Pure Spinor Field Theory Description of 10D Super-Yang-Mills and Color-Kinematics Duality

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Outline

Introduction

- A few years ago, scattering amplitudes corresponding to several gauge theories have been shown to exhibit a peculiar symmetry referred to as (CK) *color-kinematics duality* [Bern,Carrasco,Johansson '08].
- Such a symmetry is, in general, hidden from standard Lagrangian formulations, and it states that given an scattering amplitude formulated in terms of cubic graphs:

$$A = \sum_i \frac{c_i n_i}{D_i} \quad (1)$$

  it is possible to find kinematic numerators n_i satisfying the same relations obeyed by their respective color factors c_i:
  $c_i + c_j + c_k = 0 \rightarrow n_i + n_j + n_k = 0$.
- Gravity interactions then immediately follow from (1) through a *double-copy* mechanism, consisting of replacing $c_i \rightarrow n_i$ [Bern,Carrasco,Johansson '10].

- In particular, this novel approach has been used for studying the UV behaviour of maximal supergravity from CK representations of super-Yang-Mills scattering amplitudes .
- The simplest description of maximally supersymmetric Yang-Mills is realized in 10D.
- Pure spinors have been shown to elegantly describe maximally supersymmetric gauge theories from a field-theory and superstring approach [Berkovits '00;Berkovits,Howe '01;Cederwall '09].
- In particular, 10D super-Yang-Mills can be formulated in pure spinor superspace as a Chern-Simons-like theory.
- In this talk, we will use the cubic nature of the pure spinor action for computing 10D super-Yang-Mills scattering amplitudes, and study how color-kinematics duality is simply realized in this framework.

10D Super-Yang-Mills in Component Formulation

- The 10D super-Yang-Mills action in ordinary space is defined by the action

$$S = \int d^{10}x \, Tr\Big[-\frac{1}{4}F^{mn}F_{mn} - \frac{i}{2}(\chi\gamma^m\nabla_m\chi)\Big] \qquad (2)$$

  where $\nabla_m = \partial_m + a_m$, $F_{mn} = \nabla_m a_n - \nabla_n a_m$ is the field-strength associated to the gluon a_m, and χ^α is the gluino.
- We will denote $SO(1,9)$ vector/Majorana-Weyl indices by letters from the middle/beginning of the Latin/Greek alphabet.
- $SO(1,9)$ Pauli matrices are denoted by $(\gamma^m)_{\alpha\beta}$, $(\gamma^m)^{\alpha\beta}$, and they satisfy $(\gamma^{(m})^{\alpha\beta}(\gamma^{n)})_{\beta\delta} = \eta^{mn}\delta^\alpha_\delta$.
- The action (2) is invariant under standard gauge transformations and the SUSY transformations defined by

$$\delta a_m = (\epsilon\gamma_m\chi) \quad , \quad \delta\chi^\alpha = -\frac{i}{2}(\epsilon\gamma^{mn})^\alpha F_{mn} \qquad (3)$$

  where ϵ^α is an arbitrary fermionic parameter.

10D Super-Yang-Mills in Superspace

- The (N=1) 10D superspace is described by the coordinates X^m, θ^α.
- Using these coordinates, one can define the operators

$$Q_\alpha = \partial_\alpha - i(\gamma^m\theta)_\alpha\partial_m \tag{4}$$

which realize the SUSY algebra $\{Q_\alpha, Q_\beta\} = -2i(\gamma^m)_{\alpha\beta}\partial_m$.

- The supersymmetric derivatives can then be introduced as

$$D_\alpha = \partial_\alpha + i(\gamma^m\theta)_\alpha\partial_m \tag{5}$$

and shown to satisfy $\{D_\alpha, D_\beta\} = 2i(\gamma^m)_{\alpha\beta}\partial_m$,$\{D_\alpha, Q_\beta\} = 0$.

- As in the bosonic case, one can introduce a 1-form (super)connection and define

$$\nabla = d + \mathbb{A} \tag{6}$$

where $\mathbb{A}$ is Lie-algebra valued.

- The (super)field-strength is then defined in the usual way

$$\mathbb{F} = d\mathbb{A} + \mathbb{A}\wedge\mathbb{A} \tag{7}$$

- It is not hard to show that $\mathbb{F}$ must satisfy the super-Bianchi identities

$$\nabla \mathbb{F} \;=\; 0 \tag{8}$$

- Using the (super)vielbein fields $E_A{}^M$, $E_M{}^A$, one can write the superspace components of the 2-form superfield $\mathbb{F}$ as follows

$$\mathbb{F} = \frac{1}{2} E^B E^A \mathbb{F}_{AB} = d(E^B \mathbb{A}_B) + E^B E^A \mathbb{A}_A \mathbb{A}_B \tag{9}$$

where $E^A = dZ^M E_M{}^A$, which implies

$$\mathbb{F}_{AB} \;=\; 2D_{[A}\mathbb{A}_{B\}} + 2\mathbb{A}_A \mathbb{A}_B + T_{AB}{}^C \mathbb{A}_C \tag{10}$$

where $T^A = dE^A$ is the (super)torsion.

