Colour–Kinematics Duality and the Double Copy: A Lagrangian and Homotopy Algebraic Perspective

Leron Borsten

Maxwell Institute for Mathematical Sciences, Edinburgh
Heriot–Watt University, Edinburgh
Imperial College London

Rebuilding the Tower of Babel: Bringing Together the Various Languages of Color–Kinematics Duality

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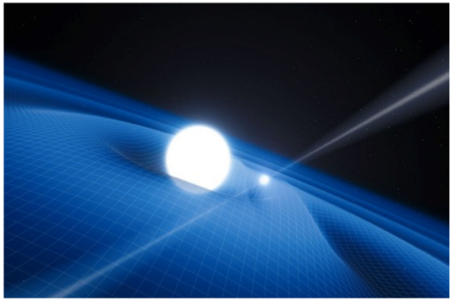
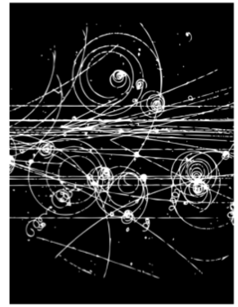
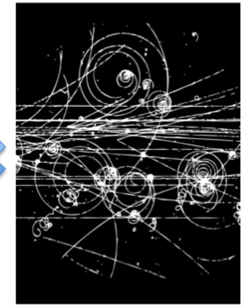
Based on joint work 2007.13803, 2102.11390, 2108.03030 and 22xx.xxxxx with Branislav Jurčo, Hyungrok Kim, Tommaso Macrelli, Christian Saemann and Martin Wolf (BJKMSW)

Gravity $=$ Gauge $\times$ Gauge



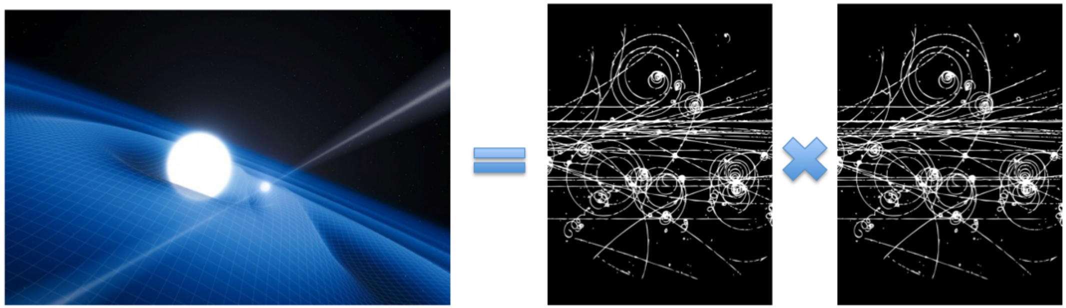




$$g_{\mu\nu} \qquad A_{\mu}{}^{a} \qquad A_{\nu}{}^{b}$$

- Is gravity the double copy of the other fundamental forces of Nature?
- Long history and many guises [Feynman; Papini; Kawai, Lewellen, Tye; Berends, Giele, Kuijf; Bern, Dixon, Dunbar, Perelstein , Rozowsky. . .]

Gravity = Gauge × Gauge



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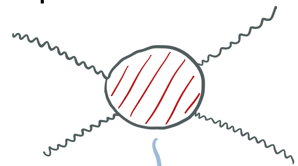
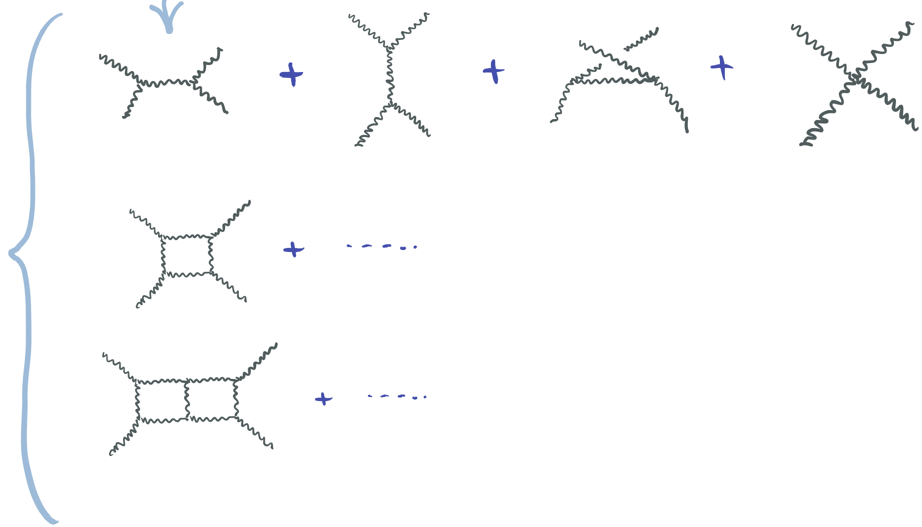
- Long history and many guises **[Feynman; Papini; Kawai, Lewellen, Tye; Berends, Giele, Kuijf; Bern, Dixon, Dunbar, Perelstein , Rozowsky. . .]**
- Renaissance: Bern–Carrasco–Johansson Colour–Kinematics (CK) duality conjecture and double copy of gauge theory and gravity scattering amplitudes **[Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]**

Scattering Amplitudes

→ Physical observables tested at particle accelerators (e.g. Large Hadron Collider)



Feynman diagram expansion



Explosion of complexity

Hidden simplicity:
On-shell amplitudes paradigm

Deep, hidden
structures

→ New insights into the underlying theories themselves

Colour–Kinematics Duality

- Amplitude for gluons to scatter schematically:

Colour numerators $c \sim f_{ab}{}^{c} f_{cd}{}^{e} \cdots$
colour/gauge group data of gluons

Kinematic numerators $n \sim \varepsilon_{\mu} p^{\mu} \cdots$
polarisation and momentum data of gluons

$$\mathcal{A}^{n,L}_{\text{gluons}} \sim \sum_i \int_{\text{loops}} \frac{c_i n_i}{d_i}$$

Sum over cubic diagrams

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Sum over cubic diagrams

Kinematic Jacobi identity

- Bern-Carrasco-Johansson CK duality conjecture 2008:

Jacobi identity

$$c_i + c_j + c_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0$$

- Proven at tree level **[Stieberger '09; Bjerrum, Bohr, Damgaard, Vanhove '09; Du, Teng '16; Bridges, Mafra '19; Mizera '19; Reiterer '19. . .]**

- Conjectured at loop level with highly non-trivial examples **[Bern, Carrasco, Johansson '08 '10; Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14. . .]**

The Double Copy Prescription

- Assuming CK duality is realised, gravity comes for free:
  [Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]

$$\mathcal{A}^{n,L}_{\text{gluons}} \sim \sum_i \int_{\text{loops}} \frac{c_i n_i}{d_i}$$

Double copy kinematics

$$c_i \longrightarrow n_i$$

$$\mathcal{A}^{n,L}_{\text{gravitons}} \sim \sum_i \int_{\text{loops}} \frac{n_i n_i}{d_i}$$

