Gravity Solutions via Twistor Theory

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The spinorial formalism

- GR typically uses the language of tensors and four-vectors.
- Alternative formulation in terms of two-component spinors $\pi^A \equiv (\pi^0, \pi^1)$, and their higher-rank generalisations. Spinor indices raised and lowered with

$$\pi_A = \epsilon_{AB}\pi^B, \quad \pi^B = \pi_A\epsilon^{AB}.$$

- Any multi-rank spinor can be decomposed into a sum of terms, each of which involves symmetric spinors, multiplying Levi-Civita symbols.
- Any symmetric spinor factorises into a symmetrised product of spinors e.g.

$$S_{AB\ldots C} = S_{(AB\ldots C)} \quad \Rightarrow \quad S_{AB\ldots C} = \alpha_{(A}\beta_B \ldots \gamma_{C)}.$$

with $\alpha_A, \ldots$ called *pricipal spinors*.

- Any tensorial quantity can be translated into the spinorial language using

$$\sigma^0_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spinorial formalism

- For a 4-vector this gives

$$V_\alpha \sigma^\alpha_{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix},$$

where the determinant of the matrix on the right-hand side is

$$\det\left(V_\alpha \sigma^\alpha_{AA'}\right) = \frac{1}{2}\left((V_0)^2 - (V_1)^2 - (V_2)^2 - (V_3)^2\right).$$

This is proportional to the norm of the 4-vector, such that the determinant vanishes if V_α is null.

- Then the matrix must factorise i.e.

$$V_\alpha V^\alpha = 0 \quad \Rightarrow \quad V_\alpha \sigma^\alpha_{AA'} = \pi_A \pi_{A'},$$

where $\pi_{A'} = (\pi_A)^*$ given that the matrix above is Hermitian.

- Conversely, given any spinor π_A, we may construct a matrix $M_{AA'} = \pi_A \pi_{A'}$, which in turn corresponds to a null 4-vector in spacetime. In particular, each of the so-called *principal spinors* appearing in the decomposition of a general symmetric tensor can be associated with a *principal null direction* in spacetime.

The spinorial formalism - gravity

- The Riemann tensor $R_{\alpha\beta\gamma\delta}$ can translate this into the spinor language as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} \to R_{AA'BB'CC'DD'} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ &+ \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\ &+ 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'}), \end{aligned}$$

- For *vacuum spacetimes*, we are left with the *Weyl tensor*: $C_{\alpha\beta\gamma\delta}$. We have the spinorial identification

$$C_{\alpha\beta\gamma\delta} \to \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}.$$

  where, Ψ_{ABCD} and $\bar{\Psi}_{A'B'C'D'}$ are the *anti-self-dual* and *self-dual* parts of the Weyl tensor respectively.
- The dynamics of the Weyl tensor is constrained by the Bianchi identity for the Riemann tensor, which leads to:

$$\nabla^{AA'}\Psi_{ABCD} = 0, \quad \nabla^{AA'}\bar{\Psi}_{A'B'C'D'} = 0.$$

- Ψ_{ABCD} is usually referred to as the *Weyl spinor*.

The spinorial formalism - various spins

- Electromagnetism:

$$F_{\alpha\beta} \to F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB},$$

where the symmetric spinors ϕ_{AB} and $\bar{\phi}_{A'B'}$ are the anti-self-dual and self-dual parts.

- The Maxwell equations then imply

$$\nabla^{AA'}\phi_{AB} = 0, \quad \nabla^{AA'}\bar{\phi}_{A'B'} = 0.$$

- General spinorial equations:

$$\nabla^{AA'}\phi_{AB\dots C} = 0, \quad \nabla^{AA'}\bar{\phi}_{A'B'\dots C'} = 0$$

where $\phi_{AB\dots C}$ is assumed symmetric, with n indices. These are known as the *massless free field equations*.

- The spin of the field is given by the number of spinor indices divided by two,

The spinorial formalism - classifying solutions

- An immediate use of the spinorial language is that it allows us to classify different types of solutions in electromagnetism and gravity in terms of the degeneracy of the spinors.
- Electromagnetism

$$\phi_{AB} = \alpha_{(A}\beta_{B)},$$

  and there are then two different "types" of field strength spinor:
  - (i) those with distinct null directions ($\alpha_A \not\propto \beta_A$);
  - (ii) those with a degenerate null direction, so that $\alpha_A \propto \beta_A$.

