THE NEWMAN-PENROSE MAP AND THE CLASSICAL DOUBLE COPY

Based on arXiv: 2006.08630, 2104.09525, 2205.xxxxx
with Gilly Elor, Michael Graesser and Gabriel Herczeg

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DOUBLE COPY

$$\text{gravity} = (\text{gauge})^2$$

- **simplifies calculations, only proven at tree level**
- **how general is it? exact solutions?**

Classical Double Copy

- **map classical gravity sol'ns to classical gauge sol'ns**
- **start with linear solutions**
  - **= Maxwell (gauge), Kerr-Schild (gravity)**

KERR-SCHILD DOUBLE COPY

- **Kerr-Schild spacetimes** $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} = \phi k_\mu k_\nu$$
- k_μ **null and geodesic** $$\eta^{\mu\nu} k_\mu k_\nu = g^{\mu\nu} k_\mu k_\nu = 0 \qquad k^\nu \partial_\nu k_\mu = k^\nu \nabla_\nu k_\mu = 0$$
- **mixed Ricci tensor is linear in** $h_{\mu\nu}$ $$R^\mu{}_\nu = \frac{1}{2}\left[\partial^\mu \partial_\alpha (h^\alpha{}_\nu) + \partial_\nu \partial^\alpha (h^\mu{}_\alpha) - \partial^2 (h^\mu{}_\nu)\right]$$

KERR-SCHILD DOUBLE COPY

- **Stationary Kerr-Schild spacetimes** $\partial_0 k_\mu = \partial_0 \phi = 0$ $k_0 = 1$

$$R^i{}_0 = \frac{1}{2}\partial_j \left[\partial^i(\phi k^j) - \partial^j(\phi k^i)\right] \qquad R^0{}_0 = \frac{1}{2}\partial_i^2 \phi$$

Double Copy		**Single Copy**		**Zeroth Copy**
$h_{\mu\nu} = \phi k_\mu k_\nu$	$k_\mu \to c^a$ →	$A^a_\mu \equiv c^a \phi k_\mu$	$k_\mu \to c^{a'}$ →	$\phi^{aa'} = c^a c^{a'} \phi$
$R_{\mu\nu} = 0$		$\partial_\mu F^{a\,\mu\nu} = 0$		$\Box\, \phi^{aa'} = 0$

Monteiro, O'Connell, White, 1410.0239

KERR-SCHILD DOUBLE COPY

- **Schwarzschild in Kerr-Schild form**

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \qquad \phi = \frac{2GM}{r} \qquad k_\mu dx^\mu = dt + dr$$

$$A_\mu dx^\mu = \phi k_\mu dx^\mu = \frac{2GM}{r}\left(dt + dr\right)$$

$$M \to Q$$

$$2G \to \frac{1}{4\pi\epsilon_0}$$

$$A_\mu dx^\mu = \frac{Q}{4\pi\epsilon_0 r}(dt + dr)$$

point charge!

Monteiro, O'Connell, White, 1410.0239

KERR-SCHILD DOUBLE COPY

Gravity/Double Copy	Gauge/Single Copy
Schwarzschild black hole	Point charge *(1410.0239)*
Kerr black hole	Rotating disk of charge *(1410.0239)*
Plane waves	Plane waves *(1410.0239)*
Kerr-Taub-NUT	Rotating dyon/Wu-Yang monopole *(1507.01869, 2001.09918)*
Photon Rocket	Lienard-Wiechert potential *(1603.05737)*
(A)dS backgrounds	(A)dS backgrounds *(1710.01953, 1711.01296)*
BTZ	Constant charge density *(1711.01296, 1904.11001)*
⋮	⋮

- Only guaranteed to solve vacuum Maxwell equations if **vacuum**, **stationary** (other examples)

SELF-DUAL DOUBLE COPY

- **Kerr-Schild metric with differential operator $\hat{k}_\mu$** $\qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} = \hat{k}_\mu \hat{k}_\nu \varphi$
- **$\hat{k}_\mu$ 'null' and 'geodesic'** $\qquad \eta^{\mu\nu}\hat{k}_\mu \hat{k}_\nu = 0 \qquad \hat{k} \cdot \partial = 0$

Double Copy

$$h_{\mu\nu} = \hat{k}_\mu \hat{k}_\nu \phi$$

$$R_{\mu\nu} = 0 \iff \partial^2\varphi + (\hat{k}^\mu \hat{k}^\nu \varphi)(\partial_\mu \partial_\nu \varphi) = 0$$

$$R_{\mu\nu\tau\lambda} = \pm\, i\epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\tau\lambda}$$

