Double-copy and unitarity of tree-level string amplitudes

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based on hep-th/2201.11575 with

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Outline

I Double-copy-like factorization on all resonances in string theory

$$A_n^{\text{open}} \sim \frac{c_\Gamma \, n_{\Gamma,\vec{m}}}{\prod_{e\in\Gamma} (P_e^2 - m_e^2)}$$

integers $\vec{m}^2 \in \frac{1}{\alpha'}\mathbb{N}^{n-3}$

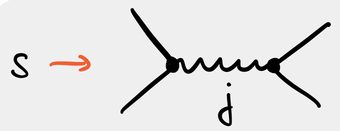
$$A_n^{\text{closed}} \sim \frac{n_{\Gamma,\vec{m}} \, \tilde{n}_{\Gamma,\vec{m}}}{\prod_{e\in\Gamma} (P_e^2 - m_e^2)}$$

cubic diagrams

II Unitarity constraints for 4-point amplitudes

coefficient $\geqslant 0$

partial waves

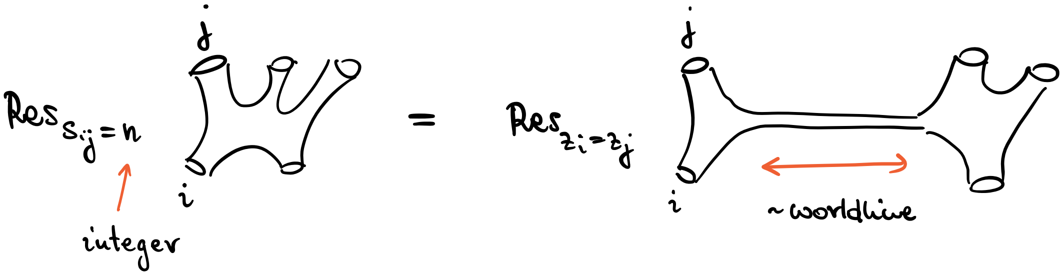


$$\sim \frac{c^2 \, G_j(\cos\theta)}{s - m^2}$$

mass level

III On-shell proof of unitarity in $D \leqslant 6$

I Unitarity cuts for tree-level strings



Computations in string perturbation theory are done on Euclidean worldsheets :

$$A_n^{\text{closed}} = \int_{\mathcal{M}_{g,n}} \langle V_1 V_2 \cdots V_n \rangle , \qquad V_i(z,\bar{z}) = f(z) f(\bar{z})\, e^{i p_i \cdot X}$$

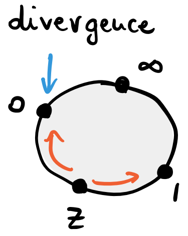
$$A_n^{\text{open}} = \int_{\Gamma \subset \mathcal{M}_{g,n}} \langle V_1 V_2 \cdots V_n \rangle , \qquad V_i(z) = f(z)\, e^{i p_i \cdot X}$$

But many physical properties, such as unitarity, are intrinsically Lorentzian ...

Price to pay : we don't know how to define the above integrals !

The simplest example is the Veneziano amplitude:

$$A_4^{\text{open}}(s,t) = \int_0^1 dz\; z^{-s-1}\,(1-z)^{-t-1}$$



mostly-minus signature

$s>0,\ t<0,\ \alpha'=1$

The s-channel poles come from $z \approx 0$, so let's set $z = e^{-\tau}$ and study the $\tau \to \infty$ behavior:

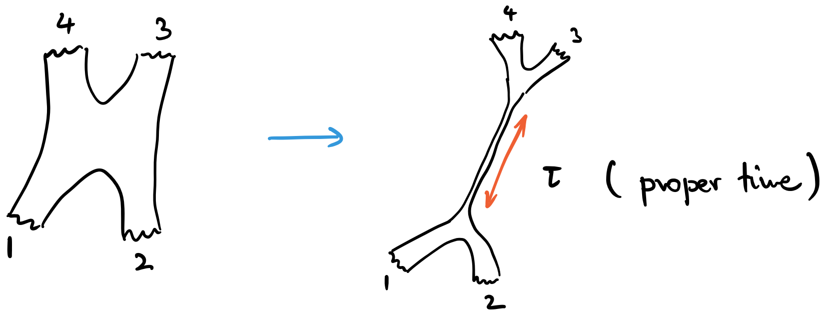
$$A_4^{\text{open}}(s,t) = \int_0^\infty d\tau\; e^{\tau s}\left(1 + \# e^{-\tau} + \# e^{-2\tau} + \dots\right)$$

massless $-\frac{1}{s}$

level-1 $-\frac{\#}{s-1}$

level-2 $-\frac{\#}{s-2}$

This is precisely the worldline limit:



Importantly, we land on Euclidean worldlines,

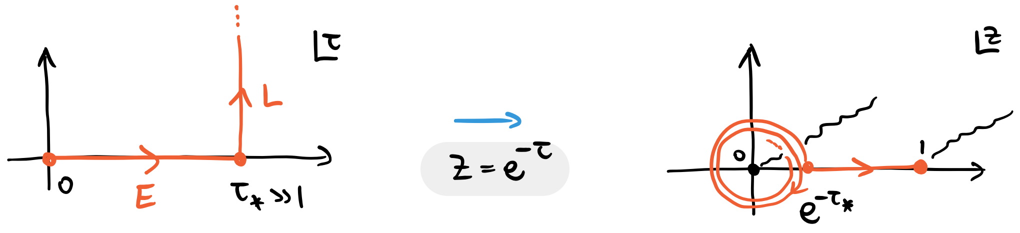
$$\frac{-1}{s-m^2} = \int_0^\infty d\tau_E \, e^{-\tau_E(s-m^2)} ,$$

as opposed to Lorentzian

$$\frac{i}{s-m^2} = \lim_{\epsilon\to 0^+} \int_0^\infty d\tau_L \, e^{i\tau_L(s-m^2+i\epsilon)}$$

The two are related by the Wick rotation, $\tau_L = i\,\tau_E$

Integrating over Lorentzian worldsheets locally close to divergences thus amounts to



$\underbrace{}$ wind an infinite number of times

[Witten '13]

We can resum this infinite tower of circles

$$\odot + e^{-2\pi i s}\odot + e^{-4\pi i s}\odot + \dots$$

$$= \frac{1}{1-e^{-2\pi i s}}\odot$$

↑ string resonances at $s \in \mathbb{Z}$.

Hence the correct contour looks like

$$A_4^{\text{open}}(s,t) = \frac{1}{e^{-2\pi i s}-1} \oint_{z=0} dz\, z^{-s-1}(1-z)^{-t-1} + \int_{\epsilon}^{1-\epsilon} \cdots - \frac{1}{e^{-2\pi i t}-1} \oint_{z=1} \cdots$$

All the singularities are now manifest and we can take cuts:

$$\text{Res}_{s=n} A_4^{\text{open}} = -\text{Res}_{z=0} \left(z^{-n-1} (1-z)^{-t-1} \right).$$

At the residue, closed-string amplitudes are holomorphically-factorized, so:

$$\text{Res}_{s=n} A_4^{\text{closed}} = \oint_{z=0} (\dots) \oint_{\bar{z}=0} (\dots)$$

$$= \left(\text{Res}_{s=n} A_4^{\text{open}} \right)^2 .$$

Such contours easily generalize to higher multiplicity, giving [SM '17]

Res (...)

$$A_n^{\text{open}} \sim \frac{c_\Gamma \, n_{\Gamma,\vec{m}}}{\prod_{e\in\Gamma} (P_e^2 - m_e^2)}$$

integers $\vec{m}^2 \in \frac{1}{\alpha'}\mathbb{N}^{n-3}$

cubic graphs

$$A_n^{\text{closed}} \sim \frac{n_{\Gamma,\vec{m}} \, \tilde{n}_{\Gamma,\vec{m}}}{\prod_{e\in\Gamma} (P_e^2 - m_e^2)}$$

→ Note that the numerators do not have to satisfy kinematic Jacobi relations.

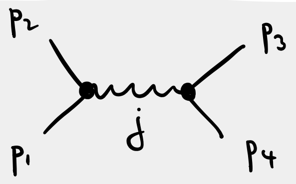
→ Residue theorem guarantees that they do so on massless poles. [SM '20]

How can we use these facts to make statements about unitarity?

II Tree-level unitarity

In principle guaranteed by the no-ghost theorem, but we'd like to prove it on shell (progress in 1970's)

We start with the simplest case of external scalars:



$$= \frac{1}{s-m^2}\;\underset{\substack{\uparrow\\ \text{Coupling}\\ \text{to spin-}j\\ \text{particle}}}{c^2}\;\underset{\substack{\uparrow\\ \text{D-dim}\\ \text{Gegenbauer polyn.}}}{G_j^D}\big(\overset{\substack{\text{scattering angle}\\ \downarrow}}{\cos\theta}\big)$$

Unitarity: $c^2 \geqslant 0$

Hence for every $n \in \mathbb{Z}$ we need to check

$$\operatorname{Res}_{s=n} A_4^{\text{open}}(s,t) = \sum_{j=0}^{\infty} \underset{\substack{\rotatebox{-90}{$\geqslant$}\\ 0}}{B_{n,j}^D}\, G_j^D(\cos\theta)$$