- 'Gluons for (almost) nothing, gravitons for free' JJ Carrasco

Generalisations, Implications and Applications

- Growing zoology of generalisations: ϕ^3 theory, Maxwell/scalar/Yang-Mills supergravity, gauged supergravity (Minkowski vacua), non-linear sigma model, pure gravity, Born-Infeld, conformal gravity, Z-theory, Navier-Stokes fluids, topologically massive Yang, Mills, geometric/world-sheet and pure spinor formalisms, ambitwistor string theories, scattering equations, non-trivial gluon and spacetime backgrounds, special Galileons, massive gravity, EFT... [Hodges '11; Broedel, Carrasco '11; Bern, Boucher–Veronneau, Johansson '11; Broedel, Dixon '12; Bargheer, He,McLoughlin '12; Huang, Johansson '12; Cachazo, He, Yuan '13 '14; Dolan, Goddard '13; Naculich '14 '15; Mason, Skinner '13; Adamo, Casali, Skinner '13; Adamo, Casali, Mason, Nekovar '17 '18; Geyer, Monteiro '18; LB '18; Geyer, Mason '19; LB-Duff-Marrani '19; Geyer, Monteiro, Stark, Muchão '21; Chiodaroli, Günaydin, Johansson, Roiban '14 '15; Johansson, Ochirov '15 '16; Chiodaroli, Günaydin, Johansson, Roiban '17; Carrasco, Mafra, Schlotterer '16; Johansson, Nohle '17; Azevedo, Chiodaroli, Johansson, Schlotterer '18; Farrow, Lipstein, McFadden '19; Momeni, Rumbutis, Tolley '20; Johnson, Jones, Paranjape '20; Zhou '21; Diwakar, Herderschee, Roiban, Teng '21; González, Momeni, Rumbutis '21; Cheung, Parra-Martínez, Sivaramakrishnan '22, González, Liang, Mark Trodden '22...]
- Uncovering new facets of perturbative quantum gravity: miraculous cancelations, anomalies, unchartered amplitudes ... [Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14; Bern, Carrasco, Chen, Edison, Johansson, Parra-Martínez, Roiban, Zeng '18; Bern, Kosower, Parra, Martínez '20...]
- Applications to classical general relativity: (non)perturbative solutions, black holes merger modelling [Monteiro, O'Connell, White '14; Cardoso, Nagy, Nampuri '16; Luna, Monteiro, Nicholson, Ochirov, O'Connell, Westerberg, White '16; Berman, Chacón, Luna, White '18; Kosower, Maybee, O'Connell '18; Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Bern, Luna, Roiban, Shen, Zeng '20; Chacón-Nagy, White '21; Adamo, Cristofoli, Ilderton '22...]

Elucidating Colour–Kinematics Duality and the Double Copy

Manifold perspectives (hopelessly partial list)

- Structural properties, e.g. geometric, graph theoretic, analytic, algebraic [Carrasco, Johansson '11; Broedel, Carrasco '11; de la Cruz, Kniss, Weinzierl '17; Mizera '19; Reiterer '19. . .]
- String theory and pure spinors [Stieberger '09; Bjerrum, Bohr, Damgaard, Vanhove '09; Mafra, Schlotterer, Stieberger '11; Broedel, Schlotterer, Stieberger '13; Mafra, Schlotterer '14 '15; Carrasco, Mafra, Schlotterer '16; Casali, Mizera, Tourkine '20; Bridges, Mafra '21. . .]
- Kinematic Algebras [Monteiro, O'Connell '11, '13; Bjerrum, Bohr, Damgaard, Monteiro, O'Connell '12; Fu, Krasnov '16; Chen, Johansson, Teng, Wang 19; Campiglia, Nagy '21; Frost, Mason '20; Brandhuber, Chen, Johansson, Travaglini, Wen '21. . .]
- Classical double copy [Monteiro, O'Connell, White '14; Luna, Monteiro, O'Connell, White '15; Cardoso, Nagy, Nampuri '16; Berman, Chacón, Luna, White '18; Bahjat, Abbas, Stark, Muchão, White '20; White '20; Chacón-Nagy-White '21, Emond, Moynihan '22. . .]
- Ambitwistors and scattering equations [Cachazo, He, Yuan '13 '14; Mason, Skinner '13; Adamo, Casali, Skinner '13; Adamo, Casali, Mason, Nekovar '17 '18; Geyer, Monteiro '18; Geyer, Mason '19; Geyer, Monteiro, Stark, Muchão '21. . .]
- KLT bootstrap [Chi, Elvang, Herderschee, Jones, Paranjape '21]
- Covariant CK duality [Cheung, Mangan '21; Moynihan '21. . .]
- Lagrangian CK duality and double copy [Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13; Cheung, Shen '16; LB, Nagy '20; Ben–Shahar, Johansson '21. . .]

Lagrangian Perspective

Action manifesting physical tree-level CK duality

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

4-point vertex auxiliary

$$
\begin{aligned}
S^{\mathrm{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int d^D x\, & \tfrac{1}{2}A_\mu \Box A^\mu + \tfrac{1}{2}g\partial_\mu A_\nu[A^\mu, A^\nu] \\
& + \tfrac{1}{2}B^{\mu\nu\kappa}\Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}}\partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + C^{\mu\nu}\Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa}\Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda}\Box \bar{C}_{\mu\nu\kappa\lambda} + \\
& + gC^{\mu\nu}[A_\mu, A_\nu] + g\partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2}\partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu}A_{\kappa]}, A_\lambda] \\
& + g\bar{C}^{\mu\nu}(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]) + \cdots
\end{aligned}
$$

Identically zero:
Colour Jacobi

5-point vertex auxiliary

- Purely cubic Feynman diagrams:

$$
A^{n,0}_{\mathrm{YM}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

[Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13; BJKMSW '21]

The Becchi–Rouet–Stora–Tyutin Complex

- Off-shell symmetry considerations lead one to naturally consider the full BRST or Batalin–Vilkovisky (BV) complex

Antifields

Nakanishi–Lautrup

Ghost

$$
\begin{tikzcd}
 & \overset{A^a_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \arrow[r, "\delta^\mu_\nu\Box-\partial_\nu\partial^\mu"] & \overset{A^{+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \arrow[ddr, "-\partial^\mu"] & \\
 & \overset{b^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} \arrow[dr, "1"] & \overset{b^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & \\
\underbrace{\overset{c^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_0} \arrow[uur, "-\partial_\mu"] & \underbrace{\overset{\bar{c}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_1} \arrow[ur, "-1"] & \underbrace{\overset{\bar{c}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_2} & \underbrace{\overset{c^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_3}
\end{tikzcd}
$$

[Anastasiou, LB, Duff, Hughes, Nagy Zoccali '14 '18; BJKMSW '20, '21, '22]

- Also suggested by original Lagrangian picture [Bern, Dennen, Huang, Kiermaier '10]
- See also the pure spinor BRST cohomology approach e.g. [Mafra, Schlotterer, Stieberger '11; Mafra, Schlotterer '14 '15; Bridges, Mafra '21]

Logical Conclusion

- CK duality can be realised as an infinite dimensional anomalous symmetry of Yang, Mills BRST action [BJKMSW '20, '21]

Kinematic algebra index
(DeWitt index over all field species
with ∞ tower of auxiliaries)

