Recent Progress on the Kinematic Algebra

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YOUNGST@RS

MITP Workshop

Based on recent work with:

Bern, Carrasco, Chiodaroli, HJ, Roiban [1909.01358, 2203.13013];
Gang Chen, HJ, Fei Teng, Tianheng Wang [1906.10683, 2104.12726];
Andi Brandhuber, Gang Chen, HJ, Gab Travaglini, Congkao Wen [2111.15649];
Maor Ben-Shahar, HJ [2112.11452]

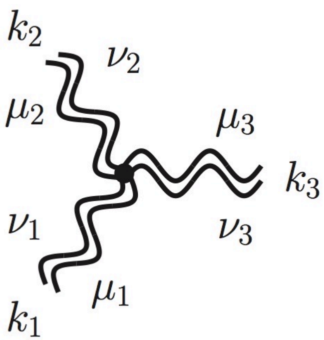
On-shell simplifications in GR

Graviton plane wave: $\varepsilon^{\mu}(p)\varepsilon^{\nu}(p)\, e^{ip\cdot x}$

Yang-Mills polarization

$$|\text{spin } 2\rangle \sim |\text{spin } 1\rangle \otimes |\text{spin } 1\rangle$$

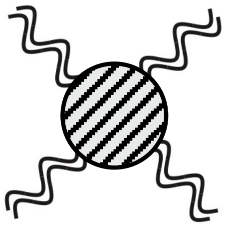
On-shell 3-graviton vertex:



$$= \quad \Big(\eta_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + \text{cyclic}\Big)\Big(\eta_{\nu_1\nu_2}(k_1 - k_2)_{\nu_3} + \text{cyclic}\Big)$$

Yang-Mills vertex

Gravity scattering amplitude:



Yang-Mills amplitude

$$M^{\text{GR}}_{\text{tree}}(1,2,3,4) = \frac{st}{u}\Big[A^{\text{YM}}_{\text{tree}}(1,2,3,4)\Big]^2$$

Gravity processes = "squares" of gauge theory ones: KLT, BCJ, CHY

Kawai-Lewellen-Tye Relations ('86)

String theory
tree-level identity:

closed string ~ (left open string) × (right open string)

$$A_n \sim \int \frac{dx_1 \cdots dx_n}{\mathcal{V}_{abc}} \prod_{1\leq i<j\leq n} |x_i - x_j|^{k_i\cdot k_j} \exp\left[\sum_{i<j}\left(\frac{\epsilon_i\cdot\epsilon_j}{(x_i-x_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(x_i-x_j)}\right)\right]\Bigg|_{\mathrm{multi-linear}}$$

KLT relations emerge after nontrivial world-sheet integral identities

Field theory limit $\Rightarrow$ gravity theory ~ (YM theory) × (YM theory)

$$M_4^{\mathrm{tree}}(1,2,3,4) = -is_{12}A_4^{\mathrm{tree}}(1,2,3,4)\,\widetilde{A}_4^{\mathrm{tree}}(1,2,4,3)$$

$$M_5^{\mathrm{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\mathrm{tree}}(1,2,3,4,5)\,\widetilde{A}_5^{\mathrm{tree}}(2,1,4,3,5)$$
$$+is_{13}s_{24}A_5^{\mathrm{tree}}(1,3,2,4,5)\,\widetilde{A}_5^{\mathrm{tree}}(3,1,4,2,5)$$

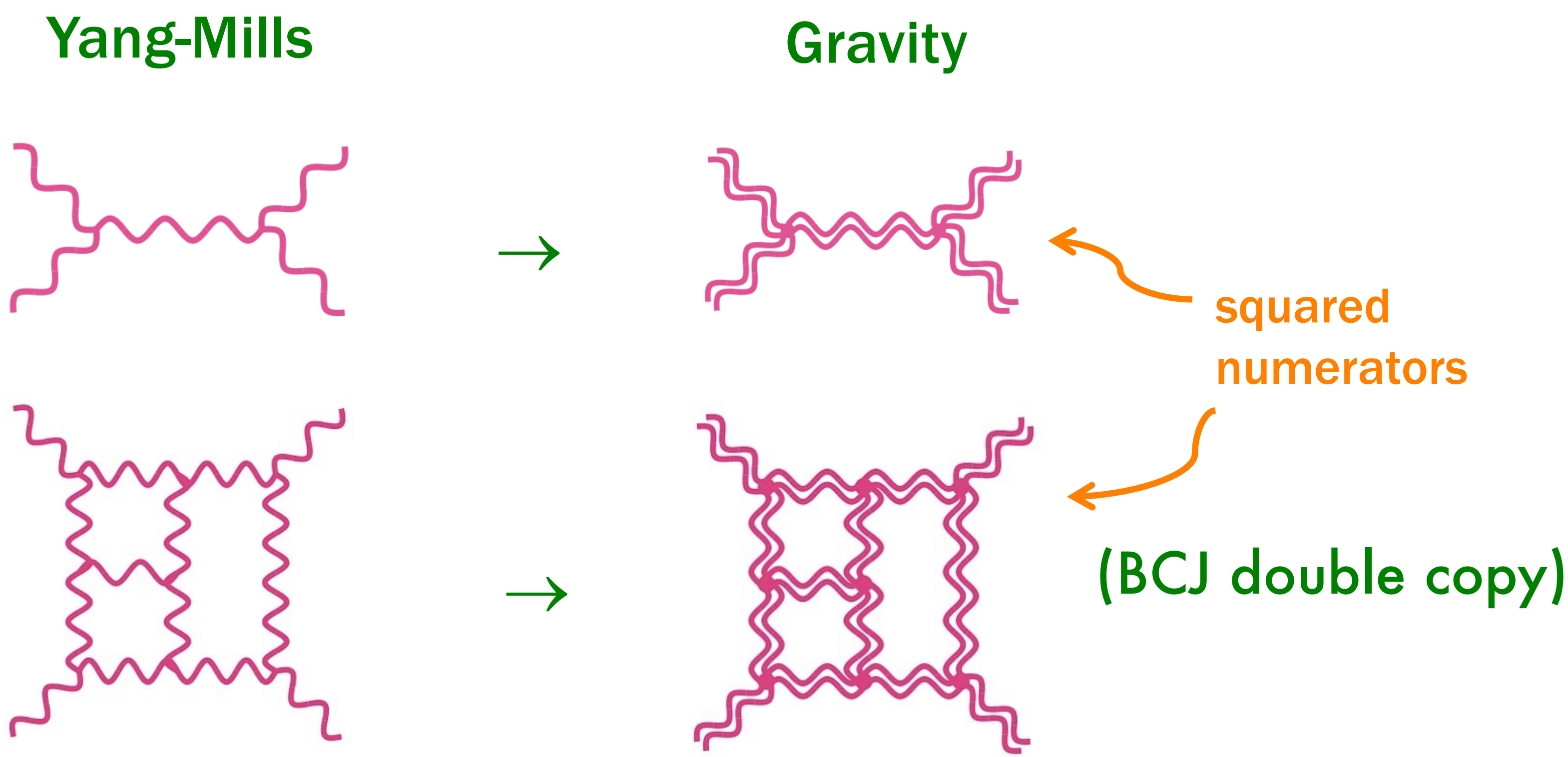
gravity states are products of YM states:

$$|2\rangle = |1\rangle \otimes |1\rangle$$
$$|3/2\rangle = |1\rangle \otimes |1/2\rangle$$

etc...