$\Longrightarrow$

Single Copy

$$A_\mu \equiv \hat{k}_\mu \phi$$

$$\partial_\mu F^{\mu\nu} = 0 \iff \partial^2\varphi = 0$$

$$F_{\mu\nu} = \pm\, i\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

Monteiro, O'Connell, White, 1410.0239, Berman, Chacón, Luna, White, 1809.04063

NEWMAN-PENROSE FORMALISM

- **Null (rather than orthonormal) tetrad**
  - ℓ, n **real,** $m, \bar{m}$ **complex conjugates**

$$g_{\mu\nu} = -\ell_\mu n_\nu - n_\mu \ell_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu$$

- **makes symmetries manifest (align one direction with KS vector)**
- **easy to write in spinor formalism (connections with twistor space)**
- **only in 4d**

NEWMAN-PENROSE + KERR-SCHILD

- **Kerr-Schild metric** $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $h_{\mu\nu} = V\ell_\mu \ell_\nu$
- ℓ_μ **null, geodesic, shear-free, expanding**
  - implied already if vacuum
  - fixes form of metric up to two functions
  - excludes Petrov type N (pp waves)
  - includes every other example so far
    - Schwarzschild, Kerr, Taub-NUT, photon rocket, ….

NEWMAN-PENROSE + KERR-SCHILD

- **Kerr-Schild metric in lightcone coordinates**

$$u = \frac{1}{\sqrt{2}}(t - z) \qquad \zeta = \frac{1}{\sqrt{2}}(x + iy)$$

$$v = \frac{1}{\sqrt{2}}(t + z) \qquad \bar{\zeta} = \frac{1}{\sqrt{2}}(x - iy)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + V\ell_\mu \ell_\nu$$

$$n_\mu dx^\mu = dv + \frac{1}{2} V \ell_\mu dx^\mu$$

$$\ell_\mu dx^\mu = du + \bar{\Phi} d\zeta + \Phi d\bar{\zeta} + \Phi\bar{\Phi} dv$$

$$-m_\mu dx^\mu = \Phi dv + d\zeta$$

$$-\bar{m}_\mu dx^\mu = \bar{\Phi} dv + d\bar{\zeta}$$

- **only depends on two functions**
  - V, **real**
  - Φ, **complex**

Elor, Farnsworth, Graesser, Herczeg, 2006.08630

NEWMAN-PENROSE + KERR-SCHILD

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$$-m_\mu dx^\mu = \Phi dv + d\zeta$$

$$-\bar{m}_\mu dx^\mu = \bar{\Phi} dv + d\bar{\zeta}$$

$$A_\mu \equiv \hat{k}_\mu \Phi \qquad \hat{k} = -\frac{Q}{2\pi\epsilon_0}(dv\partial_\zeta + d\bar{\zeta}\partial_u)$$

$$\partial^2 \Phi = 0 \quad \Longleftrightarrow \quad \partial_\mu F^{\mu\nu} = 0$$

$$F_{\mu\nu} = \pm\, i\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

Elor, Farnsworth, Graesser, Herczeg, 2006.08630

SCHWARZSCHILD

- **Schwarzschild in Kerr-Schild form**

$$u = \frac{1}{\sqrt{2}}(t - z) \qquad \zeta = \frac{1}{\sqrt{2}}(x + iy)$$

$$v = \frac{1}{\sqrt{2}}(t + z) \qquad \bar{\zeta} = \frac{1}{\sqrt{2}}(x - iy)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \qquad \phi = \frac{2GM}{r}$$

$$k_\mu dx^\mu = dt + dr$$

$$= \frac{1}{2r}\left[(\sqrt{2}r + (v-u))du - 2(\zeta d\bar{\zeta} + \bar{\zeta} d\zeta) + (\sqrt{2}r - (v-u))dv\right]$$

$$\ell_\mu dx^\mu = du + \bar{\Phi} d\zeta + \boxed{\Phi} d\bar{\zeta} + \Phi\bar{\Phi} dv \qquad \Phi = \frac{-2\zeta}{\sqrt{2}r + (v-u)} = \frac{-(x+iy)}{r+z}$$

