound on a_k we assume phase equality through ρ peak)
- Using $|F_\pi(t)|$ data from
  - NA7 experiment [Amendolia et al. PLB **138**, 454 (1984)]
  - SND experiment [Achasov et al. arXiv:hep-ex/0506076]
- Using $f_+^1(t)$ tables from [G. Höhler, Pion-nucleon scattering, Springer-Verlag, Berlin, 1983]
- For $t_0 = 0$: $a_0 \approx 2.1$ $a_1 \approx -1.4$, $a_2 \approx -1.6$, $a_3 \approx -0.9$, $a_4 \approx 0.2$ Using $|\sin(k\theta)| \leq 1$ in the integral gives $|a_k| \lesssim 2.0$ for $k \geq 1$.

Size of a_k: $t > 4m_N^2$ region

- For the region $t > 4m_N^2$ we can use $e^+e^- \to N\bar{N}$ data, e.g.

- $p - \bar{p}$: BES collaboration [Ablikim et al. arXiv:hep-ex/0506059]
- $n - \bar{n}$: FENICE experiment [Antonelli et al. NPB **517**, 3 (1998)]

- We find a very small contribution from this region

- $|\delta a_k| \lesssim 0.006 + 0.002$ for the proton
- $|\delta a_k| \lesssim 0.013 + 0.025$ for the neutron

Size of a_k: Summary

- In all of the above $|a_k| \leq 10$ appears very conservative
- In practice we find max $|a_k| \sim 2$
- Final results are presented for both $|a_k| \leq 5$ and $|a_k| \leq 10$

Results: Summary

- Proton: $Q^2 < 0.5\,\mathrm{GeV}^2$

$$r_E^p = 0.870 \pm 0.023 \pm 0.012\,\mathrm{fm}$$

- Proton and neutron data

$$r_E^p = 0.880^{+0.017}_{-0.020} \pm 0.007\,\mathrm{fm}$$

- Proton, neutron and $\pi\,\pi$ data

$$r_E^p = 0.871 \pm 0.009 \pm 0.002 \pm 0.002\,\mathrm{fm}$$

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The proton magnetic radius problem

[Zachary Epstein, GP, Joydeep Roy PRD **90**, 074027 (2014)]

The proton magnetic radius problem

- The proton *magnetic* radius

$$\langle r^2 \rangle_M^p = \frac{6}{G_M^p(0)} \frac{dG_M^p(q^2)}{dq^2}\bigg|_{q^2=0}$$

- PDG 2012:

- Recent high precision data from A1 experiment at Mainz
  $r_M^p = 0.777 \pm 0.017$ fm [Bernauer et al. PRL **105**, 242001 (2010)]
  Older data sets
- $r_M^p = 0.876 \pm 0.019$ fm [Borisyuk et al. 2010]
- $r_M^p = 0.854 \pm 0.005$ fm [Belushkin et al. 2007]
  Are we facing a magnetic radius puzzle too?

- We need a model independent extraction of r_M^p!

Bound on $|a_k|$

- Analyzing p and n data separate G_M^p and G_M^n to isospin channels

  $G_M^{(0)} = G_M^p + G_M^n$
  $G_M^{(0)}(0) = \mu_p + \mu_n \approx 0.88$
  $\Rightarrow I = 0, \quad a_0 = 0.88$

  $G_M^{(1)} = G_M^p - G_M^n$
  $G_M^{(1)}(0) = \mu_p - \mu_n \approx 4.7$
  $\Rightarrow I = 1, \quad a_0 = 4.7$

- Vector dominance ansatz:
  - $I = 0$ (ω exchange) $|a_k| \leq 1.1$
  - $I = 1$ (ρ exchange) $|a_k| \leq 5.1$

- Between $t = 4m_\pi^2$ and $t = 16m_\pi^2$ only $\pi\pi$ contributes
  $I = 1$: $|a_k| \leq 7.2$

- Above $t = 4m_N^2$ use $e^+e^- \to N\bar{N}$: negligible contribution to a_k

- Two options
  - Use $|a_k| \leq 10$ and $|a_k| \leq 15$ (default)
  - Use $|a_k/a_0| \leq 5$ and $|a_k/a_0| \leq 10$ (used as a check)

r_M^p extraction: results

- Results from model independent extraction
- Proton data :$r_M^p = 0.91^{+0.03}_{-0.06} \pm 0.02$ fm
- Proton and neutron data: $r_M^p = 0.87^{+0.04}_{-0.05} \pm 0.01$ fm
- Proton, neutron and $\pi\,\pi$ data: $r_M^p = 0.87 \pm 0.02$ fm
- Proton, neutron and $\pi\,\pi$ data: $r_M^n = 0.89 \pm 0.03$ fm

- PDG 2014:
- $r_M^p = 0.777 \pm 0.017$ fm [Bernauer et al. PRL **105**, 242001 (2010)]
- $r_M^p = 0.876 \pm 0.019$ fm [Borisyuk NPA **843**, 59 (2010)]
- $r_M^p = 0.854 \pm 0.005$ fm [Belushkin et al. PRC **75**, 035202 (2007)]

- Other non-PDG values:
- $r_M^p = 0.855 \pm 0.035$ fm [Sick Prog.Part.Nucl.Phys. **55**, 440 (2005)]
- $r_M^p = 0.86^{+0.02}_{-0.03}$ fm [Lorenz et al. EPJA **48**, 151 (2012)]
- $r_M^p = 0.78 \pm 0.08$ fm [Karshenboim PRD **90** 053013 (2014) 5]

r_M^p extraction: comments

- Our results
- do not depend on the number of parameters
- are very consistent over the range of Q^2
- barely change (less than 1 σ)
  using $|a_k| \leq 20$, or $|a_k/a_0| \leq 5$, or $|a_k/a_0| \leq 10$

- The reduction in the error bar by inclusion of $\pi\pi$ data arises from the increase in $t_{\rm cut}$ to $16m_\pi^2$ for $G_M^{(1)}$

Lessons for flavor-changing form factors

Lessons for flavor-changing form factors

- Successful use of the z expansion requires bounds on the coefficients
- For $H \to L$ *meson* form factors singularity starts at $(m_H + m_L)^2$ ($\bar{H}L$ threshold)
  Removing possible sub-threshold poles allows to bound $\sum_{k=0}^{\infty} |a_k|^2$ using unitarity, see e.g.
  [Richard J. Hill, FPCP 2006 proceedings (hep-ph/0606023)]
- For $H \to L$ *baryon* form factors singularity starts much "earlier "
  e.g. $4m_\pi^2 = 0.02$ GeV^2 instead of $4m_p^2 = 4$ GeV^2
  Need to work harder to bound the coefficients
  Still, can get good determination (a few %) of r_E^p, r_M^p, r_M^n