Squaring of YM theory – the double copy

Gravity processes = squares of gauge theory ones - entire S-matrix



E.g. pure Yang-Mills $\rightarrow$ Einstein gravity + dilaton + axion

4D YM + massless quarks $\rightarrow$ Pure 4D Einstein gravity

Example: axion-dilaton gravity

Consider double copy of *D*-dimensional pure YM:

States:

$$
\begin{cases}
(\varepsilon^h)^{ij}_{\mu\nu} = \varepsilon^{((i}_\mu \varepsilon^{j))}_\nu & \text{(graviton)} \\
(\varepsilon^B)^{ij}_{\mu\nu} = \varepsilon^{[i}_\mu \varepsilon^{j]}_\nu & (B\text{-field}) \\
(\varepsilon^\phi)_{\mu\nu} = \dfrac{\varepsilon^i_\mu \varepsilon^j_\nu \delta_{ij}}{D-2} & \text{(dilaton)}
\end{cases}
$$

Amplitudes consistent with the theory:

$$S = \int d^D x \sqrt{-g} \left[-\frac{1}{2} R + \frac{1}{2(D-2)} \partial^\mu \phi \partial_\mu \phi + \frac{1}{6} e^{-4\phi/(D-2)} H^{\lambda\mu\nu} H_{\lambda\mu\nu} \right]$$

In 4D this is axion-dilaton gravity:

$$S = \int d^4 x \sqrt{-g} \left[-\frac{1}{2} R + \frac{1}{4} \partial_\mu \phi \partial^\mu \phi + \frac{1}{4} e^{-2\phi} \partial_\mu \chi \partial^\mu \chi \right]$$

Symmetry $\begin{matrix} \chi \to -\chi \\ \phi \to -\phi \end{matrix}$ allows for consistent truncation of scalars

Example: pure GR

Pure 4D Einstein gravity: $\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{g}R$

HJ, Ochirov

Does not match YM^2 spectrum: $\mathrm{YM}\otimes\mathrm{YM} = \mathrm{GR} + \phi + a$

Deform YM theories with massless fundamental quarks

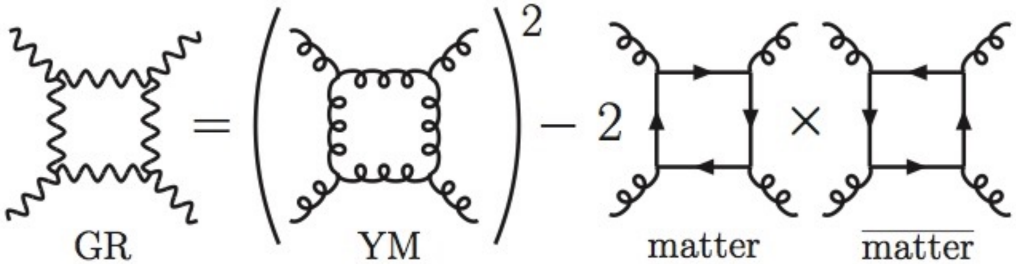
$$(\mathrm{YM} + \mathrm{quark}) \otimes (\mathrm{YM} + n_f \ \mathrm{quarks})$$
$$= \mathrm{GR} + 2(n_f + 1)\ \mathrm{scalars}$$

Anti-align the spins of the quarks → gives scalars in GR

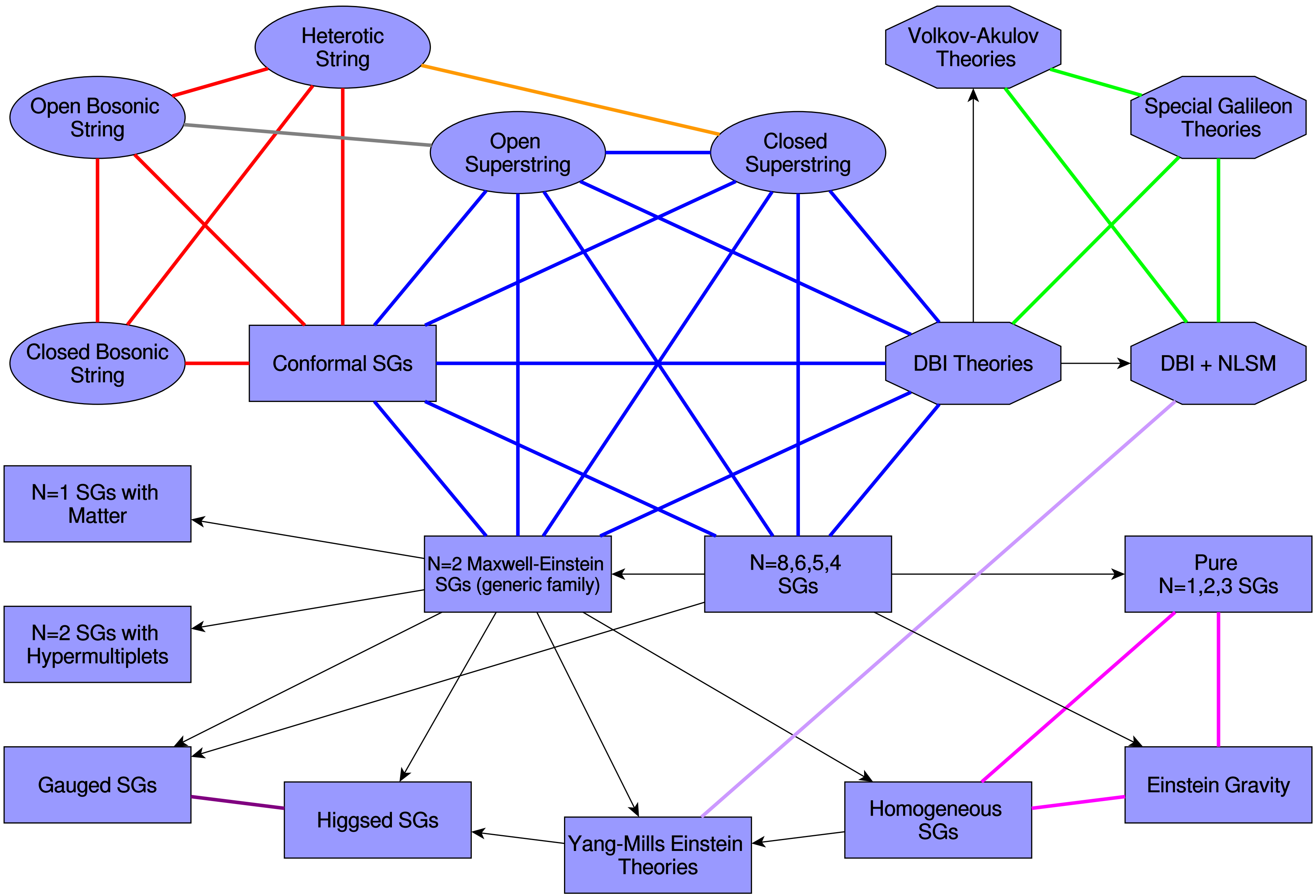
e.g. $\phi = q\otimes\bar{q} + \bar{q}\otimes\bar{q}$

$a = q\otimes\bar{q} - \bar{q}\otimes\bar{q}$

become ghosts if $n_f = -1$



Web of double-copy constructible theories



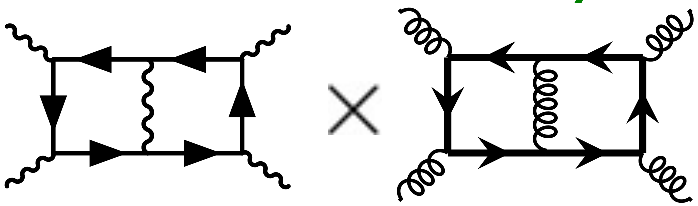
See reviews [1909.01358], [2203.13013] – Bern, Carrasco, Chiodaroli, HJ, Roiban

N=2 SUGRA double copies

N=2 SQCD on color-kinematics form
known up to two loops
→ permits the double copies:

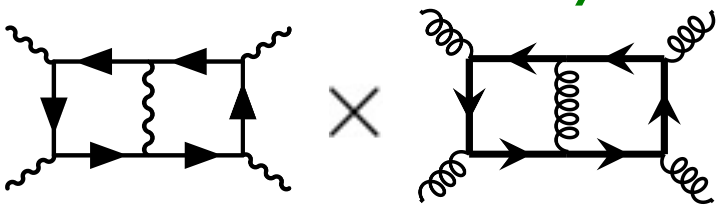
HJ, Ochirov;
Chiodaroli, Gunaydin, HJ, Roiban;
Ben-Shahar, Chiodaroli;
Mogull, Kälin, HJ

N=2 SQCD × 4D Feynman



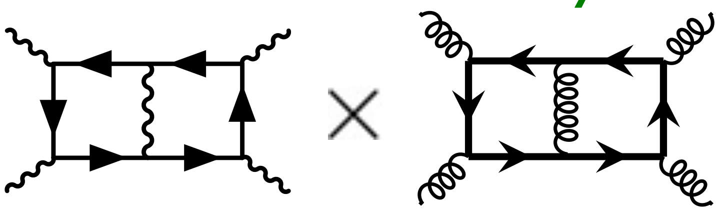
→ N=2 Luciani Model (1978)

N=2 SQCD × 5D Feynman



→ N=2 Generic non-Jordan family
Günaydin, Sierra, Townsend (1986)

N=2 SQCD × 6D Feynman



→ N=2 Generic Jordan family
Günaydin, Sierra, Townsend (1984)
(self-dual tensors in 6D)

Exception that proves the rule...

Not all gauge theories obey color-kinematics duality

Imagine the double copy:

$$\mathrm{YM} \otimes (\mathrm{YM} + \mathcal{N} \text{ adjoint fermions}) \stackrel{?}{=} \mathrm{GR} + \mathcal{N}\ \Psi_{3/2}$$

According to conventional wisdom $\Psi_{3/2}$ must be a gravitino and $\mathcal{N} \leq 8$ is the number of supersymmetries

What goes wrong? The theory

$$\mathrm{YM} + \mathcal{N} \text{ adjoint fermions} + \ldots$$

only obeys color-kinematics duality if supersymmetric → $\mathcal{N} \leq 4$

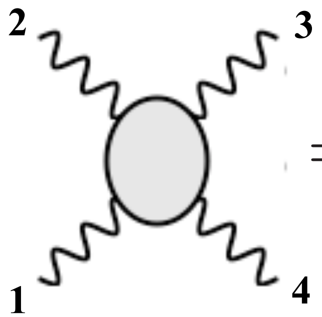
Kinematic Jacobi Id. → Fierz Id. that enforces SUSY

Chiodaroli, Jin, Roiban

The (Square-)Root of Gravity

Color-kinematics duality

Consider Yang-Mills 4p tree amplitude:



$$= \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

color factors: $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$

kinematic numerators:

$$n_s = \Big[(\varepsilon_1 \cdot \varepsilon_2) p_1^\mu + 2(\varepsilon_1 \cdot p_2)\varepsilon_2^\mu - (1 \leftrightarrow 2)\Big]\Big[(\varepsilon_3 \cdot \varepsilon_4) p_{3\mu} + 2(\varepsilon_3 \cdot p_4)\varepsilon_{4\mu} - (3 \leftrightarrow 4)\Big]$$
$$+ s\big[(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3)\big]\,,$$

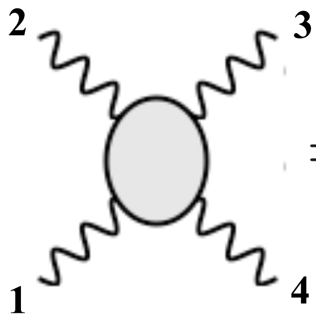
consider gauge transformation $\delta A_\mu = \partial_\mu \phi$

$$n_s\Big|_{\varepsilon_4 \to p_4} = s\Big[(\varepsilon_1 \cdot \varepsilon_2)\big((\varepsilon_3 \cdot p_2) - (\varepsilon_3 \cdot p_1)\big) + \text{cyclic}(1,2,3)\Big] \equiv s\,\alpha(\varepsilon, p)$$

(individual diagrams not gauge inv.)

Color-kinematics duality

Consider Yang-Mills 4p tree amplitude:



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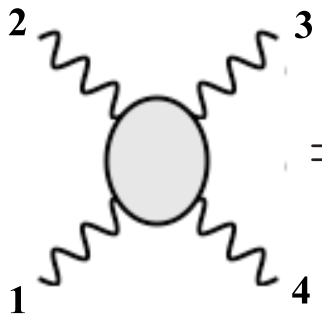
$$+ s\Big[(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3)\Big],$$

consider linearized gauge transformation $\delta A_\mu = \partial_\mu \phi$

$$\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\bigg|_{\varepsilon_4 \to p_4} = \underbrace{(c_s + c_t + c_u)}_{=0 \text{ Jacobi identity}} \alpha(\varepsilon, p)$$

Color-kinematics duality

Consider Yang-Mills 4p tree amplitude:



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kinematic numerators:

$$n_s = \Big[(\varepsilon_1\cdot\varepsilon_2)p_1^\mu + 2(\varepsilon_1\cdot p_2)\varepsilon_2^\mu - (1\leftrightarrow 2)\Big]\Big[(\varepsilon_3\cdot\varepsilon_4)p_{3\mu} + 2(\varepsilon_3\cdot p_4)\varepsilon_{4\mu} - (3\leftrightarrow 4)\Big]$$
$$+s\big[(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\big]\,,$$

$c_s + c_t + c_u = 0$ Jacobi Id. (gauge invariance)

$\Leftrightarrow$

$n_s + n_t + n_u = 0$ kinematic Jacobi Id. (diffeomorphism inv.)

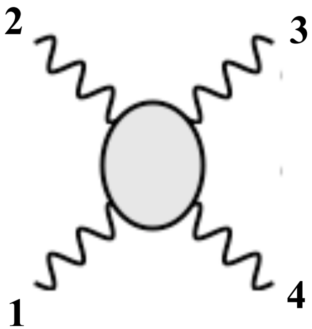
BCJ ('08)

Double copy

Color and kinematics are dual...

$$c_s + c_t + c_u = 0 \qquad \Leftrightarrow \qquad n_s + n_t + n_u = 0$$

...replace color by kinematics $\quad c_i \to n_i \quad$ BCJ double copy



$$= \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

← gravity ampl.

Properties of ampl:
- spin-2 scattering $\qquad \varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$
- 2-derivative interactions $\qquad \partial_\mu \to \partial_\mu \partial_\nu$
- diffeomorphism inv. $\qquad \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$$\left.\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right|_{\varepsilon_4^{\mu\nu} \to p_4^\mu \varepsilon_4^\nu + p_4^\nu \varepsilon_4^\mu} = 2(n_s + n_t + n_u)\,\alpha(\varepsilon, p) = 0$$

Progress on Kinematic Lie Algebra

What is the Kinematic Algebra ?

– YM numerators obey Jacobi Id. → kinematic algebra should exist ?
– Algebra *may* dramatically simplify GR calculations!

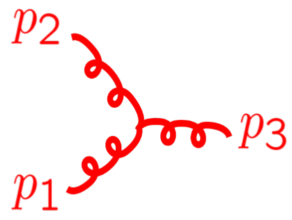
What is known?