- Therefore, one has

$$\begin{aligned} \mathbb{F}_{\alpha\beta} &= \{\nabla_\alpha, \nabla_\beta\} - 2i(\gamma^m)_{\alpha\beta}\nabla_m && (11) \\ \mathbb{F}_{m\alpha} &= [\nabla_m, \nabla_\alpha] && (12) \\ \mathbb{F}_{mn} &= [\nabla_m, \nabla_n] && (13) \end{aligned}$$

- Using similar manipulations, the Bianchi identities in component form read

$$\nabla_{[A}\mathbb{F}_{BC\}} + T_{[AB}{}^{D}\mathbb{F}_{|D|C\}} = 0 \qquad (14)$$

- Explicitly,

$$\begin{aligned} \nabla_{(\alpha}\mathbb{F}_{\beta\delta)} + 2i(\gamma^m)_{(\alpha\beta}\mathbb{F}_{\delta)m} &= 0 && (15) \\ \nabla_m\mathbb{F}_{\alpha\beta} + 2\nabla_{(\alpha}\mathbb{F}_{\beta)m} - 2i(\gamma^n)_{\alpha\beta}\mathbb{F}_{nm} &= 0 && (16) \\ 2\nabla_{[m}\mathbb{F}_{n]\alpha} + \nabla_\alpha\mathbb{F}_{mn} &= 0 && (17) \\ \nabla_{[m}\mathbb{F}_{np]} &= 0 && (18) \end{aligned}$$

- In order to describe 10D super-Yang-Mills from these identities, one needs to impose constraints.
- **Conventional constraint**:

$$(\gamma^m)^{\alpha\beta}\mathbb{F}_{\alpha\beta} \;=\; 0 \tag{19}$$

This constraint can always be satisfied by performing a field redefinition, namely $\mathbb{A}_A = (\mathbb{A}_\alpha, \mathbb{A}_m - \frac{i}{32}(\gamma_m)^{\alpha\beta}\mathbb{F}_{\alpha\beta})$.

- **Dynamical constraint**:

$$(\gamma^{mnpqr})^{\alpha\beta}\mathbb{F}_{\alpha\beta} \;=\; 0 \tag{20}$$

This is the constraint which puts the theory on-shell.

- All in all, 10D SYM is described by setting

$$\mathbb{F}_{\alpha\beta} \;=\; 0 \tag{21}$$

- The Bianchi identities then imply that

$$\nabla_\alpha \mathbb{W}^\beta = -\frac{i}{2}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} \tag{22}$$

where

$$\mathbb{F}_{m\alpha} = (\gamma_m)_{\alpha\beta}\mathbb{W}^\beta \tag{23}$$

- It is not hard to see that this equation together with the Bianchi identities imply that no new fields will appear in the θ-expansion of $\mathbb{W}^\alpha$.
- The equations of motion immediately follow from (22) and the dynamical constraint $\{\nabla_\alpha, \nabla_\beta\} = 2i(\gamma^m)_{\alpha\beta}\nabla_m$. Explicitly,

$$(\gamma^m)_{\alpha\beta}\nabla_m \mathbb{W}^\beta = 0 \tag{24}$$

$$\nabla_m \mathbb{F}^{mn} = \frac{i}{2}\gamma^n_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\} \tag{25}$$

- Using the gauge transformation $\delta\mathbb{A} = d\Lambda + \mathbb{A} \wedge \Lambda$, one can fix the so-called Harnard-Schnider gauge $\theta^\alpha \mathbb{A}_\alpha = 0$. The equations above studied then imply that

$$\mathbb{A}_\alpha = i(\gamma^m\theta)_\alpha a_m - \frac{1}{36}(\theta\gamma^{mnp}\theta)(\gamma_{mnp}\chi)_\alpha + \ldots \quad (26)$$

$$\mathbb{A}_m = a_m + i(\theta\gamma^m\chi) + \ldots \quad (27)$$

where $\delta a_m = \partial_m\lambda + [\lambda, a_m]$.