Elor, Farnsworth, Graesser, Herczeg, 2006.08630

SCHWARZSCHILD

Schwarzschild in Kerr-Schild form

$$u = \frac{1}{\sqrt{2}}(t - z) \qquad \zeta = \frac{1}{\sqrt{2}}(x + iy)$$

$$v = \frac{1}{\sqrt{2}}(t + z) \qquad \bar{\zeta} = \frac{1}{\sqrt{2}}(x - iy)$$

$$\Phi = \frac{-2\bar{\zeta}}{\sqrt{2}r + (v - u)} = \frac{-(x + iy)}{r + z}$$

$$\hat{k} = -\frac{Q}{2\pi\epsilon_0}(dv\partial_\zeta + d\bar{\zeta}\partial_u)$$

$$A_\mu dx^\mu = \hat{k}\Phi = \frac{Q}{2\sqrt{2}\pi\epsilon_0}\left(dv - \Phi d\bar{\zeta}\right)$$

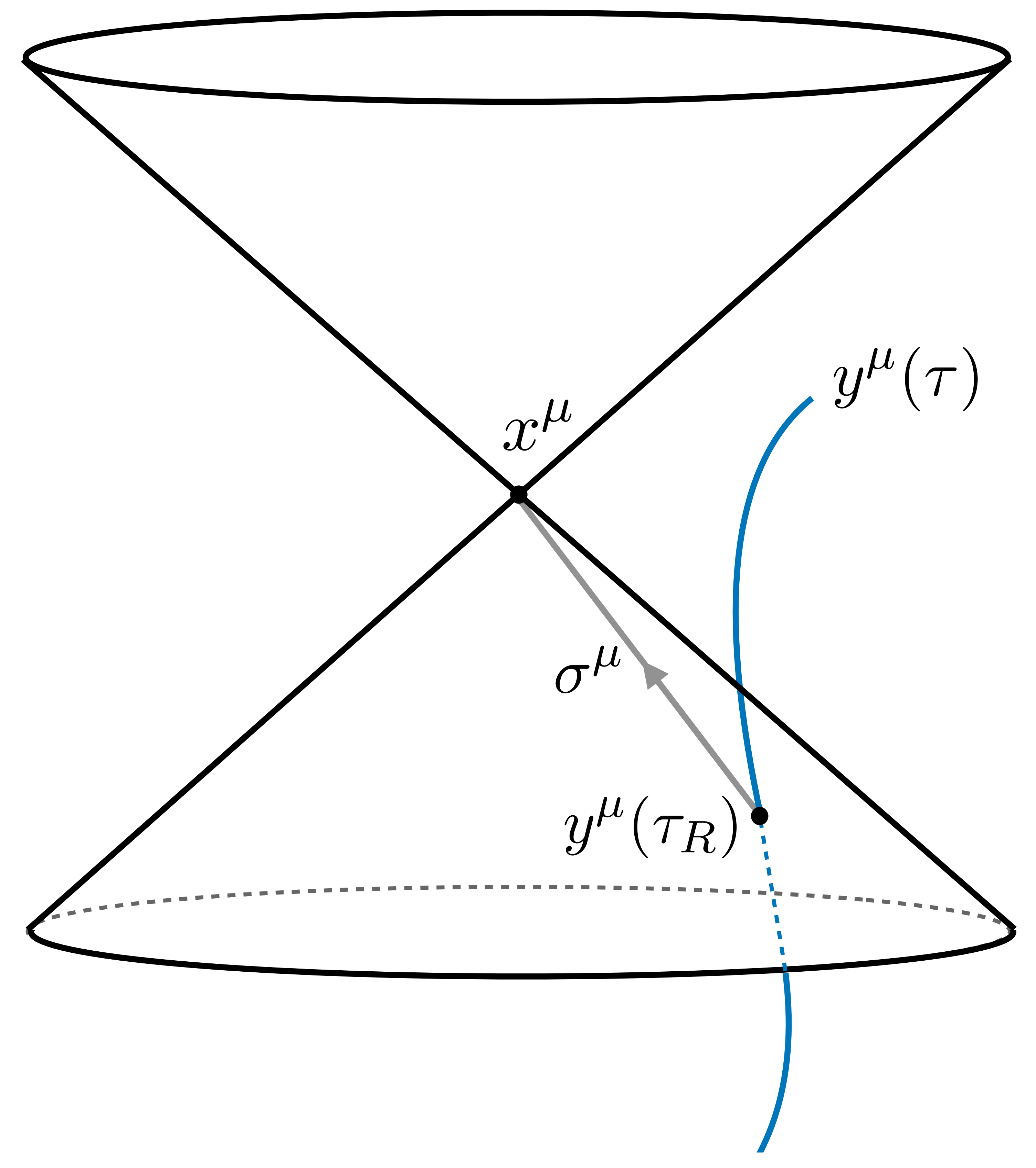
point charge + i(magnetic monopole)