Self dual YM in light-cone gauge: $u^2 = w^2 = u \cdot w = 0$ **Monteiro, O'Connell ('11)**

$$A^+ = \partial_u \Phi$$

EoM: $$\Box \Phi = -ig[\partial_w \Phi, \partial_u \Phi]$$

Feynman vertex: $$X(p_1, p_2) = p_{1w} p_{2u} - p_{2w} p_{1u}$$



Introduce generators of area-preserving diffeomorphisms:

$$L_k = e^{-ik\cdot x}(-k_w \partial_u + k_u \partial_w)$$

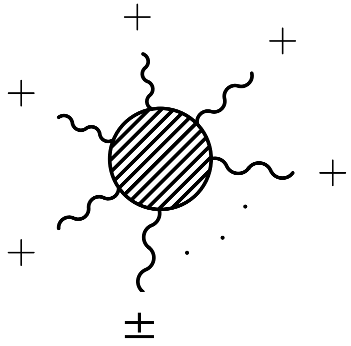
Lie Algebra: $$[L_{p_1}, L_{p_2}] = iX(p_1, p_2) L_{p_1+p_2} = iF_{p_1 p_2}{}^{k} L_k$$

YM vertex

Amplitudes in self-dual YM ?

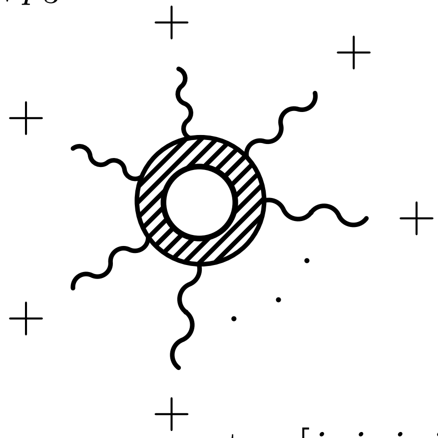
Numerators exist, but
tree amplitudes vanish:

$$N_s = [[L_{p_1}, L_{p_2}], L_{p_3}]$$
$$= -X(p_1, p_2) X(p_1 + p_2, p_3) L_{p_1+p_2+p_3}$$



$= \; 0$

Only 1-loop all-plus-heliciy
amplitudes non-vanishing:



$=$ rational fn.

Bern, Chalmers, Dixon, Kosower; Mahlon

$$A_{n;1}^{\text{gluon}}(1^+, 2^+, \ldots, n^+) \; = \; -\frac{i}{48\pi^2} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \frac{\mathrm{tr}_-[i_1 i_2 i_3 i_4]}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n\,1\rangle} + \mathcal{O}(\epsilon)$$

Self-dual gravity:

$$\mathcal{M}_n^{\text{1-loop}}(1^+, 2^+, \ldots, n^+) = -\frac{i}{(4\pi)^2 \cdot 960} \left(\frac{-\kappa}{2}\right)^n \sum_{\substack{1 \le a < b \le n \\ M, N}} h(a, M, b) h(b, N, a) \, \mathrm{tr}^3[a\, M\, b\, N]$$

Bern, Dixon, Perelstein, Rozowsky

Calculations via kinematic algebra: Boels, Iserman, Monteiro, O'Connell

Kinematic algebra encoded in Lagrangian

Attempt to construct Lagrangian that manifests color-kinematics duality

Bern, Dennen, Huang, Kiermaier ('10): add non-local operators

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \dots$$

$$\mathcal{L}'_5 \sim \mathrm{Tr}\,[A^\nu, A^\rho]\frac{1}{\Box}\big([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho]\big)$$

gives 5pts BCJ numerators, but there exist an ambiguity

$$\mathcal{D}_5 = \tfrac{-\beta}{2} g^3 f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5}\Big(\partial_{(\mu} A^{a_1}_{\nu)} A^{a_2}_\rho A^{a_3 \mu} + \partial_{(\mu} A^{a_2}_{\nu)} A^{a_3}_\rho A^{a_1 \mu} + \partial_{(\mu} A^{a_3}_{\nu)} A^{a_1}_\rho A^{a_2 \mu}\Big)\frac{1}{\Box}(A^{a_4 \nu} A^{a_5 \rho})$$

at 6pts the ambiguity contains 30 free parameters!

Introduce auxiliary fields to make Lagr. manifestly cubic:

$$\mathcal{L}_{YM} = \tfrac{1}{2} A^{a\mu}\Box A^a_\mu - B^{a\mu\nu\rho}\Box B^a_{\mu\nu\rho} - g f^{abc}(\partial_\mu A^a_\nu + \partial^\rho B^a_{\rho\mu\nu}) A^{b\mu} A^{c\nu}$$

@4pts: one aux. field
@5pts: five aux. fields

also see Tolotti & Weinzierl

Cheung-Shen Lagrangian

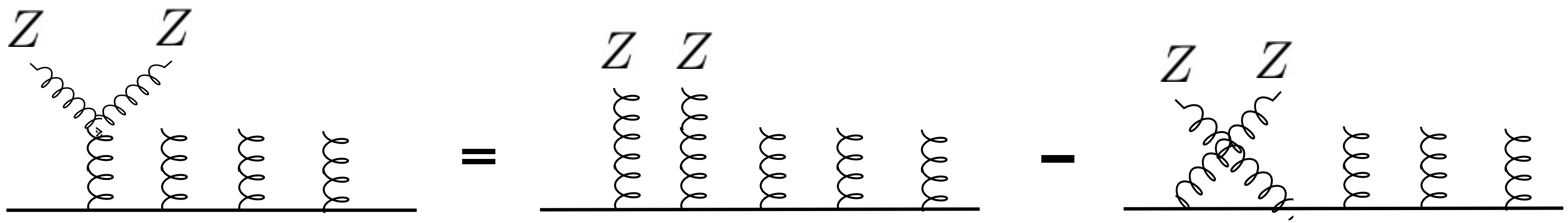
Cubic Lagrangian that manifests color-kinematics duality, gives:
→ NLSM pions at tree level
→ YM trees for MHV sector

Cheung, Shen ('16)

$$\mathcal{L}_{\mathrm{CS}} = Z^{a\mu}\Box X^a_\mu + \frac{1}{2}Y^a\Box Y^a - g f^{abc} Z^{a\mu}\left(Z^{b\nu}X^c_{\mu\nu} + Y^b\partial_\mu Y^c\right)$$

Jacobi Id. manifest:

$$X^a_{\mu\nu} = \partial_\mu X^a_\nu - \partial_\nu X^a_\mu \,;$$



NLSM pions: external states Y^a or $\partial_\mu Z^{a\mu}$

Gives all YM numerator terms of type: $n^{\mathrm{YM}} \sim \ (\varepsilon_1\cdot\varepsilon_n)\prod_{i,j}(\varepsilon_i\cdot p_j)$

sufficient for MHV amplitude: Chen, HJ, Teng, Wang

Stratify kinematic algebra by MHV sectors

Chen, HJ, Teng, Wang [1906.10683]

polarization power one: $(\varepsilon_i \cdot \varepsilon_j) \prod (\varepsilon_k \cdot p_l) \longrightarrow$ MHV

polarization power two: $(\varepsilon_{i_1} \cdot \varepsilon_{j_1})(\varepsilon_{i_2} \cdot \varepsilon_{j_2})(p_{i_3} \cdot p_{j_3}) \prod (\varepsilon_k \cdot p_l) \longrightarrow$ NMHV

The sectors map to different helicity amplitudes via the choice

$$\varepsilon_i^+ = \varepsilon_i^+(p_i, q^-) \qquad \varepsilon_j^- = \varepsilon_j^-(p_j, q^+)$$

Ref. mom. $q^- = p_{j_1} \qquad q^+ = p_{i_1}$

→ only. $\varepsilon_i^+ \cdot \varepsilon_j^- \neq 0$ for $|i| = N_+ - 1$, $|j| = N_- - 1$

Numerators stratified by MHV sectors:

$$n^{\mathrm{BCJ}} \sim \prod_{i,j}^{\min\{|i|,|j|\}} (\varepsilon_i \cdot \varepsilon_j) \prod_{k,l}^{\min\{|i|,|j|\}-1} (p_k \cdot p_l) \prod (\varepsilon \cdot p)$$