Colour metric

Kinematic metric

Colour algebra index

$$C_{ij}c_{ab}A^{ia}\Box A^{ja} + F_{ijk}f_{abc}A^{ia}A^{jb}A^{kc}$$

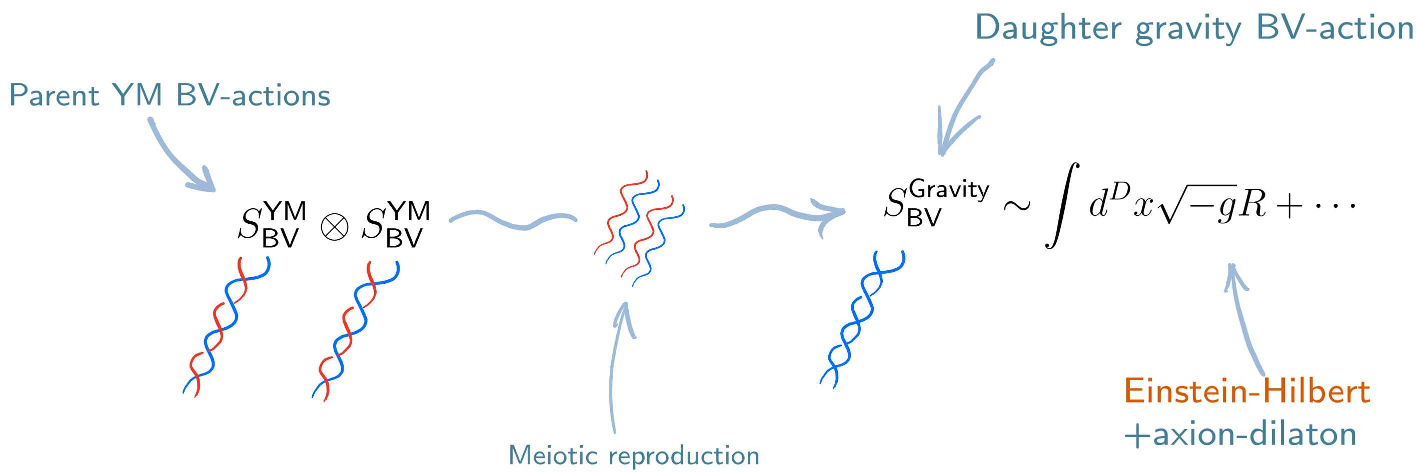
Colour structure constants

Kinematic structure constants

- Natural, but non-standard, notion of CK duality:
  - Natural: symmetry of action
  - Non-standard: loop integrands CK dual, but...
  - ...there is a CK anomaly due to Jacobian counterterms for unitarity
  - Generalised unitarity proof does not apply, at least not straightforwardly
- Double copy of BV action is manifestly valid [BJKMSW '20, '21]

Syngamic Reproduction

- Batalin–Vilkovisky (BV) double copy **[BJKMSW '20; '21, '22 (to appear)]**



(The DNA allusion doesn't actually work, but the picture conveys the idea!)

- Double copy origin of symmetries:

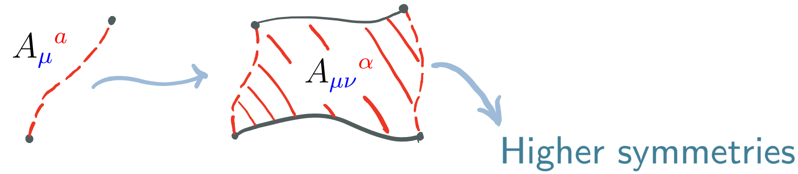
$$\underbrace{(\text{gauge, global susy, R-sym}\ldots)}_{\text{(super) Yang, Mills symmetries}} \longrightarrow \underbrace{(\text{diffeomorphism, local susy, R-sym}\ldots)}_{\text{(super)gravity symmetries}}$$

Homotopy Algebras

- Higher symmetries and gauge theories are everywhere: condensed matter, (T)QFT, SCFT, T-duality, M-theory...

$$A_\mu{}^a \longrightarrow A_\mu{}^a, A_{\mu\nu}{}^\alpha, A_{\mu\nu\rho}{}^i, \ldots$$

Tower of higher gauge fields



- Higher symmetry $\longrightarrow$ homotopy algebras: intersection of category theory, topology, geometry and algebra Mac Lane, Stasheff, Kontsevich, Baez, Markle...

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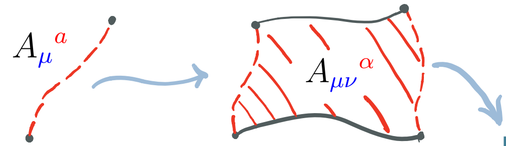


Higher symmetries

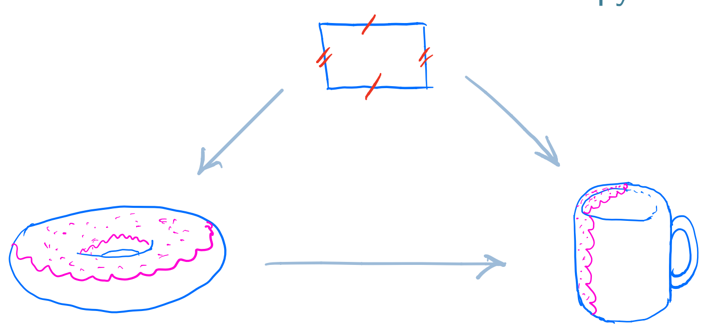
- Higher symmetry $\longrightarrow$ homotopy algebras: intersection of category theory, topology, geometry and algebra Mac Lane, Stasheff, Kontsevich, Baez, Markle...

- Generalise familiar (matrix, Lie. . .) algebras to include higher products:

$$[-,-] \qquad \longrightarrow \qquad [-],\ \ [-,-],\ \ [-,-,-],\ \ [-,-,-,-]\cdots$$

Jacobi relation

Homotopy Jacobi relations



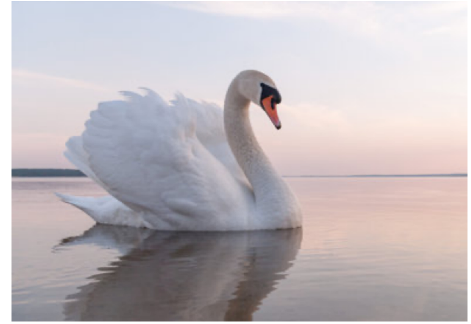
- Homotopy Lie L_∞-algebras: string field theory, quantum field theory, condensed matter/higher Berry connections...

The Homotopy Algebra of Colour-Kinematics Duality

- CK duality: kinematic algebra
  Hands on quantum field theory

- Q: but what is it?

The Homotopy Algebra of Colour-Kinematics Duality



- CK duality: kinematic algebra
  Hands on quantum field theory
- A: $BV_\infty^\square$ homotopy algebra
  Abstract mathematics See Christian's talk
  [BJKMSW '22 (to appear)]

Gluons, quarks. . .
Scattering amplitudes
Symmetries of Nature

$BV_\infty^\square$-algebras —Homotopy transfer— Exact CK duality

$BV_\infty^\square$-operad —Koszul resolution— $BV^\square$-operad

Cat(Hodge)

Symmetric monoidal category

Twisted tensor product of (co)algebras

Homotopy algebras
Koszul duality
Operads
Symmetric monoidal categories

$H^\square$ —$H^\square$-modules— Hodge Complexes

Hopf algebras with central element $\square$

Natural generalisation of familiar Hodge–de Rham complex

- CK duality is a symmetry of Nature in the same sense that a mug is a donut!

§1.

Phase 1: Tree-Level Colour-Kinematic Duality Redux

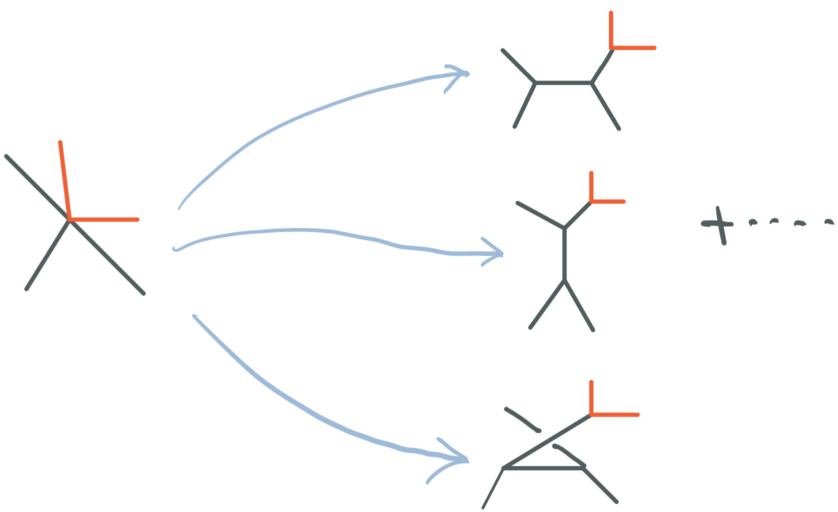
Tree-Level Colour–Kinematic Duality Redux