The spinorial formalism - classifying solutions

- For the Weyl tensor there are more possibilities. In general we have

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

then we can classify solutions as

Weyl type	Petrov label
$\{1,1,1,1\}$	I
$\{2,1,1\}$	II
$\{3,1\}$	III
$\{4\}$	N
$\{2,2\}$	D
$\{-\}$	O

Weyl Double Copy

- Given an electromagnetic field strength spinor ϕ_{AB}, one may construct a Weyl spinor according to the rule [Luna, Monteiro, Nicholson, O'Connell '18]

$$\Psi_{ABCD} = \frac{1}{S}\phi_{(AB}\phi_{CD)}$$

where S is a scalar function.

- This procedure was futher developed and shown to hold for type D and N vacuum spacetimes,and also in various dimensions [Alawadhi,Berman,Carrillo González,Dalhuisen,Easson,Emond,Han,Huang,H Godazgar,M Godazgar,Keeler, Kol,Manton,Momeni,Monga,Monteiro,Moynihan,O'Connell,Peinador Veiga,Pope,Rumbutis,Sabharwal,Sergola,Spence,Svesko]
- Why does this formula hold ?
- Can we generalise to other Petrov types of solutions?

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Twistors

- Define twistor space as the set of solutions of the *twistor equation*

$$\nabla_{A'}^{(A}\Omega^{B)} = 0$$

whose general solution in Minkowski space is

$$\Omega^A = \omega^A - ix^{AA'}\pi_{A'}.$$

- Twistors:

$$Z^\alpha = \left(\omega^A, \pi_{A'}\right) = \left(\omega^0, \omega^1, \pi_{0'}, \pi_{1'}\right).$$

- The "location" of a twistor in Minkowski space is defined to be the region in which its associated spinor field Ω^A vanishes. This implies the *incidence relation*

$$\omega^A = ix^{AA'}\pi_{A'}$$

invariant under simultaneous rescalings

$$\omega^A \to \lambda\omega^A, \quad \pi_{A'} \to \lambda\pi_{A'}, \quad \lambda \in \mathbb{C},$$

so twistor space is projective. A point $x^{AA'}$ in position space defines a complex line in twistor space, which (adding the point at infinity) can be mapped to a Riemann sphere.

Twistors - the Penrose transform

- Correspondence between solutions of the massless free field equations (with n indices) and twistor space:

$$\phi_{A'B'\dots C'}(x) = \frac{1}{2\pi i}\oint_{\Gamma} \pi_{E'}\, d\pi^{E'}\, \pi_{A'}\pi_{B'}\dots\pi_{C'}[\rho_x f(Z^{\alpha})],$$

  where the symbol ρ_x denotes that we must restrict to the line in projective twistor space corresponding to the spacetime point $x^{AA'}$. The contour Γ for this integral is defined on the related Riemann sphere. Then

$$\nabla^{AA'}\phi_{AB\dots C} = 0$$

  is satisfied automatically !
- We can deal with exact solutions which linearise the e.o.m., or with general but linearised solutions.
- The integrand (including the measure) must be homogeneous of degree zero under rescalings $\pi_{A'} \to \lambda\pi_{A'}$ (or $Z^{\alpha} \to \lambda Z^{\alpha}$). This in turn implies that the function $f(Z^{\alpha})$ must have degree $(-n-2)$, where n is the number of indices appearing on the left-hand side.

Twistors - the Penrose transform

- There are some tricks for formulating representative twistor functions for spacetime fields possessing certain properties.
- Note that the factorisation property of symmetric spinors means that if a given n-index spinor has a k-fold principal spinor $\xi_{A'}$, it will vanish if contracted with $(n-k+1)$ factors of $\xi_{A'}$, but not if only $(n-k)$ factors are contracted. See

$$\phi_{A'B'\dots F'} = \underbrace{\xi_{(A'}\xi_{B'}\dots\xi_{C'}}_{k\ \text{factors}}\underbrace{\alpha_{D'}\beta_{E'}\dots\gamma_{F')}}_{(n-k)\ \text{factors}}$$

- Contracting the Penrose Transform with m factors of $\eta^{A'}$ gives

$$\underbrace{\eta^{A'}\eta^{B'}\dots\eta^{C'}}_{m\ \text{factors}}\underbrace{\phi_{A'B'\dots C'D'\dots F'}}_{n\ \text{indices}}(x) = \frac{1}{2\pi i}\oint_{\Gamma}\pi_{E'}d\pi^{E'}[\pi\eta]^m\pi_{D'}\dots\pi_{F'}[\rho_x f(Z^\alpha)]$$

we see that the field $\phi_{A'B'\dots F'}$ has at least a $(n-m+1)$-fold principal spinor $\eta_{A'}$, if the twistor function $f(Z^\alpha)$ has a single m^{th}-order pole as $\pi_{A'}\to\eta_{A'}$, enclosed by Γ.