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{dt + dr}{r} - i(1 - \cos\theta)d\phi - \frac{\sin\theta}{1 + \cos\theta}d\theta\right)$$

Elor, Farnsworth, Graesser, Herczeg, 2006.08630

PHOTON ROCKET

- **particle moving along arbitrary timelike worldline**



$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$$

$$\phi = \frac{2GM(\tau_R)}{r} \qquad k_\mu = \frac{\sigma_\mu}{r} \qquad \sigma^\mu = x^\mu - y^\mu(\tau_R)$$

$$r = \sigma \cdot \lambda(\tau_R) \qquad \lambda^\mu = \frac{dy^\mu(\tau)}{d\tau}$$

non-vacuum: $T_{\mu\nu} = f k_\mu k_\nu$

PHOTON ROCKET

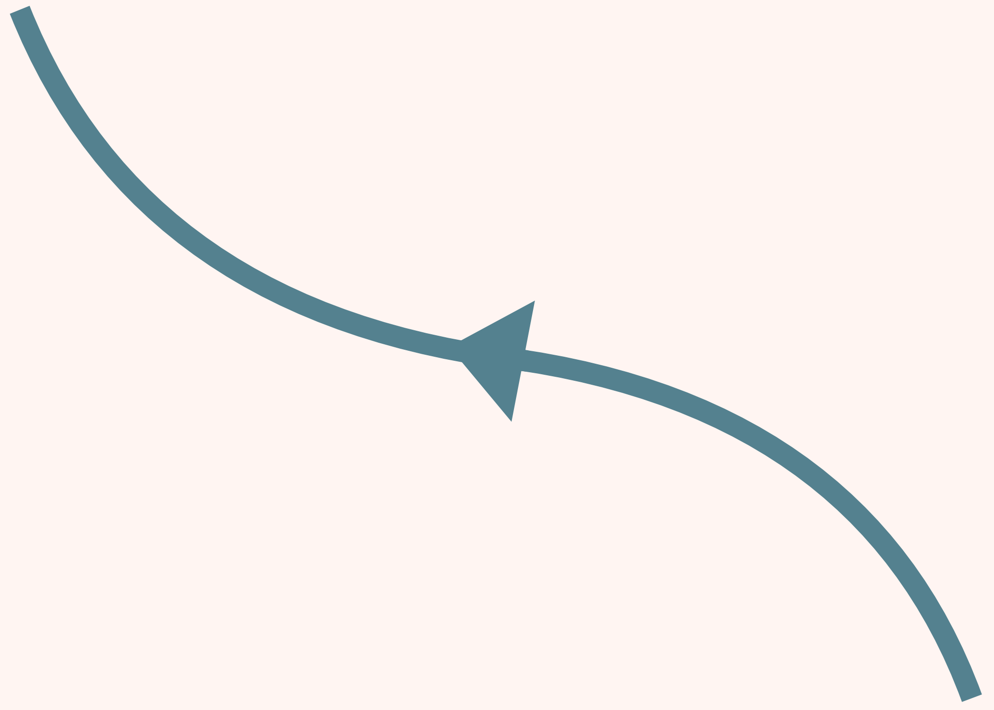
- **Kerr-Schild double copy:**

$$\phi = \frac{2GM(\tau_R)}{r} \qquad k_\mu = \frac{\sigma_\mu}{r}$$

- accelerating point charge
- no radiation
- double copy of source

$$A = \frac{Q}{4\pi\epsilon_0 r^2}\sigma_\mu dx^\mu$$

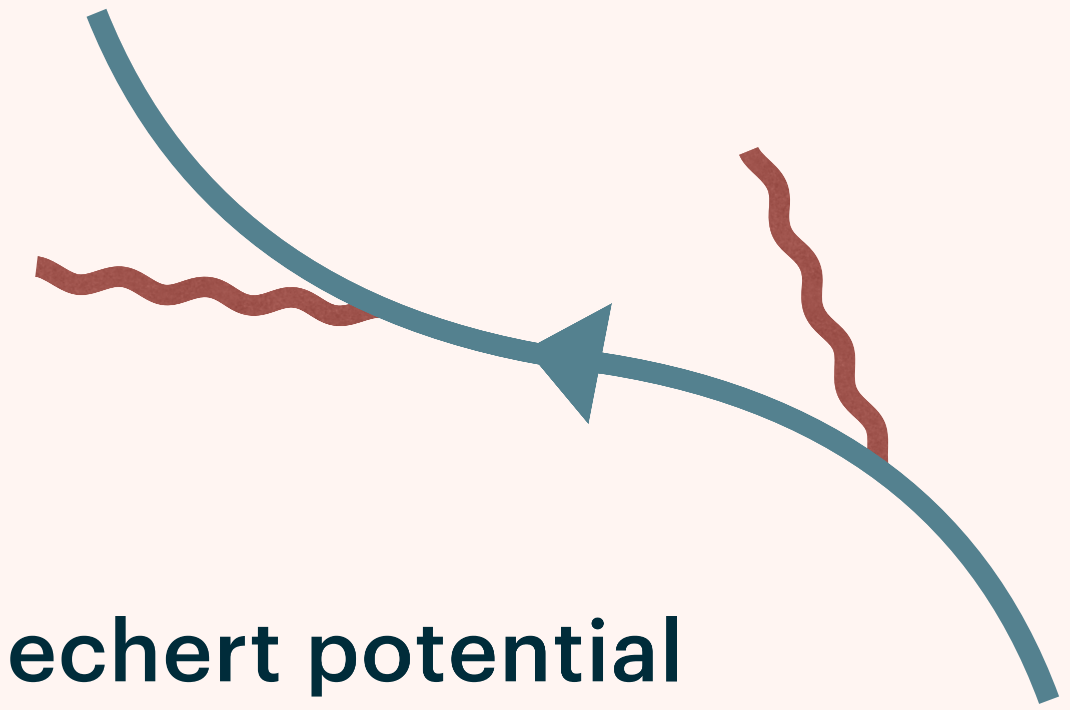
$$\partial_\mu F^{\mu\nu} = j^\nu \qquad j^\nu \propto k^\nu$$