Manifest physical tree-level CK duality

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$$A\Box A + \partial AAA + \frac{\Box}{\Box}AAAA + \frac{\partial^3}{\Box^2}AAAAA + \cdots$$

Identically zero:
Colour Jacobi



[Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13]

Colour–Kinematic Duality Redux

Manifest physical tree-level CK duality

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [BJKMSW '21]

$$
\begin{aligned}
S^{\mathrm{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int & d^D x \tfrac{1}{2} A_\mu \Box A^\mu + \tfrac{1}{2} g \partial_\mu A_\nu [A^\mu, A^\nu] \\
& \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\
& + g C^{\mu\nu}[A_\mu, A_\nu] + g \partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2} \partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\
& + g \bar{C}^{\mu\nu}\left(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\right) + \cdots
\end{aligned}
$$

[Bern, Dennen, Huang, Kiermaier '10]

- Purely cubic Feynman diagrams $\longrightarrow$

$$
A^{n,0}_{\mathrm{YM}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

Colour–Kinematic Duality Redux

The need for unphysical BRST CK duality

- Include off-shell unphysical/ghost modes in the external states, the full BRST-extended state space

$$(A_\mu{}^a, b^a, c^a, \bar{c}^a)$$

- BRST CK duality ensures consistent double copy of the BRST charge Q

- Intuition: unphysical off-shell modes propagate in the loops of Feynman diagrams

Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

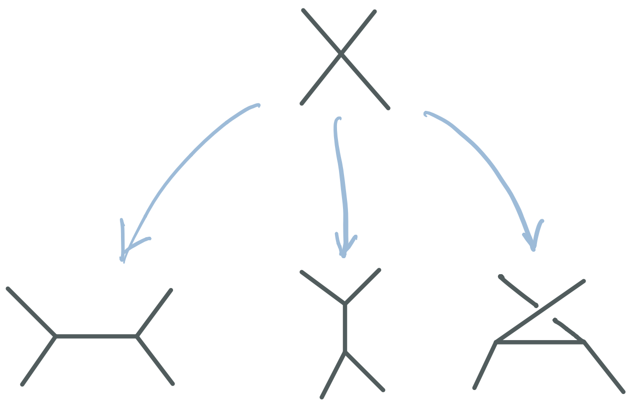
- Relax transversality $p_i \cdot \varepsilon_i \neq 0$ for external states $\Rightarrow$ CK duality fails

$$A^{n,0}_{\text{YM}} \quad \to \quad \hat{A}^{n,0}_{\text{YM}}$$

Physical amplitude

BRST-extended 'amplitude'

- By analogy with physical gluons can compensate with new vertices [BJKMSW '20]:



- Non-zero: BRST-extended amplitude changes $\hat{A}^{n,0}_{\text{YM}} \to \hat{A}'^{n,0}_{\text{YM}}$ (but physical amplitude $A^{n,0}_{\text{YM}}$ is invariant)

Colour-Kinematic Duality Redux

Tree-level CK duality for on-shell longitudinal gluons and ghosts

- New vertices are necessarily of the form

$$(\partial \cdot A)Y[A]$$

- Can add through the gauge-fixing fermion $\Psi' = \Psi - 2\xi\bar{c}Y$

$$\begin{array}{rcccc}
\text{Gauge-fixing } G[A]: & \partial \cdot A & \mapsto & G'[A] & = & \partial \cdot A - 2\xi Y \\
\text{Nakanishi-Lautrup } b: & b & \mapsto & b' & = & b + Y \\
\text{BRST action } S^{\text{YM}}_{\text{BRST}} & S^{\text{YM}}_{\text{BRST}} & \mapsto & S'^{\text{YM}}_{\text{BRST}} & = & S^{\text{YM}}_{\text{BRST}} + \int (\partial \cdot A)Y + \cdots
\end{array}$$

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\text{BRST action } S_{\text{BRST}}^{\text{YM}} & S_{\text{BRST}}^{\text{YM}} & \mapsto & S_{\text{BRST}}^{\prime\text{YM}} & = & S_{\text{BRST}}^{\text{YM}} + \int(\partial \cdot A)Y + \cdots
\end{array}$$

- Longitudinal CK duality $\Leftrightarrow$ gauge choice [BJKMSW '20, '21]

- On-mass-shell BRST Ward identities transfers CK duality onto ghosts through

$$S_{\text{YM}}^{\text{ghost}} = \int d^D x \, \mathrm{tr} \bar{c} Q(\partial^\mu A_\mu - 2\xi Y)$$

Colour-Kinematic Duality Redux

On-shell tree-level CK manifesting BRST action

- Introduce further auxiliary gluons and ghosts [BJKMSW '20, '21]:

$$
\begin{aligned}
\mathcal{L}^{\mathrm{YM}}_{\mathrm{BRST\ CK}} \;=\;& \tfrac{1}{2}A_{a\mu}\Box A^{\mu a} - \bar{c}_a\Box c^a + \tfrac{1}{2}b_a\Box b^a + \xi\, b_a\sqrt{\Box}\,\partial_\mu A^{\mu a} \\
& - K^{\mu}_{1a}\Box\bar{K}^{1a}_{\mu} - K^{\mu}_{2a}\Box\bar{K}^{2a}_{\mu} - gf_{abc}\bar{c}^a\partial^\mu(A^b_\mu c^c) \\
& - \tfrac{1}{2}B^{\mu\nu\kappa}_{a}\Box B^{a}_{\mu\nu\kappa} + gf_{abc}\Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}}\partial^\kappa B^a_{\kappa\mu\nu}\Big)A^{\mu b}A^{\nu c} \\
& - gf_{abc}\Big\{K^{a\mu}_{1}(\partial^\nu A^b_\mu)A^c_\nu + [(\partial^\kappa A^a_\kappa)A^{b\mu} + \bar{c}^a\partial^\mu c^b]\bar{K}^{1c}_{\mu}\Big\} \\
& + gf_{abc}\Big\{K^{a\mu}_{2}\Big[(\partial^\nu\partial_\mu c^b)A^c_\nu + (\partial^\nu A^b_\mu)\partial_\nu c^c\Big] + \bar{c}^a A^{b\mu}\bar{K}^{2c}_{\mu}\Big\} + \cdots
\end{aligned}
$$

Auxiliary gluon

Auxiliary ghost

- Cubic Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes, ghosts and auxiliary fields (on-shell)

Colour-Kinematic Duality Redux

Lifting to off-shell CK duality

- Off-shell momenta $p^2 \neq 0$: resulting CK duality violations are compensated by vertices $f\Box\phi$ generated by generically non-local field redefinitions:

$$\phi \mapsto \phi + f(\phi), \qquad \phi\Box\phi \mapsto \phi\Box\phi + f\Box\phi + \cdots$$

- Strictify again: off-shell tree-level BRST CK duality is rendered manifest

Colour–Kinematic Duality Redux

Lifting to off-shell CK duality

- Off-shell momenta $p^2 \neq 0$: resulting CK duality violations are compensated by vertices $f\Box\phi$ generated by generically non-local field redefinitions:

$$\phi \mapsto \phi + f(\phi), \qquad \phi\Box\phi \mapsto \phi\Box\phi + f\Box\phi + \cdots$$

- Strictify again: off-shell tree-level BRST CK duality is rendered manifest

Price to pay

- Jacobian determinants $\rightarrow$ counterterms ensuring unitarity

$$\det\left(\mathbb{1} + \frac{\delta f(\phi)}{\delta\phi}\right) = \int \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\left(\bar{\chi}_I\chi^I + \bar{\chi}_I\frac{\delta f^I}{\delta\phi^J}\chi^J\right)}$$