Twistorial Double Copy [White'20]

- Remember the general (mixed) type D Weyl double copy may be written as

$$\phi_{A'B'C'D'} = \frac{1}{\phi}\phi^{(1)}_{(A'B'}\phi^{(2)}_{C'D')}.$$

- Consider two holomorphic twistor functions $f^{(1,2)}_{\mathrm{EM}}(Z^\alpha)$ of homogeneity -4, and a further holomorphic twistor function $f(Z^\alpha)$ of homogeneity -2.
- These will necessarily correspond to electromagnetic spinors $\phi^{(1,2)}_{A'B'}$ and a scalar field ϕ in spacetime. One may then form a product

$$f_{\mathrm{grav.}}(Z^\alpha) = \frac{f^{(1)}_{\mathrm{EM}}(Z^\alpha)\,f^{(2)}_{\mathrm{EM}}(Z^\alpha)}{f(Z^\alpha)},$$

such that the function on the left-hand side necessarily has homogeneity -6, and thus potentially corresponds to a spacetime field solving the spin-2 massless free field equation i.e. to a self-dual gravity solution.

- For a suitable choice of twistor functions, this spacetime relationship is precisely the type D Weyl double copy.

Twistorial Double Copy

- Define a family of twistor functions

$$f_m(Z^\alpha) = \frac{1}{m!}\left[Q_{\alpha\beta}Z^\alpha Z^\beta\right]^{-m} = \frac{1}{m!}\left[\frac{N(x)}{(\xi-\xi_1(x))\,(\xi-\xi_2(x))}\right]^m$$

for some constant $Q_{\alpha\beta}$, using

$$Z^\alpha = \left(\omega^A, \pi_{A'}\right) \overset{\rho_X}{\to} \left(\omega^A, ix^{AA'}\pi_{A'}\right), \qquad \pi_{A'} = (1,\xi)$$

- Then carry out the Penrose transform for $m = (1,2,3)$ to get:

$$\phi = \frac{N(x)}{\xi_1-\xi_2},\;\; \phi_{A'B'} = \frac{-N^2(x)}{(\xi_1-\xi_2)^3}\alpha_{(A'}\beta_{B')},\;\; \phi_{A'B'C'D'} = \frac{N^3(x)}{(\xi_1-\xi_2)^5}\alpha_{(A'}\beta_{B'}\alpha_{C'}\beta_{D')}$$

with

$$\alpha = (1,\xi_1), \quad \beta = (1,\xi_2)$$

which reproduces the Type D Weyl Double Copy!

- Opens the door to a formulation on curved backgrounds.
- Allows us to go beyond Type D/N.

Generalising the Weyl Double Copy [Chacón,SN,White]

- Consider the homogeneity -4 functions related to two different electromagnetic spinors

$$f^{(0,2)}_{\rm EM} = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)^3} = \frac{1}{[\pi\mathcal{A}][\pi\mathcal{B}]^3} \quad\rightarrow\quad \phi^{(0,2)}_{A'B'} = \left(\frac{2}{\Omega|x-y|^2}\right)^3 \mathcal{A}_{(A'}\mathcal{A}_{B')}$$

$$f^{(1,1)}_{\rm EM} = \frac{1}{(A_\alpha Z^\alpha)^2(B_\beta Z^\beta)^2} = \frac{1}{[\pi\mathcal{A}]^2[\pi\mathcal{B}]^2} \quad\rightarrow\quad \phi^{(1,1)}_{A'B'} = -2\left(\frac{2}{\Omega|x-y|^2}\right)^3 \mathcal{A}_{(A'}\mathcal{B}_{B')}$$

as well as the homogeneity -2 function

$$f^{(0,0)} = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)} = \frac{1}{[\pi\mathcal{A}][\pi\mathcal{B}]} \quad\rightarrow\quad \phi = \frac{2}{\Omega|x-y|^2}, \qquad \Omega = A_B B^B$$

Then we can construct the twistor representative for Type II solutions

$$f^{(\rm II)}_{\rm grav.} = \frac{1}{f^{(0,0)}} f^{(1,1)}_{\rm EM}\left(-\frac{[\mathcal{C}\mathcal{B}]}{[\mathcal{A}\mathcal{B}]} f^{(0,2)}_{\rm EM} + \frac{[\mathcal{C}\mathcal{A}]}{[\mathcal{A}\mathcal{B}]} f^{(1,1)}_{\rm EM}\right).$$

which in space-time becomes

$$\Psi^{(\rm II)}_{A'B'C'D'} = \frac{1}{\phi}\left[3\frac{[\mathcal{C}\mathcal{A}]}{[\mathcal{A}\mathcal{B}]}\phi^{(0,2)}_{(A'B'}\phi^{(1,1)}_{C'D')} - 4\frac{[\mathcal{C}\mathcal{B}]}{[\mathcal{A}\mathcal{B}]}\phi^{(1,1)}_{(A'B'}\phi^{(1,1)}_{C'D')}\right]$$