→ Let's verify it directly (bosonic):

$$\operatorname{Res}_{s=n} - \frac{\Gamma(-s-1)\,\Gamma(-t-1)}{\Gamma(-s-t-2)} = \#\,(t+2)(t+3)\cdots(t+n+2) \quad \Big\} \quad \begin{aligned} x &= \cos\theta \\ &= 1+\frac{2t}{n+4}\end{aligned}$$

$$= \#\left(x-\frac{n}{n+4}\right)\left(x-\frac{n-2}{n+4}\right)\cdots\left(x+\frac{n}{n+4}\right)$$

Gegenbauer poly^s

$$\left(G_0^D,\ G_1^D,\ G_2^D,\ \dots\right) = \left(1,\ x,\ x^2 - \frac{1}{D-1},\ \dots\right)$$

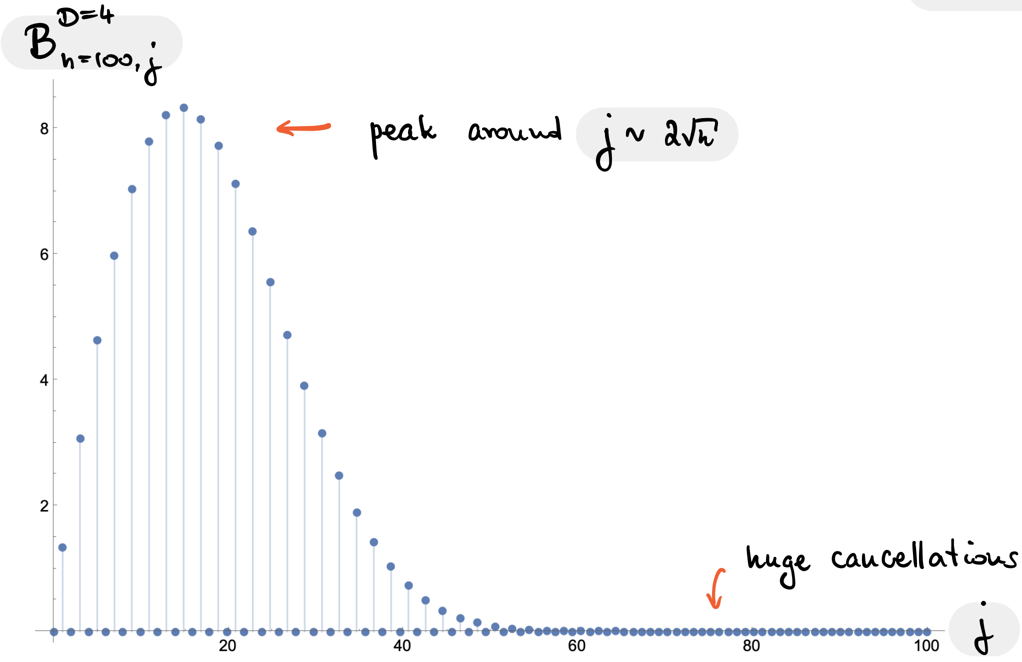
Low levels:

$n = -1$: $1 = G_0^D(x)$

$n = 0$: $x = G_1^D(x)$

$n = 1$: $(x - 1/5)(x + 1/5) = G_2^D(x) + \underbrace{\left(\frac{1}{D-1} - \frac{1}{25}\right)}_{\geq 0 \text{ provided that } D \leqslant 26} G_0^D(x)$

⋮



peak around $j \sim 2\sqrt{n}$

huge cancellations

even $n+j$ coefficients vanish

Seems extremely difficult to prove positivity in general!

→ Slightly more involved in the case of superstrings

$$A_4^{\text{open, super}}(s,t) = \mathcal{F}^4 \, \frac{\Gamma(-s)\,\Gamma(-t)}{\Gamma(1-s-t)}$$

where $\mathcal{F}^4 = \left(F_{\mu\nu} F^{\mu\nu} \right)^2 - 4\, F_{\mu\nu} F^{\nu\alpha} F_{\alpha\beta} F^{\beta\mu}$

$$F_{\mu\nu} = \sum_{i=1}^{4} \left(p_{i,\mu}\, \varepsilon_{i,\nu} - p_{i,\nu}\, \varepsilon_{i,\mu} \right).$$

At the first massive level

$$\mathcal{F}^4\Big|_{s=1} = \underbrace{\left(\frac{9}{D-1} - 1 \right)}_{\geq 0 \text{ implies } D \leq 10} G^{D,\bullet}_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}(x) + G^{D,\Box\Box}_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}(x) + G^{D,\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}}_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}(x)$$

Spinning Gegenbauers for $SO(D-1)$ irreps

Cf. $|\Box\Box\rangle = \varepsilon_{\mu\nu}\, \partial X^\mu \psi^\nu e^{ip\cdot X}$

$|\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}\rangle = \varepsilon_{\mu\nu\lambda}\, \psi^\mu \psi^\nu \psi^\lambda e^{ip\cdot X}$

$|\bullet\rangle$ from compactifications

for the experts

Long story short, positivity of the scalar Gegenbauer expansion would guarantee unitarity if $D \leqslant 10$.