- One can check that coordinate transformations on superfields indeed induce SUSY transformations. For instance,

$$\delta_Q \mathbb{W}^\alpha = \epsilon^\beta Q_\beta(\mathbb{W}^\alpha) \quad (28)$$

After expanding in components, one finds

$$\delta\chi^\alpha = -\frac{i}{2}(\epsilon\gamma^{mn})_\alpha F_{mn} \quad (29)$$

- It will be useful to write the superspace equations of motion of linearized 10D super-Yang-Mills in light-cone gauge coordinates. In a reference frame where the only non-zero component of the momentum k^m is k^+, the dynamical constraint can be written as

$$D_a\mathbb{A}_b + D_b\mathbb{A}_a = -2\sqrt{2}i\delta_{ab}\mathbb{A}^+ \quad (30)$$

$$D_a\mathbb{A}_{\dot{b}} + D_{\dot{b}}\mathbb{A}_a = 2i(\sigma^i)_{a\dot{b}}\mathbb{A}_i \quad (31)$$

$$D_{\dot{a}}\mathbb{A}_{\dot{b}} + D_{\dot{b}}\mathbb{A}_{\dot{a}} = 0 \quad (32)$$

where a, $\dot{a}$, i are respectively $SO(8)$ Weyl, anti-Weyl and vector indices.
- Using the SUSY algebra, one then learns that $\mathbb{A}_a$ is pure gauge ($\delta\mathbb{A}_\alpha = D_\alpha\Lambda$), and so

$$D_a\mathbb{A}_{\dot{a}} = 2i(\sigma^i)_{\dot{a}a}\mathbb{A}_i \quad (33)$$

with $\mathbb{A}_{\dot{b}} = \mathbb{A}_{\dot{b}}(\theta)$.
- Similarly, one can use the e.o.m $\mathbb{F}_{m\alpha} = (\gamma_m\mathbb{W})_\alpha$ to show that

$$D_a\mathbb{A}_i = -\frac{\sqrt{2}}{2}k^+(\sigma^i)_{a\dot{a}}\mathbb{A}_{\dot{a}} \quad (34)$$

10D Brink-Schwarz Superparticle

- The 10D Brink-Schwarz superparticle action is defined by [Brink, Schwarz '81]

$$S = \int d\tau \left[P_m \Pi^m - \frac{e}{2} P^2 \right] \quad (35)$$

where $\Pi^m = \dot{X}^m + i(\theta\gamma^m\dot{\theta})$.

- The action (35) is invariant under worldline reparametrizations, (global) SUSY transformations

$$\delta\theta^\alpha = \epsilon^\alpha \ , \ \ \delta X^m = -i(\epsilon\gamma^m\theta) \ , \ \ \delta P^m = 0 \ , \ \ \delta e = 0 \quad (36)$$

and the so-called (local) kappa symmetry [Siegel '83]

$$\delta\theta^\alpha = (\gamma^m\kappa)^\alpha P_m \ , \ \delta X^m = i(\delta\theta\gamma^m\theta) \ , \ \delta P^m = 0 \ , \ \delta e = -4i\dot{\theta}^\alpha\kappa_\alpha \quad (37)$$

- The system can be easily shown to be contrained by $d_\alpha = p_\alpha + i(\gamma^m\theta)_\alpha P_m$ with algebra

$$\{d_\alpha, d_\beta\} \;=\; -2(\gamma^m)_{\alpha\beta}P_m \tag{38}$$

- Since $P^2 = 0$, d_α contains 8 first-class and 8 second-class constraints.
- It turns out that there is no simple way to separate these constraints out in a Lorentz covariant manner.
- However, we can compute the physical spectrum in a simple way by gauge-fixing the kappa symmetry.
- In the semi-light-cone gauge $(\gamma^+\theta) = 0$, the BS action takes the form

$$S \;=\; \int d\tau\left[P_m\dot{X}^m + \frac{i}{2}S^a\dot{S}^a - \frac{e}{2}P^2\right] \tag{39}$$

where $S^a = 2^{\frac{1}{4}}(P^+)^{\frac{1}{2}}(\gamma^-\theta)^a$.
- This system is constrained by $\tilde{d}_a = p_a + \frac{i}{2}S_a$ with algebra

$$\{\tilde{d}_a, \tilde{d}_b\} \;=\; \delta_{ab} \tag{40}$$

- Using the usual Dirac procedure one then learns that $\{S^a, S^b\}_D = \delta^{ab}$
- Using the triality property of $SO(8)$, one can show that the physical states realizing this algebra are given by

$$S_a|\dot{a}>= \frac{1}{\sqrt{2}}(\sigma^i)_{a\dot{a}}|i> \quad , \quad S_a|i>= \frac{1}{\sqrt{2}}(\sigma_i)_{a\dot{a}}|\dot{a}> \tag{41}$$

- These are the same $SO(8)$ equations of motion found in the superspace description of (linearized) 10D super-Yang-Mills.