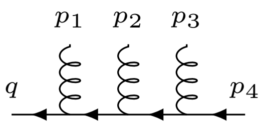
Note: standard Feynman rules compute the MHV sector → unique sector

Kinematic algebra beyond MHV sector

Chen, HJ, Teng, Wang

Consider BCJ numerator in massless QCD

$$n\Bigg($$

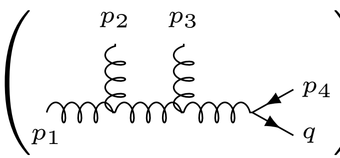


$$\Bigg) = \bar{v}\slashed{\varepsilon}_1(\slashed{p}_1+\slashed{q})\slashed{\varepsilon}_2(\slashed{p}_{12}+\slashed{q})\slashed{\varepsilon}_3 u + \frac{1}{3}(p_1+q)^2\,\bar{v}[\slashed{\varepsilon}_1,\slashed{\varepsilon}_2]\slashed{\varepsilon}_3 u$$

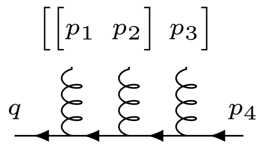
$$= \bar{v}\slashed{\varepsilon}_1\slashed{p}_1\slashed{\varepsilon}_2\slashed{p}_{12}\slashed{\varepsilon}_3 u + \mathcal{O}(q)\,,$$

Can be used to get pure YM numerator

$$n(123;\bar{v}u) \equiv n\Bigg($$



$$\Bigg) = n\Bigg($$



$$\Bigg)$$

$$= \bar{v}\slashed{\varepsilon}_1\slashed{p}_1\slashed{\varepsilon}_2\slashed{p}_{12}\slashed{\varepsilon}_3 u - \bar{v}\slashed{\varepsilon}_2\slashed{p}_2\slashed{\varepsilon}_1\slashed{p}_{12}\slashed{\varepsilon}_3 u - \bar{v}\slashed{\varepsilon}_3\slashed{p}_3\slashed{\varepsilon}_1\slashed{p}_{13}\slashed{\varepsilon}_2 u + \bar{v}\slashed{\varepsilon}_3\slashed{p}_3\slashed{\varepsilon}_2\slashed{p}_{23}\slashed{\varepsilon}_1 u + \mathcal{O}(q)$$

→ Motivate buliding blocks using tensor curents of the Clifford algebra

$$\bar{v}\slashed{\varepsilon}_i\cdots\slashed{p}_k\cdots\slashed{\varepsilon}_j\cdots\slashed{p}_l u$$

$$\bar{v}\gamma^{\mu_i}\cdots\gamma^{\mu_k}\cdots\gamma^{\mu_j}\cdots\gamma^{\mu_l}u \quad \equiv \quad J^{\mu_i\cdots\mu_k\cdots\mu_j\cdots\mu_l}$$

Current algebra

Chen, HJ, Teng, Wang

Formalize using current algebra tools

off-shell vs. on-shell currents $J^{(w)}_{\mathfrak{a}_1\otimes\mathfrak{a}_2\otimes\cdots\otimes\mathfrak{a}_m}(p) \to \bar{v}(q)\not{\mathfrak{a}}_1\not{\mathfrak{a}}_2\cdots\not{\mathfrak{a}}_m u(p)$

linearity of tensors $J^{(w)}_{\cdots\otimes(x\mathfrak{a}_i+y\mathfrak{a}'_i)\otimes\cdots}(p) = xJ^{(w)}_{\cdots\otimes\mathfrak{a}_i\otimes\cdots}(p) + yJ^{(w)}_{\cdots\otimes\mathfrak{a}'_i\otimes\cdots}(p)$

Clifford alg. $J^{(w)}_{\cdots\mathfrak{a}_i\otimes\mathfrak{a}_j\otimes\mathfrak{a}_k\otimes\mathfrak{a}_l\otimes\cdots}(p) + J^{(w)}_{\cdots\mathfrak{a}_i\otimes\mathfrak{a}_k\otimes\mathfrak{a}_j\otimes\mathfrak{a}_l\otimes\cdots}(p) = (2\mathfrak{a}_j\cdot\mathfrak{a}_k)J^{(w)}_{\cdots\otimes\mathfrak{a}_i\otimes\mathfrak{a}_l\otimes\cdots}(p)$

Pre-numerators: $N(\sigma) = J_{\hat{\varepsilon}_1}(p_1)\star J_{\varepsilon_{\sigma_2}}(p_{\sigma_2})\star\cdots\star J_{\varepsilon_{\sigma_{n-1}}}(p_{\sigma_{n-1}})$

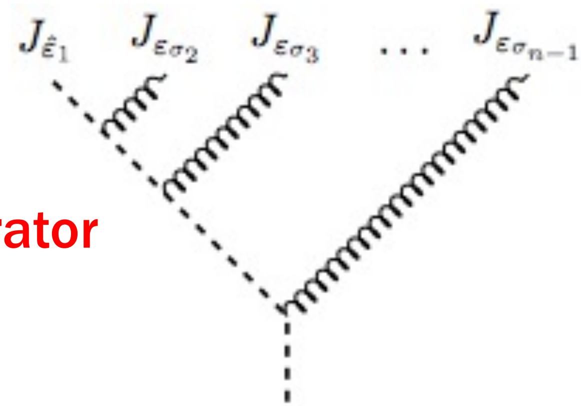
Fusion product $J^{(w)}_{\mathfrak{a}_1\otimes\cdots\otimes\mathfrak{a}_m}(p)\star J_{\varepsilon_i}(p_i) = \sum_{k=1}^{m+2}\sum_{\mathfrak{a}'_1,\ldots,\mathfrak{a}'_k}\sum_{w'} f^{\mathfrak{a}'_1\ldots\mathfrak{a}'_k}_{\mathfrak{a}_1\ldots\mathfrak{a}_m;\varepsilon_i}(p,p_i;w,w')J^{(w')}_{\mathfrak{a}'_1\otimes\cdots\otimes\mathfrak{a}'_k}(p+p_i)$

BCJ numerator from nested commutator: $N([\cdots[[[[1,2],3],4],5],\ldots,m])$

Kinematic algebra at the Next-to-MHV level (NMHV)

Chen, HJ, Teng, Wang [1906.10683]

Simplify by considering YM-scalar numerator:



$$J_{\hat{\varepsilon}_1}(p) \star J_{\varepsilon_i}(p_i) = \varepsilon_i \cdot p \, J_{\hat{\varepsilon}_1}(p+p_i) - \frac{1}{2} J_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes (p+p_i)}(p+p_i)$$

vector generator

tensor generator

$$J_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes p}(p) \star J_{\varepsilon_k}(p_k) = \frac{1}{2} p^2 J^{(1)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_k}(p+p_k)$$

$$\begin{aligned} J^{(1)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_j}(p) \star J_{\varepsilon_k}(p_k) &= (\varepsilon_k \cdot p) J^{(2)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_j}(p+p_k) \\ &\quad + (\varepsilon_j \cdot p) J^{(1)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_k}(p+p_k) - (\varepsilon_i \cdot p) J^{(1)}_{\hat{\varepsilon}_1 \otimes \varepsilon_j \otimes \varepsilon_k}(p+p_k) \\ J^{(2)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_j}(p) \star J_{\varepsilon_k}(p_k) &= (\varepsilon_k \cdot p) J^{(2)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_j}(p+p_k) \, . \end{aligned}$$

Five types of tensors: $J_{\hat{\varepsilon}_1}\,, \quad J_{\varepsilon_i}\,, \quad J_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes p}\,, \quad J^{(1)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_j}\,, \quad J^{(2)}_{\hat{\varepsilon}_1 \otimes \varepsilon_i \otimes \varepsilon_j}$