- accelerating point charge
- add in radiation by hand
- vacuum solution

$$A + A_{rad} = \frac{Q}{4\pi\epsilon_0 r}\lambda_\mu dx^\mu$$

$$\partial_\mu F^{\mu\nu} = 0$$

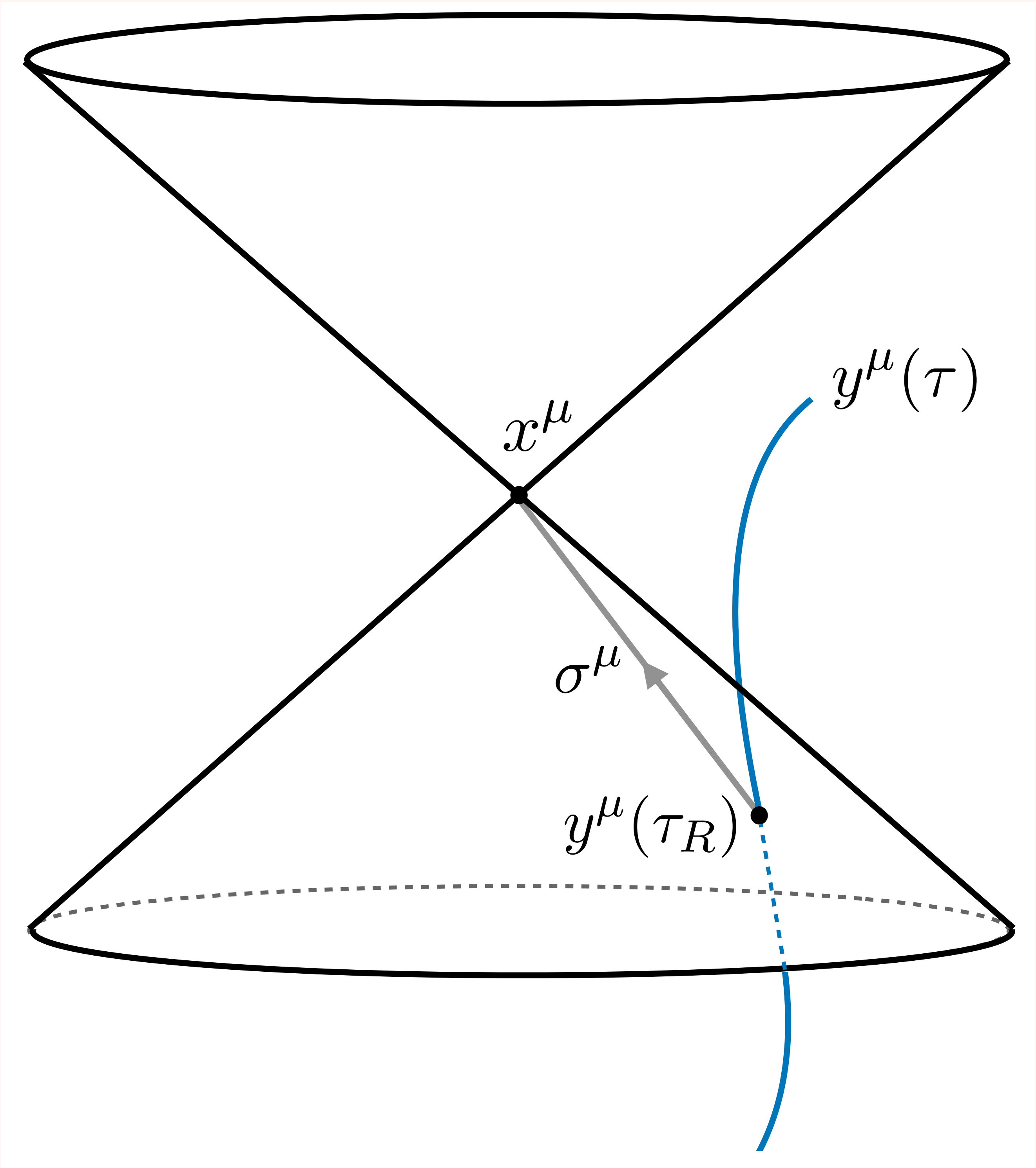