10D Pure Spinor Superparticle

- Using BRST symmetry arguments, one can show that the 10D BS superparticle in semi-light-cone gauge is physically equivalent to the pure spinor description of the superparticle [Berkovits '01] described by the action

$$S = \int d\tau \left[P_m \dot{X}^m + p_\alpha \dot{\theta}^\alpha + w_\alpha \dot{\lambda}^\alpha - \frac{1}{2} P^2 \right] \tag{42}$$

  where λ^α is a pure spinor satisfying $\lambda\gamma^m\lambda = 0$, and w_α is its conjugate momentum defined up to the gauge transformation $\delta w_\alpha = (\gamma^m \lambda)_\alpha r_m$ for any r_m; and the BRST operator $Q = \lambda^\alpha d_\alpha$.
- In this pure spinor framework, the problem of covariant quantization is then translated into a cohomological problem.

- To find the physical spectrum, we write down the wavefunction in a coordinate representation as

$$\Psi(x,\theta,\lambda) \;=\; \Psi_0(x,\theta)+\Psi_1(x,\theta,\lambda)+\ldots \tag{43}$$

where the subscript indicates the polynomial degree in λ.
- Let us focus on the ghost number one sector, that is $\Psi_1(x,\theta,\lambda)=\lambda^\alpha V_\alpha(x,\theta)$.
- The physical state conditions then require

$$\begin{aligned} Q\Psi_1=0 \;&\rightarrow\; \lambda^\alpha\lambda^\beta D_\alpha V_\beta=0 \\ &\rightarrow\; (\gamma^{mnpqr})^{\alpha\beta}D_\alpha V_\beta=0 \end{aligned} \tag{44}$$

and also,

$$\delta\Psi_1=Q\Omega \;\rightarrow\; \delta V_\alpha=D_\alpha\Omega \tag{45}$$

for some arbitrary parameter Ω.
- Eqn. (44) is nothing but the dynamical constraint previously studied, and eqn. (45) is the usual gauge transformation for the fermionic gauge potential.

- Therefore we identify

$$V_\alpha(x,\theta) \;=\; \mathbb{A}_\alpha(x,\theta) \tag{46}$$

- In this manner the 10D pure spinor superparticle describes 10D super-Maxwell in a manifestly supersymmetric way through the ghost number one state

$$\Psi_1(x,\theta,\lambda) \;=\; \lambda^\alpha \mathbb{A}_\alpha(x,\theta) \tag{47}$$

- One can use the gauge transformation (45) to fix the HS gauge: $\theta^\alpha \mathbb{A}_\alpha = 0$, in which $\mathbb{A}_\alpha$ looks like

$$\begin{aligned}\mathbb{A}_\alpha(x,\theta) \;=\;& \frac{1}{2}(\theta\gamma^m)_\alpha a_m - \frac{1}{3}(\theta\gamma^m)_\alpha(\chi\gamma^m\theta)\\ &-\frac{1}{32}(\theta\gamma_p)_\alpha(\theta\gamma^{mnp}\theta)F_{mn} + \frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\partial_n\chi\gamma_p\theta)\\ &+\frac{1}{1152}(\gamma_m\theta)_\alpha(\theta\gamma^{mrs}\theta)(\theta\gamma_s{}^{pq}\theta)\partial_r F_{pq} + \ldots \end{aligned} \tag{48}$$

- It turns out that one also finds the ghost, antifields and ghost antifield of 10D super-Maxwell in its BV description at ghost numbers 0, 2 and 3, respectively.
- Therefore, the 10D pure spinor superparticle describes 10D super Maxwell in a manifestly super-Poincaré covariant way.
- In order to construct a well-defined measure in pure spinor superspace and negative ghost number operators, one needs to introduce the so-called non-minimal pure spinor variables [Berkovits '05] consisting of two pairs of conjugate variables $(\bar{\lambda}_\alpha, \bar{w}^\alpha)$, (r_α, s^α), where $\bar{\lambda}_\alpha$ is a pure spinor and r_α is a fermionic spinor satisfying $(\bar{\lambda}\gamma^m r) = 0$.
- The non-minimal pure spinor superparticle is then described by the action

$$S = \int d\tau \left[P_m \dot{X}^m + p_\alpha \dot{\theta}^\alpha + w_\alpha \dot{\lambda}^\alpha - \frac{1}{2}P^2 + \bar{w}^\alpha \dot{\bar{\lambda}}_\alpha + s^\alpha \dot{r}_\alpha \right] \tag{49}$$

and the BRST operator $Q = \lambda^\alpha d_\alpha + \bar{w}^\alpha r_\alpha$, so that the BRST cohomology does not change.