→ Finally, the closed (super) string amplitudes

$$A_4^{\text{closed, super}}(s,t) = \mathcal{R}^4 \, \frac{\Gamma(-s)\,\Gamma(-t)\,\Gamma(-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

$$\mathcal{R}^4 = (\mathcal{F}^4)^2$$

double-copy on each massive residue :

$$\operatorname{Res}_{s=n} A_4^{\text{closed}}(s,t) = \left(\operatorname{Res}_{s=n} A_4^{\text{open}}(s,t)\right)^2$$

$$= \left(\sum_{j=0}^{\infty} B_{n,j}^{D} \; G_j^{D}(x)\right)^2$$

But $G_j^D(x)\, G_{j'}^D(x) =$ positive sum of $G_{j''}^D(x)$

Less constraining than open-string unitarity!

Virasoro-Shapiro amplitude seems unitary up to $D \leqslant 72$

III Proof of unitarity for $D \leq 6$

We first bring the partial-wave coefficients to the double-contour form (superstring):

$$B^D_{j,n} = \# \overbrace{\oint_{\tau=0} \frac{d\tau}{2\pi i}}^{\text{unitarity cut}} \oint_{v=0} \frac{dv}{2\pi i} \frac{(v-\tau)^j}{(\tau v)^{j+\frac{D-2}{2}} (e^v - e^\tau)^n} .$$

$\uparrow$

$\sim$ worldline length $1-z = e^\tau$

for $n+j$ odd

The physical origin of the $\tau \leftrightarrow v$ symmetry is not understood

We can view the integrand as a generating function. Change of variables to $\tau = \log(1-x)$, $v = \log(1-y)$:

$$B^D_{n,j} = \# \oint_{x=0} \frac{dx}{2\pi i} \oint_{y=0} \frac{dy}{2\pi i} \frac{1}{(1-x)(1-y)[\log(1-x)\log(1-y)]^{\frac{D-2}{2}}} \times \left[\frac{\frac{1}{\log(1-x)} - \frac{1}{\log(1-y)}}{x-y} \right]^j \times \frac{1}{(x-y)^{n-j}} .$$

Each of the three factors has a positive Taylor expansion in $D \leq 6$:

$$\rightarrow \quad \frac{1}{(1-x)\left[-\log(1-x)\right]^{\frac{D-2}{2}}} = x^{-\frac{D}{2}}\left(x + \frac{\overbrace{6-D}^{D\leq 6}}{4}x^2 + \dots\right)$$

$$\rightarrow \quad \frac{\frac{1}{\log(1-x)} - \frac{1}{\log(1-y)}}{x-y} = \frac{1}{xy} + \sum_{m=0}^{\infty} \overset{\overset{0}{\nearrow}}{C_m} \sum_{k=0}^{m-1} x^{m-1-k}\, y^{k}$$

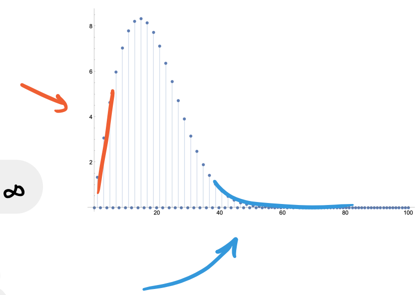
$$\rightarrow \quad \frac{1}{(x-y)^{n-j}} = \sum_{m=0}^{\infty} \binom{n-j+m-1}{m} y^{m}\, x^{-n+j-m}$$

In bosonic string, positivity is manifest for $D \leq 10$

Finally, we can use the double-contour representation to study different limits:

→ Fixed spin: $j = \text{fixed}$, $n \to \infty$

$$B^{D}_{n,j} \sim \frac{1}{n^{D/2}(\log n)^{\frac{D-2}{2}}}$$



→ Regge limit: $\Delta = n-j$ fixed, $n \to \infty$

$$B^{D}_{n,j} \sim \left(\frac{e}{4}\right)^{n} n^{\frac{\Delta-1}{2}}.$$

Thanks!