Closed form for numerators:

vector: $N^{(1)}(123\cdots n) = \left(\prod_{j=2}^{n-1} \varepsilon_j \cdot p_{1\cdots j-1} \right) J_{\hat{\varepsilon}_1}$

tensor: $N^{(2)}(123\cdots n) = \frac{1}{2} \sum_{i=2}^{n-2} \sum_{\substack{\ell,m=i \\ m>\ell}}^{n-1} (-1)^{\ell-i-1} s_{1\cdots i} \left(\prod_{j \in \mathsf{S}_{i\ell m}} \varepsilon_j \cdot p_{1\cdots j-1} \right) \det(\mathbf{P}_{[i,\ell-1]}) J_{\hat{\varepsilon}_1 \otimes \varepsilon_\ell \otimes \varepsilon_m}$

Kinematic algebra for NMHV

Chen, HJ, Teng, Wang [2104.12726]

Full NMHV sector algebra generatred by 13 currents

$$\begin{aligned} &\text{vector:} && J_{\varepsilon_i}\,, \quad J^{(2)}_{\varepsilon_i}\,, \quad J^{(1)}_{p_i}\,, \quad J^{(2)}_{p_i} \\ &\text{tensor:} && J_{\varepsilon_i\otimes\varepsilon_j\otimes p}\,, \quad J_{p_i\otimes\varepsilon_j\otimes p}\,, \quad J^{(1)}_{\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_k}\,, \quad J^{(2)}_{\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_k} \\ &&& J^{(1)}_{p_i\otimes\varepsilon_j\otimes\varepsilon_k}\,, \ldots\,, J^{(5)}_{\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_k}\,. \end{aligned}$$

Again closed formula for pre-numerator

$$\mathcal{N}(1,2,\ldots,n) = \Big(1+\frac{1}{2}Q_n\Big)\mathcal{N}_V^{(1)} + \frac{1}{4}(1+Q_n)\mathcal{N}_T^{(2)} + \mathcal{N}_V^{(2)}$$

Explored pure-gauge BCJ numerators (non-uniqueness of NMHV)

crossing symmetry	Total d.o.f. $N^{\text{gauge}} \sim (\varepsilon\cdot\varepsilon)^2(p\cdot p)\prod\varepsilon\cdot p$			
	$n=4$	$n=5$	$n=6$	$n=7$
S_{n-2}	1	36	760	16583
S_{n-1}	0	8	148	2734
S_n	0	1	25	381

Non-uniqueness of numerators

Amplitudes and numerators linearly related

$$A(1,\sigma,n) = \sum_{\rho\in S_{n-2}} m(\sigma|\rho) N(1,\rho,n)$$

Propagator matrix $m(\sigma|\rho)$ (bi-adjoint scalar ampl.)
has only rank $(n-3)!$

Give rise to BCJ amplitude relations, and generalized gauge freedom

$$N(1,2,\ldots,n-1,n) \sim N(1,2,\ldots,n-1,n) + N^{\text{gauge}}(1,2,\ldots,n-1,n)$$

Pure gauge numerators: $\displaystyle\sum_{\rho\in S_{n-2}} m(\sigma|\rho) N^{\text{gauge}}(1,\rho,n) = 0$

However, for off-shell kinematics (or massive fund. matter) the kernel becomes trivial → can be exploited to obtain gauge inv. numerators

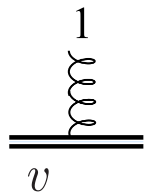
Hopf algebra structure and heavy mass EFT

Gauge invariant BCJ numerators from heavy-quark limit

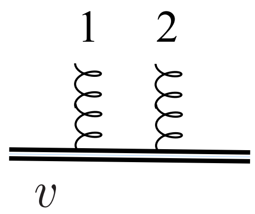
Brandhuber, Chen, HJ, Travaglini, Wen '21

$$F_i^{\mu\nu} := p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu$$
$$V_\tau^{\mu\nu} := v^\mu \sum_{j\in\tau} p_j^\nu$$

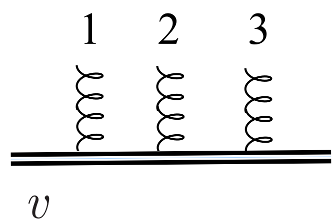
$$\mathcal{N}(1,v) = v\cdot\varepsilon_1\,,$$



$$\mathcal{N}(12,v) = -\frac{v\cdot F_1\cdot F_2\cdot v}{2v\cdot p_1}\,,$$



$$\mathcal{N}(123,v) = \frac{v\cdot F_1\cdot F_2\cdot F_3\cdot v}{3v\cdot p_1} - \frac{v\cdot F_1\cdot F_2\cdot V_{12}\cdot F_3\cdot v}{3v\cdot p_1 v\cdot p_{12}} - \frac{v\cdot F_1\cdot F_3\cdot V_1\cdot F_2\cdot v}{3v\cdot p_1 v\cdot p_{13}}$$

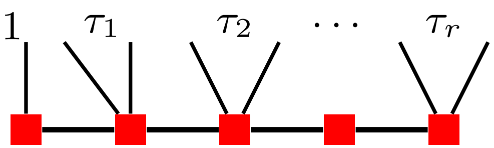


YM numerators at any multiplicity given by an associative Hopf algebra

$$\mathcal{N}(12\ldots n{-}2,v) := \langle T_{(1)}\star T_{(2)}\star\cdots\star T_{(n-2)}\rangle$$

Quasi-shuffle product: $T_{(12)}\star T_{(3)} = -T_{(123)} + T_{(12),(3)} + T_{(13),(2)}$

$$\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)}\rangle :=$$



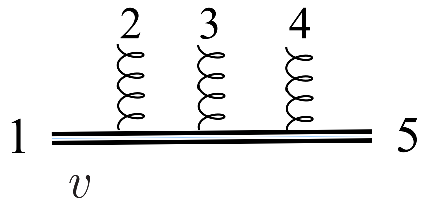
$$= \frac{v\cdot F_{1\tau_1}\cdot V_{\Theta(\tau_2)}\cdot F_{\tau_2}\cdots V_{\Theta(\tau_r)}\cdot F_{\tau_r}\cdot v}{(n{-}2)v\cdot p_1 v\cdot p_{1\tau_1}\cdots v\cdot p_{1\tau_1\tau_2\cdots\tau_{r-1}}}$$

Heavy mass numerators for PM calculations

Efficient calculations for Post Minkovskian corrections:

Brandhuber, Chen, Travaglini, Wen [2108.04216]

$$A_5^{\mathrm{YM-M}}(234,v) = \frac{\mathcal{N}_5([[2,3],4],v)}{s_{234}s_{23}} + \frac{\mathcal{N}_5([2,[3,4]],v)}{s_{234}s_{34}}$$