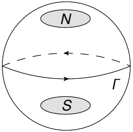
so that

$$\Psi^{(\rm II)}_{A'B'C'D'} = \frac{2}{[\mathcal{A}\mathcal{B}]^5}\mathcal{A}_{(A'}\mathcal{A}_{B'}\mathcal{B}_{C'}\mathcal{F}_{D')}, \quad \mathcal{F}_{A'} = 3\frac{[\mathcal{C}\mathcal{A}]}{[\mathcal{A}\mathcal{B}]}\mathcal{B}_{A'} + 2\frac{[\mathcal{C}\mathcal{B}]}{[\mathcal{A}\mathcal{B}]}\mathcal{A}_{A'}$$

Twistorial Double Copy- More general solutions

- One can also comstruct Type I solutions.
- We have generalised the Weyl Double copy beyond Type D/N - it is now a sum of products.
- The twistor language reduces the problem to finding combinations with the correct pole structure - this are then guaranteeed to satisfy the e.o.m. when we go to spacetime by performing the Penrose transform!

A problem with the definition



- Remember the Penrose transform

$$\phi_{A'B'\dots C'}(x) = \frac{1}{2\pi i}\oint_{\Gamma} \pi_{E'} d\pi^{E'} \pi_{A'}\pi_{B'}\dots\pi_{C'}[\rho_x f(Z^\alpha)],$$

- Notice that we are free to modify the twistor function according to the equivalence relation

$$f(Z^\alpha) \to f(Z^\alpha) + f_N(Z^\alpha) + f_S(Z^\alpha), \quad (1)$$

where $f_N(Z^\alpha)$ ($f_S(Z^\alpha)$) contains singularities only in the northern (southern) hemisphere.

- This is not respected by the double copy !
- How to choose canonical representatives ?

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Set-up

- On a complex manifold, we can define (p,q) forms

$$\omega = \omega_{a_1 \dots a_p \bar{a}_1 \dots \bar{a}_q} dz^{a_1} \wedge \dots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{a}_1} \wedge \dots \wedge d\bar{z}^{\bar{a}_q}$$

- On twistor space we define the Dolbeault operator $\bar{\partial} = d\bar{Z}^\alpha \frac{\partial}{\partial \bar{Z}^\alpha}$ and use it to define holomorphic quantities h by $\bar{\partial} h = 0$. It satisfies $\bar{\partial}^2 = 0$.
- Penrose transform (Dolbeault version):

$$\phi_{A'B'\dots C'}(x) = \frac{1}{2\pi i} \int_X \langle \pi d\pi \rangle \wedge \pi_{A'} \pi_{B'} \dots \pi_{C'} f(Z)|_X,$$

- Note that $d\pi_{A'}$ is a (1,0), so $f(Z)$ must be a (0,1)-form.
- Imposing $\bar{\partial} f = 0$ ensures the e.o.m. for ϕ!
- Note we have the redundancy

$$f \to f + \bar{\partial} g$$

where $\bar{\partial} g$ is globally defined on X, so f is an element of the Dolbeault cohomology group $H^{p,q}_{\bar{\partial}}(\mathcal{M}) = \frac{Z^{p,q}_{\bar{\partial}}(\mathcal{M})}{B^{p,q}_{\bar{\partial}}(\mathcal{M})}$, with $(p,q) = (0,1)$.

From Čech to Dolbeault double copy

- On a local patch on the Riemann sphere, we can construct Dolbeault representatives for the scalar and gauge field via

$$f = \bar{\partial}\check{f}, \quad f_{\mathrm{EM}}(Z) = \bar{\partial}\check{f}_{\mathrm{EM}}$$

  where $\check{f}$ and $\check{f}_{\mathrm{EM}}$ are the Čech representatives (then extend to the full Riemnn sphere via a partition of unity).
- Note that these are not redundancies, due to the presence of the poles in the Čech representative (remember $\frac{\partial}{\partial \bar{z}} \frac{1}{z} \propto \delta^{(2)}(z)$)!
- We can immediately translate the double copy into the Dolbeault laguage via

$$f_{\mathrm{grav.}} = \bar{\partial}\left[\frac{\check{f}^{(1)}_{\mathrm{EM}}\check{f}^{(2)}_{\mathrm{EM}}}{\check{f}}\right]$$

- Works for a wide set of solutions.
- It will also inherit the issue related to freedom of choice of representative - how to make a canonical choice?