- No reason to think such terms will preserve CK duality: in this sense, our off-shell CK duality is anomalous on the physical Hilbert space

- Two-loop CK duality with cubic Feynman rule compatible numerators is impossible [Bern, Davies, Nohle '15]

- Here, we understand this impossibility as a CK duality anomaly

Colour–Kinematic Duality Redux

Summary: manifest off-shell BRST-Lagrangian CK duality

- YM BRST-action with manifest off-shell CK duality

$$S^{\text{YM}}_{\text{BRST CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

- Rendered cubic with infinite tower of aux. fields

$$A^{ia} = (A_\mu{}^a, b^a, \bar{c}^a, c^a, \underbrace{B_{\mu\nu\rho}{}^a, \bar{K}_\mu{}^a, \ldots}_{\text{auxiliaries}})$$

- c_{ab}, f^{abc} colour metric and structure constants

- C_{ij}, F^{ijk} kinematic metric and structure constants: differential operators that satisfy the same identities as c_{ab}, f^{abc} as operator equations

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^{a}_{c)d} = 0 \qquad f_{[ab|d} f^{d}_{c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad C_{i(j} F^{i}_{k)l} = 0 \qquad F_{[ij|l} F^{l}_{|k]m} = 0$$

Phase 2: Syngamy

BRST Lagrangian Syngamy

Syngamic reproduction of factorable theories

Parent theories

$$c_{IJ}\phi^I \,\Box\, \phi^J + f_{IJK}\phi^I\phi^J\phi^K$$

$$\tilde c_{\tilde I\tilde J}\tilde\phi^{\tilde I} \,\Box\, \tilde\phi^{\tilde J} + \tilde f_{\tilde I\tilde J\tilde K}\tilde\phi^{\tilde I}\tilde\phi^{\tilde J}\tilde\phi^{\tilde K}$$

Factorisation

$$C_{ij}c_{ab}\Phi^{ia}\Box\Phi^{jb} + F_{ijk}f_{abc}\Phi^{ia}\Phi^{jb}\Phi^{kc}$$

$$\tilde c_{\tilde a\tilde b}\tilde C_{\tilde\imath\tilde\jmath}\Phi^{\tilde a\tilde\imath}\Box\tilde\Phi^{\tilde a\tilde\jmath} + \tilde f_{\tilde a\tilde b\tilde c}\tilde F_{\tilde\imath\tilde\jmath\tilde k}\tilde\Phi^{\tilde a\tilde\imath}\tilde\Phi^{\tilde b\tilde\jmath}\tilde\Phi^{\tilde c\tilde k}$$

Daughter theories

$$C_{ij}\tilde C_{\tilde\imath\tilde\jmath}\Phi^{i\tilde\imath}\Box\Phi^{j\tilde\jmath} + F_{ijk}\tilde F_{\tilde\imath\tilde\jmath\tilde k}\Phi^{i\tilde\imath}\Phi^{j\tilde\jmath}\Phi^{k\tilde k}$$

$$\tilde c_{\tilde a\tilde b}C_{ij}\Phi^{\tilde a i}\Box\Phi^{\tilde a j} + \tilde f_{\tilde a\tilde b\tilde c}F_{ijk}\Phi^{\tilde a i}\Phi^{\tilde b j}\Phi^{\tilde c k}$$

$$c_{ab}\tilde C_{\tilde\imath\tilde\jmath}\Phi^{a\tilde\imath}\Box\Phi^{a\tilde\jmath} + f_{abc}\tilde F_{\tilde\imath\tilde\jmath\tilde k}\Phi^{a\tilde\imath}\Phi^{b\tilde\jmath}\Phi^{c\tilde k}$$

$$c_{ab}\tilde c_{\tilde a\tilde b}\Phi^{a\tilde a}\Box\Phi^{a\tilde b} + f_{abc}\tilde f_{\tilde a\tilde b\tilde c}\Phi^{a\tilde a}\Phi^{b\tilde b}\Phi^{c\tilde c}$$

BRST Double Copy of Yang, Mills

- Double copy of Yang, Mills BRST-action:

$$A^{ia} = (A_\mu{}^{a}, \dots) \qquad S^{\mathrm{YM}}_{\mathrm{CK}} = C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk}\, f_{abc} A^{ia} A^{jb} A^{kc}$$

Kalb–Ramond 2-form

$$A^{i\tilde{\imath}} = (h_{\mu\nu}, B_{\mu\nu}, \varphi, \dots) \qquad S^{\mathcal{N}=0}_{\mathrm{DC}} = C_{ij}\, C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk}\, F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Graviton Dilaton

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=0}_{\mathrm{DC}} = & \left.\begin{aligned} &\tfrac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \tfrac{1}{2} \varpi_\mu \Box \varpi^\mu + \xi^2 (\partial^\mu \varpi_\mu)^2 + \tfrac{1}{2} \pi \Box \pi \\ &- 2\xi \varpi^\nu \Box^{\frac{1}{2}} \partial^\mu h_{\mu\nu} - 2\xi \pi \Box^{\frac{1}{2}} \partial_\mu \varpi^\mu + 2\xi^2 \pi \partial_\mu \partial_\nu h^{\mu\nu} \end{aligned}\right\} \text{Graviton-dilaton} \\
\text{KR 2-form } & \left\{\begin{aligned} &- 2\bar{X}_\mu \Box X^\mu - \delta \Box \delta - 2\bar{\beta} \Box \beta - 2\xi \beta \Box^{\frac{1}{2}} \partial_\mu \bar{X}^\mu + 2\xi \bar{\beta} \Box^{\frac{1}{2}} \partial_\mu X^\mu \\ &+ \tfrac{1}{2} B_{\mu\nu} \Box B^{\mu\nu} - 2\bar{\Lambda}_\mu \Box \Lambda^\mu + \alpha_\mu \Box \alpha^\mu + \xi^2 (\partial^\mu \alpha_\mu)^2 + \varepsilon \Box \varepsilon - \bar{\lambda} \Box \lambda - 2\bar{\gamma} \Box \gamma \\ &- 2\xi \alpha^\nu \Box^{\frac{1}{2}} \partial^\mu B_{\mu\nu} - 2\xi \gamma \Box^{\frac{1}{2}} \partial_\mu \bar{\Lambda}^\mu + 2\xi \bar{\gamma} \Box^{\frac{1}{2}} \partial_\mu \Lambda^\mu + \cdots \end{aligned}\right.
\end{aligned}$$

- Canonical field redefinition to Fierz-Pauli + Kalb-Ramond + dilaton action

$$QB = d\Lambda, \qquad Q\Lambda = d\lambda, \qquad Q\lambda = 0$$

- Cubic Einstein-Hilbert BRST action explicitly recovered **[LB, Nagy '20]**

BRST Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- How do we know $S^{\mathcal{N}=0}_{\text{DC}}$ is equivalent to true $S^{\mathcal{N}=0}_{\text{BRST}}$?