- In this extended superspace, one can then define the non-degenerate measure $[dZ] = d^{10}x\, d^{16}\theta\, [d\lambda]\, [d\bar{\lambda}]\, [dr]\, \mathcal{N}$, where

$$\begin{aligned} [d\lambda]\, \lambda^{\beta}\lambda^{\delta}\lambda^{\gamma} &= (\epsilon T^{-1})^{\beta\delta\gamma}_{\alpha_1\ldots\alpha_{11}} d\lambda^{\alpha_1}\ldots d\lambda^{\alpha_{11}} \\ \left[d\bar{\lambda}\right] \bar{\lambda}_{\beta}\bar{\lambda}_{\delta}\bar{\lambda}_{\gamma} &= (\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\delta\gamma} d\bar{\lambda}_{\alpha_1}\ldots d\bar{\lambda}_{\alpha_{11}} \\ [dr] &= (\epsilon T^{-1})^{\beta\delta\gamma}_{\alpha_1\ldots\alpha_{11}} \bar{\lambda}_{\beta}\bar{\lambda}_{\delta}\bar{\lambda}_{\gamma}(\frac{\partial}{\partial r_{\alpha_1}})\ldots(\frac{\partial}{\partial r_{\alpha_{11}}}) \end{aligned} \quad (50)$$

  where the tensors $(\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\delta\gamma}$, $(\epsilon T^{-1})^{\beta\delta\gamma}_{\alpha_1\ldots\alpha_{11}}$ are symmetric and gamma-traceless in (β,δ,γ), and antisymmetric in $(\alpha_1,\ldots\alpha_{11})$.
- $\mathcal{N} = e^{-\{Q,\bar{\lambda}\theta\}}$ is a regularization factor preventing the appearance of undetermined expressions coming from the zero mode integrations of non-compact bosonic and fermionic variables.
- Due to the existence of the operator $\xi = \frac{\bar{\lambda}\theta}{\lambda\bar{\lambda}+r\theta}$, integrands will be restricted to diverge slower than $\lambda^{-8}\bar{\lambda}^{-11}$.

- Using this measure and the antifield symmetry exhibited by the pure spinor superfield Ψ, one can define a pure spinor antibracket as follows [Cederwall '09]

$$(A, B) = \int \frac{\delta_R A}{\delta \Psi(Z)} [dZ] \frac{\delta_L B}{\delta \Psi(Z)} \tag{51}$$

- A pure spinor master action is then defined by

$$(S, S) = 0 \tag{52}$$

- It is not hard to see that the action

$$S = \int [dZ] (\Psi Q \Psi) \tag{53}$$

reproduces the equations of motion and gauge transformations of 10D super-Maxwell.

- The non-abelian generalization of (53) can be shown to be

$$S = \int [dZ]\, Tr \left[\frac{1}{2}\Psi Q \Psi + \frac{g}{3}\Psi\Psi\Psi \right] \tag{54}$$

- The equations of motion and gauge transformations obtained from (54) read

$$Q\Psi + g\Psi\Psi = 0 \quad , \quad \delta\Psi = Q\Lambda + g[\Psi, \Lambda] \tag{55}$$

  where Λ is an arbitrary gauge parameter.
- These equations are nothing but the equations of motion describing 10D super-Yang-Mills studied in the first part of this talk.

The b-ghost

- The use of non-minimal variables allows us to construct the so-called b-ghost satisfying the property $\{Q, b\} = \frac{P^2}{2}$.
- Explicitly, this operator reads [Berkovits '05]

$$b = \frac{(\bar{\lambda}\gamma^m d)}{2(\lambda\bar{\lambda})}P_m + \frac{(\bar{\lambda}\gamma^{mnp} r)[-(d\gamma_{mnp}d) + 24N_{mn}P_p]}{192(\lambda\bar{\lambda})^2} - \frac{(r\gamma^{mnp} r)(\bar{\lambda}\gamma_m d)N_{np}}{16(\lambda\bar{\lambda})^3} - \frac{(r\gamma^{mnp} r)(\bar{\lambda}\gamma^{pqr} r)N_{mn}N_{qr}}{128(\lambda\bar{\lambda})^4} \quad (56)$$

- This expression can be rewritten in the more compact way

$$b = \frac{1}{2}\left[P^m \mathbf{A}_m - d_\alpha \mathbf{W}^\alpha - \frac{1}{2}N^{mn}\mathbf{F}_{mn}\right] \quad (57)$$

where $\mathbf{A}_m$, $\mathbf{W}^\alpha$, $\mathbf{F}_{mn}$ are negative ghost number operators defined by the 10D super-Maxwell equations of motion [Cederwall,Karlsson '11].