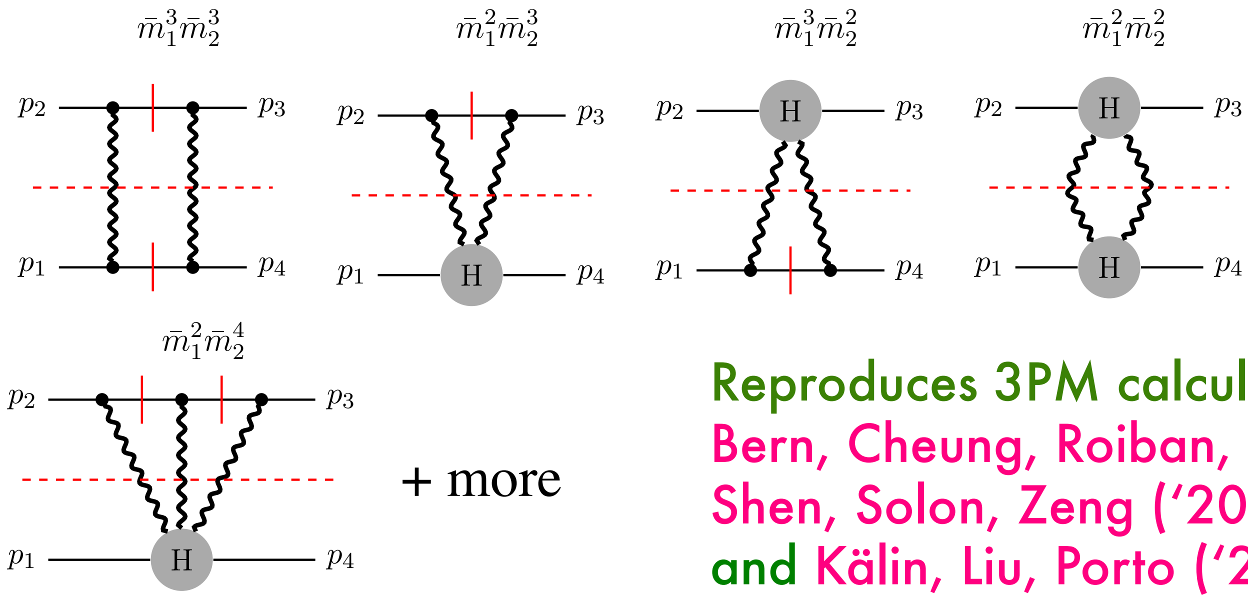


Double copy for massive scalar (Schwarzschild BH):

$$A_5^{\mathrm{GR-M}}(234,v) = \frac{\left[\mathcal{N}_5([[2,3],4],v)\right]^2}{s_{234}s_{23}} + \frac{\left[\mathcal{N}_5([[2,4],3],v)\right]^2}{s_{234}s_{24}} + \frac{\left[\mathcal{N}_5([[3,4],2],v)\right]^2}{s_{234}s_{34}}$$

2PM:

3PM:



+ more

Reproduces 3PM calculations of Bern, Cheung, Roiban, Shen, Solon, Zeng ('20) and Kälin, Liu, Porto ('20)

First complete Kinematic Lie Algebra

Recent surprise:

Ben-Shahar, HJ

A complete QFT with straightforward kinematic algebra at tree and loop level.

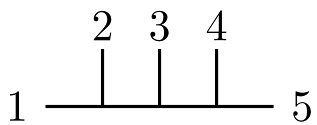
Generators $$L^\mu(p) = e^{ip\cdot x}\Delta^{\mu\nu}\partial_\nu$$

3D transversality "projector" $$\Delta^{\mu\nu}(p) = i\epsilon^{\rho\mu\nu}p_\rho$$

Infinite-dimensional kinematic Lie algebra $$[L^\mu(p_1), L^\nu(p_2)] = F^{\mu\nu}{}_\rho L^\rho(p_1+p_2)$$

Kinematic structure constants $$F^{\mu_1\mu_2}{}_\nu(p_1,p_2) = \Delta^{\rho\mu_1}(p_1)\epsilon_{\rho\nu\sigma}\Delta^{\sigma\mu_2}(p_2)$$

BCJ numerators



$$= \mathrm{tr}\Big([[[L^{\mu_1}(p_1), L^{\mu_2}(p_2)], L^{\mu_3}(p_3)], L^{\mu_4}(p_4)], L^{\mu_5}_{\mathrm{amp}}(p_5)\Big)$$

$$= F^{\mu_1\mu_2}{}_\nu F^{\nu\mu_3}{}_\rho F^{\rho\mu_4\mu_5}\delta^3(p_1+p_2+p_3+p_4+p_5)\ ,$$

Lie algebra of 3D volume-preserving diffeomorphisms!

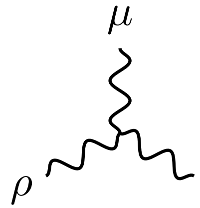
Chern-Simons theory – off-shell C/K duality

Ben-Shahar, HJ

Pure Chern-Simons theory (tree-level action)

$$S = \frac{k}{4\pi}\int \mathrm{Tr}\Big(A\wedge dA + \frac{2i}{3}A\wedge A\wedge A\Big)$$

cubic Feynman rules: $A_\mu \sim\!\sim\!\sim A_\rho = -\frac{\epsilon_{\mu\nu\rho}p^\nu}{p^2}$

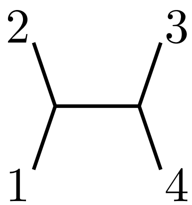


$= -\frac{\epsilon^{\mu\nu\rho}}{\sqrt{2}}$

in Lorenz gauge these obeys color-kinematics duality off shell !

Chern-Simons is toplogical → amplitudes vanish,
but off-shell correlation fn's are non-zero

Correlators automatically in Lorenz gauge $A^\mu = \frac{b^{\mu\nu}}{\Box}J_\nu \quad\Rightarrow\quad \partial_\mu A^\mu = 0$



$$= \epsilon^{\varepsilon_1\varepsilon_2\nu}\epsilon_{\nu p_{12}\mu}\epsilon^{\mu\varepsilon_3\varepsilon_4}$$

$$= \varepsilon_4\cdot p_3\epsilon^{\varepsilon_1\varepsilon_2\varepsilon_3} - \varepsilon_1\cdot p_2\epsilon^{\varepsilon_2\varepsilon_3\varepsilon_4} - \varepsilon_2\cdot p_1\epsilon^{\varepsilon_3\varepsilon_1\varepsilon_4} - \varepsilon_3\cdot p_4\epsilon^{\varepsilon_1\varepsilon_2\varepsilon_4}$$

Superfield notation

Ben-Shahar, HJ

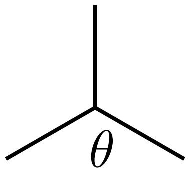
Quantum CS action includes Faddeev-Popov ghosts.

Can be packaged into superfield: $\Psi = c + \theta_\mu A^\mu + \theta_\mu \theta_\nu C^{\mu\nu} + \theta_1\theta_2\theta_3 a$

Pure Chern-Simons theory (quantum action)

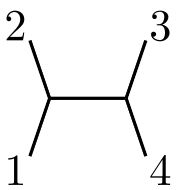
$$S = \frac{k}{2\pi}\int d^3x d^3\theta \,\mathrm{Tr}\Big(\frac{1}{2}\Psi Q \Psi + \frac{i}{3}\Psi\Psi\Psi\Big)$$

Feynman rules: $\theta \xrightarrow[p]{\rightarrow} \tilde\theta \;=\; \frac{p\cdot\vartheta}{p^2}\delta^3(\theta-\tilde\theta)$



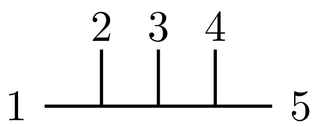
$= \; i\int d^3\theta$

again obeys color-kinematics duality!



$$= \int d^3\theta \Psi_1\Psi_2 \int d^3\tilde\theta \, p_{34}\cdot\vartheta\, \delta^3(\theta-\tilde\theta)\Psi_3(\tilde\theta)\Psi_4(\tilde\theta) \;\; = i\int d^3\theta b(\Psi_1\Psi_2)\Psi_3\Psi_4$$

5pts:



$= i\int d^3\theta\, b(b(\Psi_1\Psi_2)\Psi_3)\Psi_4\Psi_5$ similarly at higher pts

Poisson algebra & diffeomorphisms

Ben-Shahar, HJ

Represent propagator numerator with differential operator :

$$b = \frac{\partial}{\partial\theta^\mu}\partial_\mu \equiv \vartheta^\mu\partial_\mu$$

acting on product of fields gives

$$b(\Psi_1\Psi_2) \overset{\text{Lorenz}}{=} \vartheta^\mu\Psi_1\partial_\mu\Psi_2 - \partial_\mu\Psi_1\vartheta^\mu\Psi_2 \equiv \{\Psi_1, \Psi_2\}_{\mathrm{P}}$$