BRST Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- How do we know $S^{\mathcal{N}=0}_{\text{DC}}$ is equivalent to true $S^{\mathcal{N}=0}_{\text{BRST}}$?
- Semi-classical equivalence (requires physical tree-level CK duality)

$$
\begin{tikzcd}
F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \arrow[r, "\text{Double Copy}"] \arrow[d, "\text{Feynman rules}"'] & F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \arrow[d, "\text{Feynman rules}"] \\
\sum \frac{nc}{d} \arrow[r] & \sum \frac{n\tilde{n}}{d}
\end{tikzcd}
$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern, Dennen, Huang, Kiermaier '10] for gravitons up to 6 points

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- Cf. [Bern, Dennen, Huang, Kiermaier '10] for gravitons up to 6 points

- Quantum consistency: is there some Q such that

$$QS^{\mathcal{N}=0}_{\text{DC}} = 0, \qquad Q^2 = 0$$

BRST Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to true $S_{\text{BRST}}^{\mathcal{N}=0}$?
- Semi-classical equivalence (requires physical tree-level CK duality)

$$\begin{tikzcd}
F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \arrow[rr, "\text{Double Copy}"] \arrow[d, "\text{Feynman rules}"] & & F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \arrow[d, "\text{Feynman rules}"] \\
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- Cf. [Bern, Dennen, Huang, Kiermaier '10] for gravitons up to 6 points
- Quantum consistency: is there some Q such that

$$QS_{\text{DC}}^{\mathcal{N}=0} = 0, \qquad Q^2 = 0$$

Answer: double-copy operator Q_{DC} (requires off-shell BRST CK duality)

BRST Lagrangian Syngamy

Double copy of BRST charge

- Double copy of BRST-action implies double copy BRST operator Q_{DC}

$$QA^{ia} = Q^i{}_j A^{ja} + Q^i{}_{jk} f^a{}_{bc} A^{jb} A^{kc} \qquad \tilde{Q}\tilde{A}^{\tilde{a}\tilde{\imath}} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}} \tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} \tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \tilde{A}^{\tilde{b}\tilde{\jmath}} \tilde{A}^{\tilde{c}\tilde{k}}$$

$$Q_{\mathrm{DC}} = \underbrace{Q^i{}_j A^{j\tilde{\imath}} + Q^i{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{i\tilde{\jmath}} + F^i{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_R}$$

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$$Q_{\mathrm{DC}} = \underbrace{Q^i{}_j A^{j\tilde{\imath}} + Q^i{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{i\tilde{\jmath}} + F^i{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_R}$$

- Since F^{ijk} satisfy the same identities as f^{abc} and $QS^{\mathrm{YM}}_{\mathrm{BRST}} = 0$, $Q^2 = 0$ can only rely on generic properties of f^{abc}:

$$Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0, \qquad Q^2_{\mathrm{DC}} = 0$$

- Semi-classical equivalence + $Q_{\mathrm{DC}} \Rightarrow$ quantum equivalence

BRST Lagrangian Syngamy

Double Copy Symmetries

- Yang-Mills gauge $\Rightarrow$ diffeomorphisms and 2-form gauge symmetries:

$$Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

$$Q_{\text{2-form}}B = \Lambda, \quad Q_{\text{2-form}}\Lambda = \lambda \quad Q_{\text{2-form}}\lambda = 0$$

- Double copy of symmetries generalises, e.g.

$$\text{global susy} \quad \times \quad \text{gauge} \quad \rightarrow \quad \text{local susy}$$

See also [Anastasiou, LB, Duff, Hughes, Nagy '14]

Super Yang, Mills and Supergravity

BRST-Lagrangian double copy

- Irreducible super Yang–Mills multiplets are CK duality respecting: same game
  Cf. for example [Bjerrum–Bohr, Damgaard, Vanhove '09]

- (Type I super Yang, Mills)^2 = Type IIA/B supergravity

$$A^{Ia} = (A^{ia}, \Psi^{xa}) = (A_\mu{}^a, \underset{\text{Gluino}}{\psi_\alpha{}^a}, \text{ghosts}, \text{aux})$$

$$A^{J\tilde{J}} = (h_{\mu\nu}, B_{\mu\nu}, \phi, \underbrace{\Psi_{\alpha\nu}, \Psi_{\mu\beta}}_{\text{Gravitini}}, \underset{\text{RR field strengths}}{F_{\alpha\beta}}, \text{ghosts}, \text{aux})$$

- Local NS-R sector susy follows from super Yang, Mills factors

$$\mathcal{Q}_\alpha A_\mu{}^a = \delta^a{}_b \gamma_{\mu\alpha}{}^\beta \psi_\beta{}^b + \cdots \quad \longrightarrow \quad \mathcal{Q}_\alpha h_{\mu\nu} = \gamma_{(\mu\alpha}{}^\beta \Psi_{\beta\nu)} + \cdots$$

- Super $\eta, \bar{\eta}$ and Nielsen, Kallosh χ ghosts

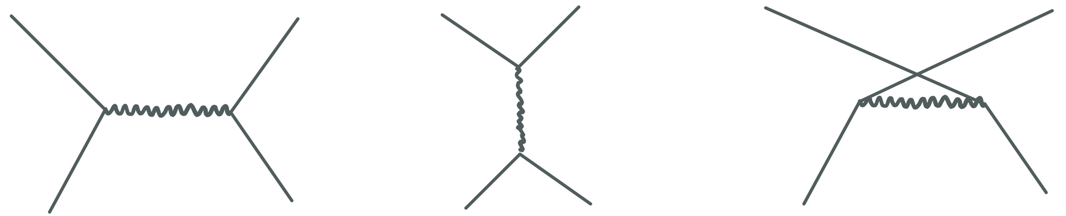
$$\bar{c} \otimes \psi \sim \bar{\eta}\,, \quad c \otimes \psi \sim \eta\,, \quad b \otimes \psi \sim \chi$$

Cf. [Anastasiou, LB, Duff, Hughes, Nagy '14]

Super Yang, Mills and Supergravity

CK duality: the mother of all symmetries

- CK duality $\Rightarrow$ supersymmetry [Chiodaroli, Jin, Roiban '13]



$$\frac{i}{2}\left(\frac{(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4)c_s}{s}+\frac{(\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4)c_s}{t}+\frac{(\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4)c_s}{u}\right)$$

- CK duality requires Fierz identity: $\mathcal{N}=1$ super Yang–Mills in $D=3,4,6,10$

$$(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4)+(\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4)+(\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4)=0$$

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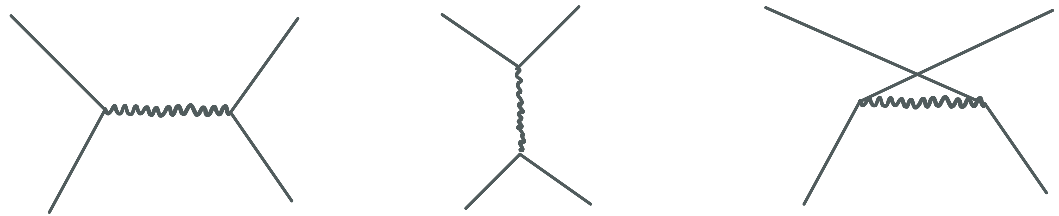

$$\frac{i}{2}\left(\frac{(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4)c_s}{s}+\frac{(\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4)c_s}{t}+\frac{(\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4)c_s}{u}\right)$$

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- Off-shell CK duality implies supersymmetry directly (exercise!)

$$S^{\mathrm{SYM}}_{\mathrm{BRST}}=C_{IJ}c_{ab}A^{Ia}\Box A^{Ja}+F_{IJK}f_{abc}A^{Ia}A^{Jb}A^{Kc}$$

$$\delta_\epsilon A^{ia}=F^i{}_{xy}\Psi^{xa}\epsilon^y,\quad \delta_\epsilon\Psi^{xa}=F^x{}_{jy}A^{ja}\epsilon^y$$

Super Yang, Mills and Supergravity

Ramond-Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives field strengths $F_{\alpha\beta}$, not potentials:
  [Nagy '14]

  - Representation theory

$$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$

  - The BRST transformation of the gluino has no linear contribution, $Q_{\mathbf{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential

  - R-R background fields couple to worldsheet through field strengths

Super Yang, Mills and Supergravity

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  - R-R background fields couple to worldsheet through field strengths

- Type IIA/B action can be written in terms of field strengths alone for RR sector:

$$F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star F_4 + B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + B_2 \wedge B_2 \wedge F_2 \wedge \tilde{F}_4 - \frac{1}{3} B_2 \wedge B_2 \wedge B_2 \wedge F_2 \wedge F_2$$