- Explicitly,

$$\begin{aligned}
[Q, \mathbf{A}_\alpha] &= -d_\alpha + (\lambda\gamma^m)_\alpha \mathbf{A}_m \\
\{Q, \mathbf{A}_m\} &= P_m + (\lambda\gamma_m \mathbf{W}) \\
[Q, \mathbf{W}^\alpha] &= \frac{1}{4}(\lambda\gamma^{mn})^\alpha \mathbf{F}_{mn} \\
\{Q, \mathbf{F}_{mn}\} &= -2(\lambda\gamma_{[m}\partial_{n]}\mathbf{W})
\end{aligned} \tag{58}$$

- These equations are solved by

$$\mathbf{A}_\alpha = \frac{1}{4(\lambda\bar{\lambda})}\left[N^{mn}(\gamma_{mn}\bar{\lambda})_\alpha + J\bar{\lambda}_\alpha\right] \tag{59}$$

$$\mathbf{A}_m = \frac{(\bar{\lambda}\gamma_m d)}{2(\lambda\bar{\lambda})} + \frac{(\bar{\lambda}\gamma_{mnp} r)}{8(\lambda\bar{\lambda})^2} N^{np} \tag{60}$$

$$\mathbf{W}^\alpha = \frac{(\gamma^m\bar{\lambda})^\alpha}{2(\lambda\bar{\lambda})}\mathbf{\Delta}_m \tag{61}$$

$$\mathbf{F}_{mn} = -\frac{(r\gamma_{mn}\mathbf{W})}{2(\lambda\bar{\lambda})} \tag{62}$$

where $\mathbf{\Delta}_m = -P_m + \frac{(r\gamma_m d)}{2(\lambda\bar{\lambda})} + \frac{(r\gamma_{mnp} r)}{8(\lambda\bar{\lambda})^2} N^{np}$.

- Up to BRST-exact and shift-symmetry terms, these operators reproduce their respective superfields after acting on the on-shell superfield $\Psi^{(1)} = \lambda^\alpha \mathbb{A}_\alpha$. Indeed, one can show that [Berkovits,MG '18;MG,Ben-Shahar '21]

$$\hat{\mathbf{A}}_\alpha \Psi^{(1)} = \mathbb{A}_\alpha + (\lambda\gamma^m)_\alpha \sigma_m \quad (63)$$

$$\hat{\mathbf{A}}_m \Psi^{(1)} = \mathbb{A}_m - (\lambda\gamma^m\rho) + Q\sigma_m \quad (64)$$

$$\hat{\mathbf{W}}^\alpha \Psi^{(1)} = \mathbb{W}^\alpha - Q\rho^\alpha + (\gamma^{mn}\lambda)^\alpha s_{mn} + \lambda^\alpha s \quad (65)$$

$$\hat{\mathbf{F}}_{mn} \Psi^{(1)} = \mathbb{F}_{mn} - 4Qs_{mn} + (\lambda\gamma_{[m} g_{n]}) + (\lambda\gamma_{mn} g) \quad (66)$$

where the superfields σ_m, ρ^α, s_{mn}, s, g_n^α, g, are functions of non-minimal variables and the 10D super-Maxwell superfields.

Berends-Giele Currents

- Scattering amplitudes can be computed after an appropriate gauge-fixing procedure.
- Since Ψ describes fields and antifields, non-standard gauge-fixing mechanisms are needed.
- Using inspiration from string field theory, we will impose the condition $b_0\Psi = Q\Xi$, for some Ξ. In this manner, the gauge-fixed action looks like

$$S = \int [dZ]\, Tr\left[\frac{1}{2}\Psi Q\Psi + \frac{g}{3}\Psi\Psi\Psi - e(b_0\Psi - Q\Xi)\right] \qquad (67)$$

- The equations of motion can then be computed to be

$$Q\Psi + \Psi\Psi - gb_0(e) = 0 \quad , \quad b_0\Psi = Q\Xi \qquad (68)$$

- After applying b_0 on both sides of the first equation and using the nilpotency of the b-ghost [Chandia '10, Jusinskas '13], we find

$$\frac{1}{2}\Box\Psi + b_0(\Psi\Psi) \;=\; 0 \tag{69}$$

- We solve this equation through the perturbiner method. The superfield Ψ is then expanded into n-particle superfields as

$$\Psi = \sum_P \Psi_P T^P e^{k_P\cdot x} = \sum_i \Psi_i T^{a_i} e^{k_i\cdot x} + \sum_{i,j} \Psi_{ij} T^{a_i} T^{a_j} e^{k_{ij}\cdot x} + \ldots \tag{70}$$

where P stands for non-empty words in particle labels, $i = 1, \ldots, n$, $T^P = T^{a_1} \ldots T^{a_i}$, with T^{a_i} the Lie generators, and $k_P = k_1 + \ldots + k_i$.

- After plugging (70) into (69), one finds that

$$QV_i = 0 \;\;,\;\; b_0 V_i = Q\Xi_i \;\;,\;\; \Psi_P = -\frac{1}{s_P}\sum_{QR=P} b_0(\Psi_Q\Psi_R) \tag{71}$$

where $s_P = \frac{1}{2}k_P^2$, and the sum runs over all the possible deconcatenations of P into the non-empty ordered words Q and R.