Liénard-Wiechert potential

Luna, Monteiro, Nicholson, O'Connell, White, 1603.05737

PHOTON ROCKET

Newman-Penrose map



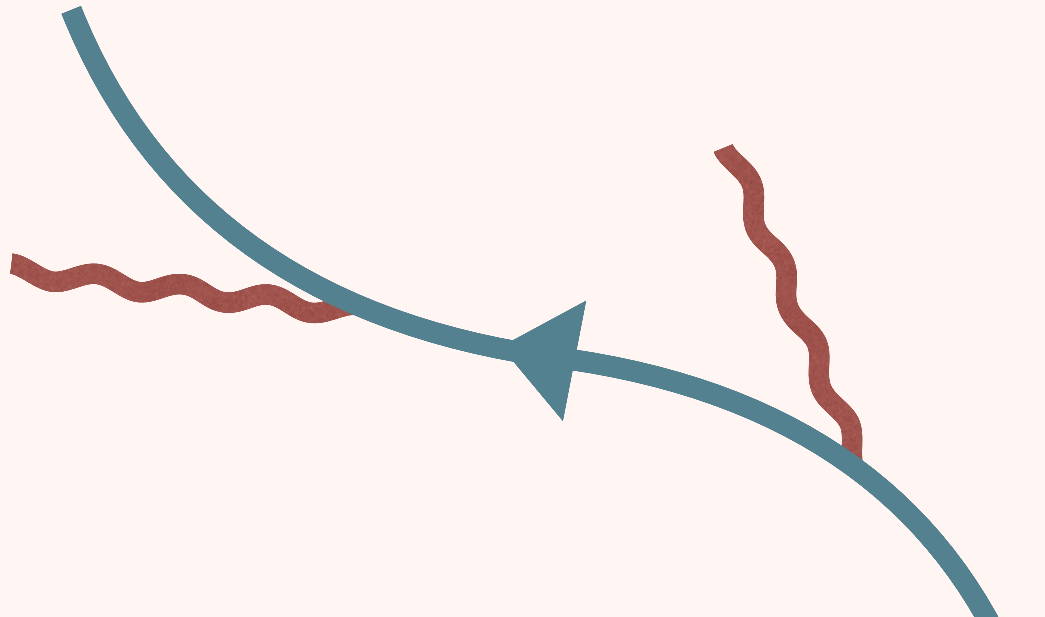
$$y^\mu(\tau_R) = (u_0, v_0, \zeta_0, \bar{\zeta}_0)$$