Super Yang, Mills and Supergravity

Sen's mechanism from double copy Ramond-Ramond sector

- Double copy RR field strengths are elementary fields:

$$
\begin{aligned}
\mathcal{L}^{\mathrm{RR}}_{\mathrm{DC}} &= \overline{F}^{\alpha\beta}\,\tfrac{1}{\Box}\,\partial\!\!\!/_{\alpha}{}^{\alpha'}\,\partial\!\!\!/_{\beta}{}^{\beta'}\,F_{\alpha'\beta'} + \cdots \\
&\to -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\Box B + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B
\end{aligned}
$$

Super Yang, Mills and Supergravity

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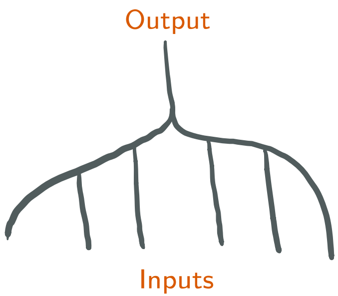
&\to -\tfrac{1}{2}F\wedge\star F-\xi B\wedge\mathrm{d}F+\tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B
\end{aligned}
$$

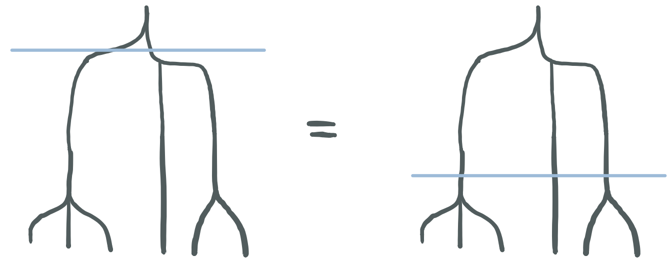
- Sen's mechanism [Sen '15] generalized to arbitrary (as opposed to self-dual) field strengths [BJKMSW '21]
- Sen's mechanism was motivated by IIB string field theory, where the R, R sector is naturally given in terms of bispinors - natural double copy shadow

Phase 3: Homotopy CK Duality

Operads and homotopy algebras

- Operads: encode types of algebras (symmetric operad: monoid in the monoidal category of $\mathbb{S}$-modules)





- Powerful abstract reasoning for deducing concrete statements (e.g. Koszul duality)

- Given a chain complex with algebraic structure, can this structure be transferred to homotopy equivalent chain complexes?

- Yes, with richer algebraic structure of higher operations: homotopy algebras

Operads and homotopy algebras

- I will only leave the door to ∞-algebras ajar: you should push it open!
- Hossenfelder warned us physicists to not get "Lost in Math", but getting lost can be fun! (and is the only way to discover something truly unexpected)

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- You won't be lost in the dark, there are excellent books:

  'Operads in algebra, topology and physics', Martin Markl, Steve Shnider and Jim Stasheff

  'Algebraic Operads', Jean-Louis Loday and Bruno Vallette

  'Algebraic Structure of String Field Theory', Martin Doubek, Branislav Jurčo, Martin Markl and Ivo Sachs

Homotopy algebras

- Informally: generalise familiar (associate, commutative, Lie. . .) algebras to include higher products satisfying higher relations up to homotopies

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- Lie algebras $\to$ L_∞-algebras [Zwiebach '93; Hinich, Schechtman '93]:

Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \ldots$
Relations Antisymmetry + Jacobi	Relations Graded antisymmetry + homotopy Jacobi

$$[[x,y],z] + (-1)^{x(y+z)}[[y,z],x] + (-1)^{y(x+z)}[[x,z],y] = -[[x,y,z]]$$

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- Associative algebras $\to$ A_∞-algebras [Stasheff '63]
- Commutative algebras $\to$ C_∞-algebras [Kadeishvili '88]
- BV algebras $\to$ BV_∞-algebras [Galvez-Carrillo, Tonks, Vallette '09]
- Lie algebras $\to$ EL_∞-algebras [LB-Kim-Saemann '21]

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- Lie algebras $\to$ EL_∞-algebras [LB-Kim-Saemann '21]
- Homotopy Maurer-Cartan theory of L_∞-algebras $\to$ Batalin–Vilkovisky Formalism See e.g. [Jurčo, Raspollini, Saemann, Wolf '18]

Batalin–Vilkovisky Formalism

- Extend the space of BRST fields

$$\mathfrak{F}_{\mathsf{BV}} \;=\; T^*[1]\mathfrak{F}_{\mathsf{BRST}}$$

- Fields Φ^A local coordinates on $\mathfrak{F}_{\mathsf{BRST}}$, antifields Φ_A^+ are fibre coordinates

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$$\mathfrak{F}_{\mathsf{BV}} \;=\; T^*[1]\mathfrak{F}_{\mathsf{BRST}}$$

- Fields Φ^A local coordinates on $\mathfrak{F}_{\mathsf{BRST}}$, antifields Φ^+_A are fibre coordinates
- Using canonical symplectic structure on $\mathfrak{F}_{\mathsf{BV}}$ we extend Q_{BRST} and S_{BRST} to Q_{BV} and S_{BV} ($\{-,-\}$ is degree -1 $\to$ Gerstenhaber algebra)

$$Q_{\mathsf{BV}} \;=\; \{S_{\mathsf{BV}}, -\}, \quad Q_{\mathsf{BV}} S_{\mathsf{BV}} = 0, \quad S_{\mathsf{BV}}|_{\mathfrak{F}_{\mathsf{BRST}}} \;=\; S_{\mathsf{BRST}}, \quad \pi_* Q_{\mathsf{BV}} \;=\; Q_{\mathsf{BRST}}$$

$$S_{\mathsf{BV}}[\phi, \phi^+] \;= S_{\mathsf{classical}}[\phi] + \phi^+_A (Q_{\mathsf{BV}}\phi)^A$$

- Gauge-fixing fermion Ψ gives symplectomorphism

$$\Phi \mapsto \Phi + \{\Psi, \Phi\}, \quad \Phi^+_A \mapsto \Phi^+_A \;+\; \frac{\delta}{\delta \Phi^A}\Psi$$

$$S_{\mathsf{BV}}[\phi, \phi^+] \;\mapsto S^{\Psi}_{\mathsf{BRST}}[\phi] + \phi^+_A (Q_{\mathsf{BV}}\phi)^A$$

Homotopy Algebras and BV Lagrangian Field Theories

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathsf{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*) := \bigoplus_{p=1}^{\infty} \mathsf{Sym}^p(\mathfrak{g}[1]^*)$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c , \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

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- Any BV field theory with Q_{BV} corresponds to a cyclic L_∞-algebra in the CE picture, see e.g. [Jurčo, Raspollini, Saemann, Wolf '18]

- The Yang-Mills theory $\mathfrak{L}^{\mathrm{YM}}$ complex (i.e. just μ_1 considered)

$$\begin{array}{ccccccc}
\mathfrak{L}_0^{\mathrm{YM}} & \oplus & \mathfrak{L}_1^{\mathrm{YM}} & \oplus & \mathfrak{L}_2^{\mathrm{YM}} & \oplus & \mathfrak{L}_3^{\mathrm{YM}} \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathrm{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathrm{Id}} & b^+ & &
\end{array}$$

Factorisation of Yang-Mills

$\mathfrak{L}^{\mathsf{YM}}$ Factorises [BJKMSW '21]

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{g}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

$$\mathfrak{g} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar} \longrightarrow \mathfrak{kinematics} \otimes_\tau \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$$

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Michel Reiterer [1912.03110]

Remarkable proof of on-shell tree-level CK duality for physical gluons via a $BV_\infty^\square$-algebra (deforming $BV^\square$-algebras [Galvez–Carrillo, Tonks, Vallette '09]):