Jaobi identity follows from Poisson bracket

$$b(b(\Psi_1\Psi_2)\Psi_3) + \text{cyclic}(1,2,3) = \{\{\Psi_1, \Psi_2\}_{\mathrm{P}}, \Psi_3\}_{\mathrm{P}} + \text{cyclic}(1,2,3) = 0$$

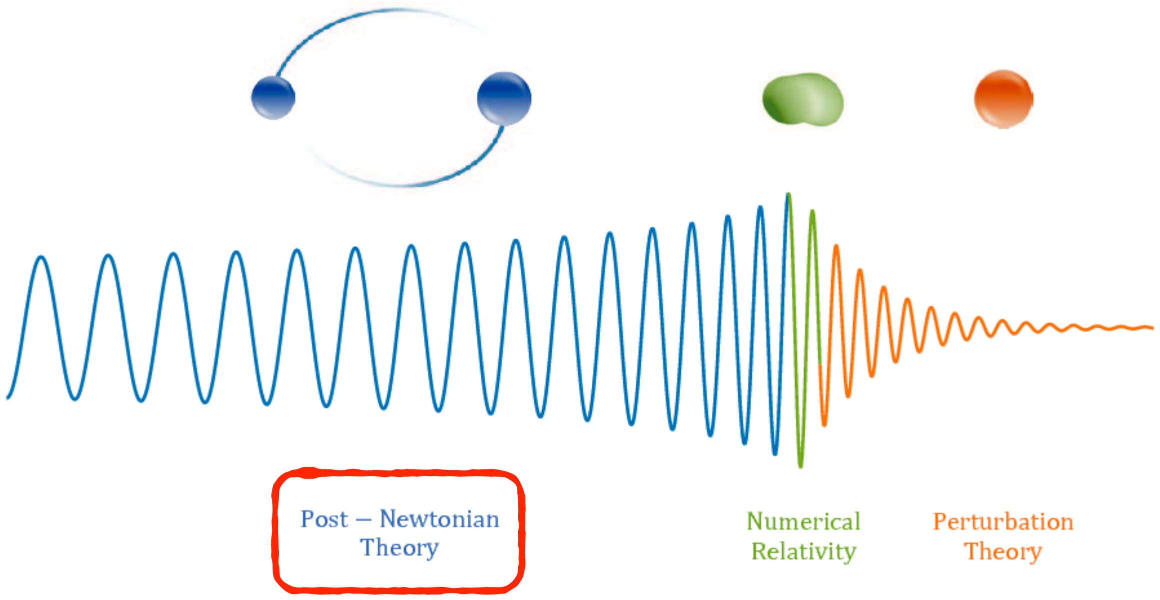
Lie algebra generators: $L_\Psi \equiv \vartheta^\mu\Psi\partial_\mu - \partial_\mu\Psi\vartheta^\mu$

Lie algebra: $[L_{\Psi_1}, L_{\Psi_2}] = L_{b(\Psi_1\Psi_2)}$

Volume-preserving diffeomorphisms (bosonic & fermonic volume)

Double copy and black hole amplitudes

Double copy and gravitational waves



Explicit PM calculations done using double copy:

Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove (‘18)
Bern, Cheung, Roiban, Shen, Solon, Zeng (‘19)+ Ruf, Parra-Martinez (‘21)
Brandhuber, Chen, Travaglini, Wen (21)

Some methods developed for PM calc. using double copy:

Bjerrum-Bohr, Cristofoli, Damgaard, Gomez+Brown;
Cristofoli, Gonzo, Kosower, O’Connell;
Maybee, O’Connell, Vines; Luna, Nicholson, O'Connell, White; …

AHH amplitudes $\leftrightarrow$ Kerr BH?

Arkani-Hamed, Huang, Huang ('17) wrote down natural higher-spin ampl's:

Gauge th 3pt:

$$A(1\phi^s, 2\bar{\phi}^s, 3A^+) = mx\frac{\langle\mathbf{12}\rangle^{2s}}{m^{2s}}\,, \qquad A(1\phi^s, 2\bar{\phi}^s, 3A^-) = \frac{m}{x}\frac{[\mathbf{12}]^{2s}}{m^{2s}}$$

Gravity 3pt:

$$M(1\phi^s, 2\bar{\phi}^s, 3h^+) = im^2x^2\frac{\langle\mathbf{12}\rangle^{2s}}{m^{2s}}\,, \qquad M(1\phi^s, 2\bar{\phi}^s, 3h^-) = i\frac{m^2}{x^2}\frac{[\mathbf{12}]^{2s}}{m^{2s}}$$

Shown to reproduce Kerr by: Guevara, Ochirov, Vines ('18)

Gravity Compton ampl.
via BCFW recursion ?

$$M(1\phi^s, 2\bar{\phi}^s, 3h^+, 4h^+) = i\frac{\langle\mathbf{12}\rangle^{2s}[34]^4}{m^{2s-4}s_{12}t_{13}t_{14}}$$

$$M(1\phi^s, 2\bar{\phi}^s, 3h^-, 4h^+) = i\frac{[4|p_1|3\rangle^{4-2s}([4\mathbf{1}]\langle 3\mathbf{2}\rangle + [4\mathbf{2}]\langle 3\mathbf{1}\rangle)^{2s}}{s_{12}t_{13}t_{14}}\,.$$

spurious pole for $s > 2$

Kerr Black hole amplitudes

The AHH ampl's for $s \leq 2$ admit double copies to any multiplicity

$$(\text{YM} + \text{scalar}) \otimes (\text{YM} + \text{scalar}) = (\text{GR} + \text{scalar})$$

$$(\text{YM} + \text{scalar}) \otimes (\text{YM} + \text{fermion}) = (\text{GR} + \text{fermion})$$

$$(\text{YM} + \text{scalar}) \otimes (\text{YM} + \text{W-boson}) = (\text{GR} + \text{Proca})$$

$$(\text{YM} + \text{W-boson}) \otimes (\text{YM} + \text{fermion}) = (\text{GR} + \text{massive gravitino})$$

$$(\text{YM} + \text{W-boson}) \otimes (\text{YM} + \text{W-boson}) = (\text{GR} + \text{massive KK graviton})$$

Lagrangians unique: have no non-minimal terms beyond cubic order in fields

Can be used for $(S^{\mu})^{\leq 4}$ PM/PN calculations.
Compton $(S^{\mu})^{4}$ yet to be confirmed via other methods (BHPT, worldline).

Ochirov, HJ; Chiodaroli, HJ, Pichini; Chiodaroli, Gunaydin, HJ, Roiban, [...]

Summary & Outlook

- Color-kinematics duality lies at the root of gravity:
  → Diffeomorphism inv. from YM numerator relations
  → makes perturbative GR more manageable!
  → allows for simpler classification of gravity theories

- Kinematic algebra is a well-hidden gem of YM (and GR)
  → Notoriously difficult, after 15 years no complete YM algebra
  → Much progress in last 3 years:
  → Stratification by sectors: NMHV algebra via current algebra
  → Gauge invariant numerators ←→ heavy mass EFT ←→ Hopf algebra
  → First complete (quantum) kinematic algebra Chern-Simons theory
  → Not discussed: Covariant color-kinematics duality Cheung & Mangan

- Explored amplitudes for massive spinning matter → Kerr BH ?
  → Double copy works well up to spin-2 (KK graviton)

<u>Take-home message from this MITP workshop:</u>
work on double copy & CK duality has significantly broadened in the last few years, and I look forward to hearing all the new perspectives!