- The color-ordered amplitudes are then calculated in the usual way

$$A_{1\dots n} \;=\; (-1)^n \sum_{P,Q=(1,\dots,n-1)} \langle \Psi_P \Psi_Q V_n \rangle \tag{72}$$

- The product $\Psi_P \Psi_Q$ is just $\Psi_{1,\dots,n-1}$ with the outer propagator $\frac{b_0}{\Box}$ stripped off, which is the LSZ reduction in pure spinor superspace.
- One can compute the action of b_0 on $V_i = \lambda^\alpha \mathbb{A}_{i\,\alpha}$ to learn that $b_0 V_i = Q\Lambda_i$, where Λ_i is a complicated function depending on non-minimal variables. Therefore, external states can be represented by the superfields $V_i = \lambda^\alpha \mathbb{A}_{i\,\alpha}$.

Examples

- Let us illustrate these ideas with a few examples. The 3-point function can easily be computed from (72) as

$$A_3 = \langle V_1 V_2 V_3 \rangle \tag{73}$$

which matches the pure spinor prescription of open superstrings for 3 massless external states.

- The 4-point function requires a bit more of work. The prescription (72) tells us that

$$A_4 = \frac{1}{s_{12}}\langle b_0(V_1 V_2) V_3 V_4 \rangle + \frac{1}{s_{23}}\langle V_1 b_0(V_2 V_3) V_4 \rangle \tag{74}$$

- Using eqn. (57) one can show that

$$b_0(V_1 V_2) = V_{12} + Q\Lambda_{12} \tag{75}$$

where Λ_{12} depends on non-minimal variables, $V_{12} = \lambda^\alpha \mathbb{A}_{12\,\alpha}$, and $\mathbb{A}_{12\,\alpha} = \frac{1}{2}\Big[(k_2 \cdot \mathbb{A}_1)\mathbb{A}_{2\,\alpha} - (k_1 \cdot \mathbb{A}_2)\mathbb{A}_{1\,\alpha} + (\gamma^p \mathbb{W}_1)_\alpha \mathbb{A}_{2\,p} - (\gamma^p \mathbb{W}_2)_\alpha \mathbb{A}_{1\,p}\Big]$ is the 2-particle superfield [Mafra, Schlotterer '14].

- In this manner the 4-point amplitude takes the form

$$A_4 = \langle M_{12} V_3 V_4 \rangle + \langle V_1 M_{23} V_4 \rangle \tag{76}$$

where $M_{ij} = \frac{V_{ij}}{s_{ij}}$, which matches the expression obtained from CFT techniques in the open pure spinor superstring.

- Analogously, the 5-point function can be determined as follows

$$\begin{aligned} A_5 = & -\frac{\langle V_1 b_0(V_2 b_0(V_3 V_4)) V_5 \rangle}{s_{234} s_{34}} - \frac{\langle V_1 b_0(b_0(V_2 V_3) V_4) V_5 \rangle}{s_{234} s_{23}} \\ & -\frac{\langle b_0(V_1 V_2) b_0(V_3 v_4) V_5 \rangle}{s_{12} s_{34}} - \frac{\langle b_0(V_1 b_0(V_2 V_3)) V_4 V_5 \rangle}{s_{123} s_{23}} \\ & -\frac{\langle b_0(b_0(V_1 V_2) V_3) V_4 V_5 \rangle}{s_{123} s_{12}} \end{aligned} \tag{77}$$

- After careful algebraic manipulations, one can show that

$$b_0(b_0(V_i V_j) V_k) = V_{ijk} + s_{ij} T_{ijk} + s_{ij} \Lambda_{ij} V_k + Q\Lambda_{[ij]k} \tag{78}$$

where $V_{ijk} = \lambda^\alpha \mathbb{A}_{ijk\,\alpha}$, $\mathbb{A}_{ijk\,\alpha}$ is the 3-particle superfield [Mafra, Schlotterer '14], and $T_{ijk} = \Lambda_i V_j V_k - \Lambda_{ij} V_k + (cyclic)$.

- After plugging (78) into (77), one learns that

$$A_5 = \langle M_{123} V_4 V_5 \rangle + \langle M_{12} M_{34} V_5 \rangle + \langle V_1 M_{234} V_5 \rangle \qquad (79)$$

where $M_{ijk} = \frac{1}{s_{ijk}} \left(\frac{V_{ijk}}{s_{ij}} - \frac{V_{jki}}{s_{jk}} \right)$. Notice the exact cancellations of the terms T_{ijk} and $\Lambda_{ij} V_k$ as a consequence of their symmetry properties. This expression coincides with that found from the pure spinor worldsheet approach.