Harmonic Representative in Euclidean Space

- Using [Woodhouse'85], and working in Euclidean signature, with the conjugation relation

$$\pi_{A'} = (c, d) \to \hat{\pi}_{A'} = (-\bar{d}, \bar{c})$$

the Dolbeault operator can be recast as

$$\bar{\partial} = \bar{e}^2 \bar{\partial}_2 + \bar{e}^A \bar{\partial}_A$$

with

$$\left\{ \bar{\partial}_2 = \langle \pi \hat{\pi} \rangle \pi^{A'} \frac{\partial}{\partial \hat{\pi}^{A'}},\ \bar{\partial}_A = \pi^{A'} \frac{\partial}{\partial x^{AA'}} \right\},\ \left\{ \bar{e}^2 = \frac{\langle \hat{\pi} d\hat{\pi} \rangle}{\langle \pi \hat{\pi} \rangle^2}, \quad \bar{e}^A = \frac{\hat{\pi}_{A'} dx^{AA'}}{\langle \pi \hat{\pi} \rangle} \right\}$$

- Then on the Riemann sphere we have

$$f|_X = f_2 \bar{e}^2$$

Harmonic Representative in Euclidean Space

- We are restricting to the Riemann sphere.
- Using the Hodge decomposition on a compact manifold we can write f as

$$f = \bar{\partial}\alpha + \bar{\partial}^{\dagger}\beta + \gamma$$

where γ is the *harmonic part* of and satisfies $\bar{\partial}\gamma = \bar{\partial}^{\dagger}\gamma = 0$.

- Remembering that $\bar{\partial}f = 0$ and positive-definiteness of the inner product

$$0 = \langle \beta, \bar{\partial}\bar{\partial}^{\dagger}\beta \rangle = \langle \bar{\partial}^{\dagger}\beta, \bar{\partial}^{\dagger}\beta \rangle \geq 0$$

we find that $\bar{\partial}^{\dagger}\beta = 0$.

- We impose the condition:

$$\bar{\partial}^{\dagger}f = 0$$

and this similarly sets $\bar{\partial}\alpha = 0$, so we have found a unique harmonic representative for f!

Harmonic Representative in Euclidean Space

- Prescription for constructing harmonic representative [Woodhouse'85]: Starting with a spacetime field $\phi_{A'B'\ldots C'}$, we can construct a twistor function on X as follows:

$$\Phi_\phi = \frac{1}{\langle\pi\hat{\pi}\rangle^{2n+1}}\phi_{A'B'\ldots C'}(x)\hat{\pi}^{A'}\hat{\pi}^{B'}\ldots\hat{\pi}^{C'}.$$

  One may then construct the (0,1) form

$$f_\phi = \hat{\partial}\Phi_\phi = \frac{2n+1}{\langle\pi\hat{\pi}\rangle^{2n}}\phi_{A'B'\ldots C'}\hat{\pi}^{A'}\pi^{B'}\ldots\pi^{C'}\bar{e}^2,$$

  where we have introduced the operator

$$\hat{\partial} \equiv d\hat{\pi}^{A'}\frac{\partial}{\partial\pi'_A},$$

- Double copy

$$f_{\rm grav.} = \hat{\partial}\left(\frac{\Phi^{(1)}_{\rm EM}\,\Phi^{(2)}_{\rm EM}}{\Phi}\right)$$

- Different form from before but gives same result in spacetime !

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Various double copies

- Other methods for choosing representatives: in purely radiative spacetime [Adamo,Kol] using the characteristic data at future null infinity $\mathcal{I}^+$ [Mason], and also via the link with scattering amplitudes [Guevara].
- Incoporate non-abelian gauge theory and expand the set of solutions vith Ward correspondence, and expansion around self-dual sector.
- Connection to other versions of the double copy?

Thank You !

Various double copies

- Different formulations
  - (1) self-dual fields and symmetries
  - (2) exact solutions, twistorial formulaiton
  - (3) scattering amplitudes (BCJ rules)
  - (4) linearised approximation (convolutions)
- (1)-(2) perturbation around self-dual sector
- (1)-(3) replacement rules
- (1)-(4) checked explicitly on the overlap
- (2)-(3) checked at linear level, from 3-point amplitude with probe particle
- (2)-(4) to do
- (3)-(4) product in momentum space → convolution in position space