$$\sigma_\mu dx^\mu = (u - u_0)dv + (v - v_0)du - (\zeta - \zeta_0)d\bar{\zeta} - (\bar{\zeta} - \bar{\zeta}_0)d\zeta$$

$$\Phi = -\frac{\zeta - \zeta_0}{v - v_0} \qquad \hat{k} = -\frac{Q}{2\pi\epsilon_0}\left(dv\partial_\zeta + d\bar{\zeta}\partial_u\right)$$

real part + gauge transformation:

$$A = \hat{k}\Phi = \frac{Q}{4\pi\epsilon_0 r}\lambda_\mu dx^\mu$$



Liénard-Wiechert potential

Elor, Farnsworth, Graesser, Herczeg, 2006.08630

KERR-TAUB-NUT

- **double Kerr-Schild form in (2,2) signature**

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu + \psi \ell_\mu \ell_\nu$$

$$\phi = \frac{2GMr}{r^2 - a^2\cos^2\theta} \qquad \psi = \frac{2Na\cos\theta}{r^2 - a^2\cos^2\theta}$$

$$k_\mu dx^\mu = dt + dr - a\sin^2\theta d\varphi$$

$$\ell_\mu dx^\mu = dt + dr + \frac{r^2 - a^2\cos^2\theta}{a\sin\theta}d\theta + \frac{(r^2 - a^2)}{a}d\varphi$$

$$\Phi_k = \frac{-2Xr}{(r-a)(\sqrt{2}r + (v-u))}$$

$$\Phi_\ell = \frac{2Xr}{(r+a)(\sqrt{2}r - (v-u))}$$

$\lim a \to 0:$

$$A = \hat{k}_M\Phi_k + \hat{k}_N\Phi_\ell \sim \frac{Q_e + iQ_m}{4\pi}\left(\frac{dt}{r} - i(1-\cos\theta)d\varphi\right) + A_{gauge}$$

NEWMAN-PENROSE MAP

Gravity Solution	Kerr-Schild	Newman-Penrose
Schwarzschild black hole	Point charge	✓
Kerr black hole	Rotating disk of charge	✓
Plane waves (type N)	Plane waves	N/A
Kerr-Taub-NUT	Rotating dyon/Wu-Yang monopole	✓
Photon Rocket	Lienard-Wiechert potential	✓
(A)dS backgrounds	(A)dS backgrounds	✓
BTZ	Constant charge density	N/A

- applies to **non-vacuum**, **non-stationary** solutions

2006.08630
2205.xxxxx

NEWMAN-PENROSE MAP

- **Kerr-Schild spacetime with expanding SNGC**

$$g_{\mu\nu} = \eta_{\mu\nu} + V\ell_\mu \ell_\nu$$

- **function Φ appears in construction of preferred null tetrad**

$$\ell_\mu dx^\mu = du + \bar{\Phi} d\zeta + \boxed{\Phi} d\bar{\zeta} + \Phi\bar{\Phi} dv \qquad \partial^2 \Phi = 0$$

- **construct gauge field $A = \hat{k}\Phi$**
  - **will satisfy vacuum Maxwell equations**

$$\hat{k} = -\frac{Q}{2\pi\epsilon_0}(dv\partial_\zeta + d\bar{\zeta}\partial_u)$$

- **different normalization for ℓ_μ?**
- **other forms of $\hat{k}$?**

Elor, Farnsworth, Graesser, Herczeg, 2006.08630

NP MAP FROM TWISTOR

- **Start with null twistor** $Z^a = \left(\omega^A, \pi_{A'}\right)$ $\ell_{AA'}\bar{\pi}^A \pi^{A'} = 1$
  - Z^a **in** $\mathbb{PT}_0 \leftrightarrow$ **totally null** α**-plane in** $\mathbb{CM}$ **with real null geodesic**
- **tangent bi-vector to** α**-plane** $= \tau$

$$\ell_\mu dx^\mu = du + \cdots \qquad \ell_\mu dx^\mu = dv + \cdots$$

$$\tau \propto \partial_u \wedge \partial_\zeta \qquad \tau \propto \partial_v \wedge \partial_{\bar{\zeta}}$$

$$\tau^{\mu\rho}\eta_{\rho\nu}dx^\nu\partial_\mu \propto -\left(d\bar{\zeta}\partial_u + dv\partial_\zeta\right) \qquad \tau^{\mu\rho}\eta_{\rho\nu}dx^\nu\partial_\mu \propto -\left(d\zeta\partial_v + du\partial_{\bar{\zeta}}\right)$$