- Using similar arguments, one can show that the N-point amplitude calculated via the formula (72) matches that obtained from the open pure spinor superstring [Mafra, Schlotterer, Stieberger '11].
- In particular, this implies that the kinematic numerators described by the b-ghost nested expressions match those obtained from string theory up to generalized gauge transformations, that is contact terms which decouple from the physical amplitude.

Siegel gauge and Color-Kinematics Duality

- One particular gauge choice in string field theory is the so-called Siegel gauge: $b_0\Psi = 0$ [Siegel '85].
- The pure spinor propagator can be rewritten in the more convenient way

$$b_0 = -\boldsymbol{\Delta}^m \mathbf{A}_m \tag{80}$$

- Eqn. (80) can then be used to show that

$$\begin{aligned} b_0(V_1 V_2) &= (b_0 V_1)V_2 - V_1(b_0 V_2) + \hat{\boldsymbol{\Delta}}^m V_1 \hat{\mathbf{A}}_m V_2 - \hat{\mathbf{A}}_m V_1 \hat{\boldsymbol{\Delta}}^m V_2 \\ b_0(V_1 V_2 V_3) &= b_0(V_1 V_2)V_3 + b_0(V_2 V_3)V_1 + b_0(V_3 V_1)V_2 \\ &\quad -(b_0 V_1)V_2 V_3 + V_1(b_0 V_2)V_3 - V_1 V_2 b_0(V_3) \end{aligned} \tag{81}$$

- Therefore, if external states are in the Siegel gauge, one finds that

$$\begin{aligned} b_0(\mathcal{V}_1\mathcal{V}_2) &= \hat{\mathbf{\Delta}}^m\mathcal{V}_1\hat{\mathbf{A}}_m\mathcal{V}_2 - \hat{\mathbf{A}}_m\mathcal{V}_1\hat{\mathbf{\Delta}}^m\mathcal{V}_2 \\ b_0(\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3) &= b_0(\mathcal{V}_1\mathcal{V}_2)\mathcal{V}_3 + b_0(\mathcal{V}_2\mathcal{V}_3)\mathcal{V}_1 + b_0(\mathcal{V}_3\mathcal{V}_1)\mathcal{V}_2 \end{aligned} \quad (82)$$

- The first equation suggest the interpretation of a Poisson bracket structure for the b-ghost, $b_0(\mathcal{V}_1\mathcal{V}_2) = \{\mathcal{V}_1, \mathcal{V}_2\}$, where

$$\{X, Y\} = \hat{\mathbf{\Delta}}^m X\hat{\mathbf{A}}_m Y - \hat{\mathbf{A}}_m X\hat{\mathbf{\Delta}}^m Y \quad (83)$$

- After applying b_0 on both sides of the second equation in (82), one concludes that

$$b_0(b_0(\mathcal{V}_1\mathcal{V}_2)\mathcal{V}_3) + b_0(b_0(\mathcal{V}_2\mathcal{V}_3)\mathcal{V}_1) + b_0(b_0(\mathcal{V}_3\mathcal{V}_1)\mathcal{V}_2) = 0 \quad (84)$$

Note the role of cyclicity in this identity.

- Eqn.(84) can be written in terms of the Poisson bracket (83) as

$$\{\{\mathcal{V}_1, \mathcal{V}_2\}, \mathcal{V}_3\} + \{\{\mathcal{V}_2, \mathcal{V}_3\}, \mathcal{V}_1\} + \{\{\mathcal{V}_3, \mathcal{V}_1\}, \mathcal{V}_2\} = 0 \quad (85)$$

  which is nothing but the Jacobi identity of the algebra (83).
- In this manner, Siegel gauge external states will automatically obey color-kinematics duality.
- In addition, Siegel gauge numerators satisfies crossing-symmetry as a consequence of the commutation of b_0 with the regularization factor $\mathcal{N}$.
- Hence, the Siegel gauge is special in that it allows us to manifest both color-kinematics duality and crossing symmetry at the level of individual numerators.

Conclusions

- 10D super-Yang-Mills possesses a Chern-Simons-like description in pure spinor superspace.
- In this pure spinor field theory approach, the b-ghost plays an importat role for computing kinematic numerators and scattering amplitudes.
- The gauge $b_0\Psi = Q\Xi$ allows us to use the standard on-shell superfields $V = \lambda^\alpha \mathbb{A}_\alpha$, and show the perturbiner method describes the same amplitudes obtained from the open pure spinor superstring in the field theory limit.
- The nested b-ghost kinematic numerators are related to those obtained from CFT techniques by generalized gauge transformations.
- The Siegel gauge $b_0\mathcal{V} = 0$ naturally realizes both color-kinematics duality and crossing symmetry.