- **coordinate indep:** $\hat{k} = \dfrac{Q}{2\pi\epsilon_0}\tau^{\mu\rho}\eta_{\rho\nu}dx^\nu\partial_\mu$ $\Phi = \dfrac{1}{\omega^A\bar{\pi}_A}\ell_{BB'}\bar{\omega}^{B'}\bar{\pi}^B$ $A = \hat{k}\Phi$

Farnsworth, Graesser, Herczeg, 2104.09525

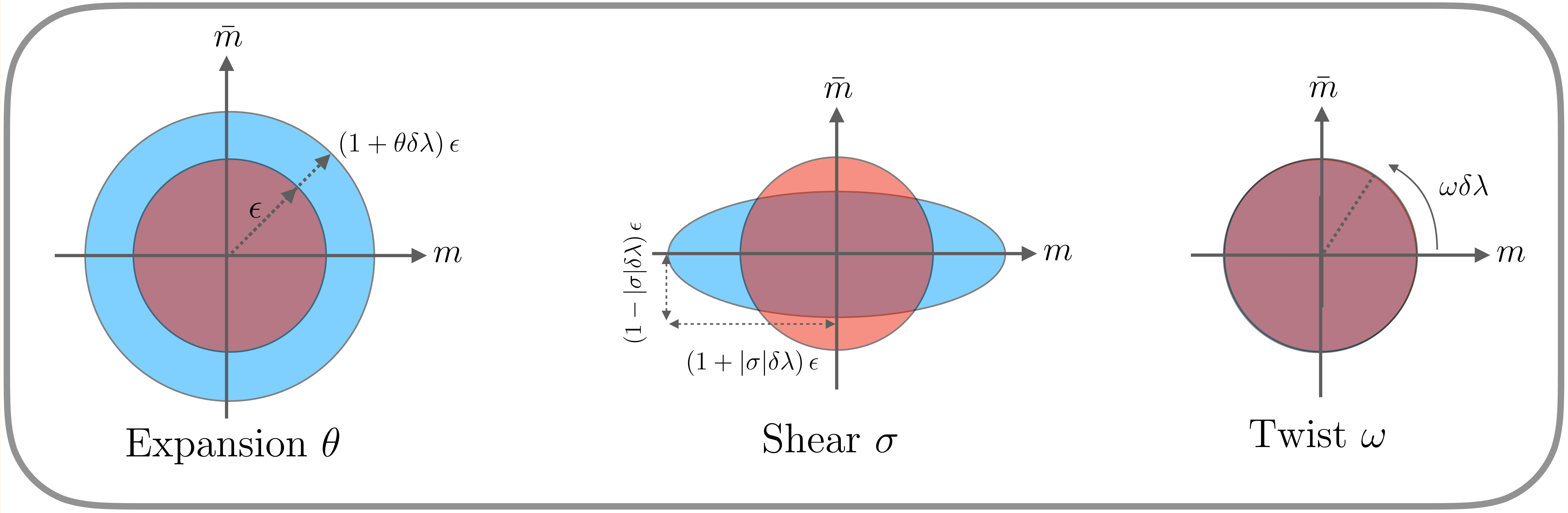
NEWMAN-PENROSE MAP

- applies to general class of metrics
  - including non-vacuum, non-stationary
  - maps to SD gauge solution
- must be in 4d
- no explicit double copy structure
- also has twistor formulation
  - gives a coordinate independent definition for $\hat{k}\Phi$

CONCLUSIONS

- Several different versions of classical double copy (Kerr-Schild, self-dual, Newman-Penrose map, Weyl)
  - agree (mostly) where they overlap
  - secretly the same? use twistors to unite?
- gauge invariance?
- position vs. momentum space?
- go beyond linear order?

THANK YOU!


$\bar{m}$
$(1+\theta\delta\lambda)\,\epsilon$
ϵ
m
Expansion θ
$\bar{m}$
m
$(1-|\sigma|\delta\lambda)\,\epsilon$
$(1+|\sigma|\delta\lambda)\,\epsilon$
Shear σ
$\bar{m}$
$\omega\delta\lambda$
m
Twist ω