1. There is a degree -1 unary map h on Zeitlin-Costello complex $\mathfrak{ZC}$

$$d^2 = 0, \quad h^2 = 0, \quad dh + hd = \square \quad \text{(plus some other conditions)}$$

2. There is a $BV_\infty^\square$-algebra on $\mathfrak{ZC}$ that deforms the BV_∞-algebra
3. Every such h is second order (in the graded sense) up to homotopy
4. Minimal model + strictification implies CK duality of physics tree-level S-matrix

Factorisation of Yang-Mills

- BRST-Lagrangian CK duality $\Leftrightarrow$ $BV^{\square}$-algebra [BJKMSW '22 (to appear)]

$$\mathfrak{L}_{\mathsf{CK}}^{\mathsf{YM}} = \mathfrak{g} \otimes \mathfrak{B}^{\mathsf{YM}}$$

For $\mathfrak{H}$ a cocommutative Hopf algebra with central element $\square$, a $BV^{\square}$-algebra is a graded $\mathfrak{H}$-module with

$$d^2 = 0, \qquad h^2 = 0, \qquad dh + hd = \square$$

$$\begin{aligned} d(x \cdot y) - dx \cdot y - (-1)^x x \cdot dy &= 0 \\ h(x \cdot y) - hx \cdot y - (-1)^x x \cdot hy &= [x, y] \\ d[x, y] - [dx, y] - (-1)^x [x, dy] &= \square(x \cdot y) - \square x \cdot y - x \cdot \square y \end{aligned}$$

and h is second order w.r.t $(- \cdot -)$

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- $BV^{\square}_{\infty}$-algebra (generalising Reiterer) is the homotopy algebra in the category of Hodge Complexes and has array of higher brackets with homotopy relations amongst them, e.g.:

$$m^0_n(x_1, \dots x_n) \quad m^0_{p,q}(x_1, \dots x_n) \quad m^0_{p,q,r}(x_1, \dots x_n)$$

$$[[x, y], z] \pm [[y, z], x] \pm [[z, x], y] = (\square_{xy} + \square_{yz} + \square_{xz}) m^0_{1,2}(x, y, z)$$

- More structure than mere C_∞

Chern-simons and $BV^{\square}$

- Chern–Simons theory has off–shell CK duality **[Ben–Shahar, Johansson '21]**
- $\Rightarrow$ there is a Chern–Simons $BV^{\square}$-algebra **[BJKMSW '22 (to appear)]**

$$
\begin{array}{ccccccc}
c & & A & & A^{+} & & c^{+} \\
\mathfrak{L}^{\mathrm{CS}} = \Omega^0(M)\otimes\mathfrak{g} & \xrightarrow{\mathrm{d}} & \Omega^1(M)\otimes\mathfrak{g} & \xrightarrow{\mathrm{d}} & \Omega^2(M)\otimes\mathfrak{g} & \xrightarrow{\mathrm{d}} & \Omega^3(M)\otimes\mathfrak{g}
\end{array}
$$

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$$\mathfrak{L}^{\mathsf{CS}} = \mathfrak{g}\otimes\mathfrak{B}^{\mathsf{CS}}$$

$$\mathfrak{B}^{\mathsf{CS}} = \begin{tikzcd} \Omega^0 \arrow[r, bend left, "\mathrm{d}"] & \Omega^1 \arrow[l, bend left, "\mathrm{d}^{\dagger}"] \arrow[r, bend left, "\mathrm{d}"] & \Omega^2 \arrow[l, bend left, "\mathrm{d}^{\dagger}"] \arrow[r, bend left, "\mathrm{d}"] & \Omega^3 \arrow[l, bend left, "\mathrm{d}^{\dagger}"] \end{tikzcd}$$

$$d(A) = \mathrm{d}A, \quad h(A) = \mathrm{d}^{\dagger}A, \quad A\cdot B = A\wedge B$$

$$\mathrm{d}\mathrm{d}^{\dagger} + \mathrm{d}^{\dagger}\mathrm{d} = \square$$

Homotopy algebra of CK duality

- Kinematic (graded) Lie algebra of Yang–Mills with infinite tower of auxiliaries is given by $(- \cdot -)$ of $\mathfrak{B}^{\mathsf{YM}}$

- Integrating out auxiliaries $\rightarrow$ kinematic $BV^{\square}_{\infty}$-algebra with higher products:

$m^0_n(x_1, \ldots x_n)$	Colour-stripped vertices of gauge-fixed action
$m^0_{p,q}(x_1, \ldots x_n)$	Tolotti-Weinzerl corrections for tree on-shell CK duality
$m^0_{p,q,r}(x_1, \ldots x_n)$	Field red. vertices correcting for off-shell CK duality

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- Integrating out auxiliaries $\to$ kinematic $BV_\infty^\square$-algebra with higher products:

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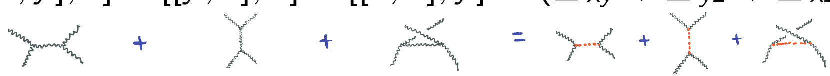
m^0_n(x_1,\ldots x_n) &\quad \text{Colour-stripped vertices of gauge-fixed action}\\
m^0_{p,q}(x_1,\ldots x_n) &\quad \text{Tolotti-Weinzerl corrections for tree on-shell CK duality}\\
m^0_{p,q,r}(x_1,\ldots x_n) &\quad \text{Field red. vertices correcting for off-shell CK duality}
\end{aligned}$$

- Axioms (homotopy ‘Jacobi’ relations) $\Rightarrow$ off-shell CK duality

$$[[x,y],z] \pm [[y,z],x] \pm [[z,x],y] = (\square_{xy} + \square_{yz} + \square_{xz}) m^0_{1,2}(x,y,z)$$



- Purely tree-level calculations and one identity at any order:

$$\sum_{p+q=n+2} n\text{-point tree with two internal } (p\text{-ary and } q\text{-ary}) \text{ vertices}$$

$$= n\text{-point tree with one internal } (n\text{-ary}) \text{ vertex}$$

Where next?

- What is the relationship between the kinematic $BV^{\square}$-algebras and the other kinematic algebra structures in the literature, e.g. see Hadleigh and Henrik's talks

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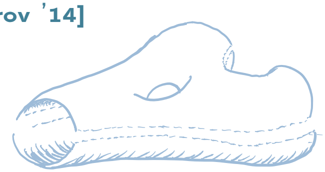
- AdS space [Farrow, Lipstein, McFadden '19; Zhou '21; Diwakar, Herderschee, Roiban, Teng '21 ...]
  $\to$ Hopf algebra of universal enveloping algebra of AdS isometries

- Bagger-Lambert-Gustavsson [Bargheer, He, McLoughlin '12; Huang, Johansson '12]
  $\to$ m-ary $BV^{\square}$ operads
- Matter coupling [Johansson, Ochirov '14]
  $\to$ many-sorted $BV^{\square}$ operads



- String theory

$$\{d, h\} = \square \quad \longrightarrow \quad \{Q, b_0\} = L_0$$

Cf. BV_∞ structure on TVOA [Galvez–Carrillo, Tonks, Vallette '09] lifting the BV-algebra structure on the BRST (co)homology [Lian-Zuckerman '93]

Thanks for listening

Homotopy Maurer-Cartan theory

- Inner product $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on dgLa $(\mathfrak{g}, d)$:

$$\langle x, dy \rangle = (-)^{1+x+y+xy} \langle y, dx \rangle, \qquad \langle x, [y, z] \rangle = (-)^{z(x+y)} \langle z, [x, y] \rangle$$

- Cyclic structure $\langle -, - \rangle : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$ on L_∞-algebra $(\mathfrak{L}, \mu_i)$:

$$\langle x_1, \mu_i(x_2, \dots x_{i+1}) \rangle = (-)^{i+i(x_1+x_{i+1})+x_{i+1}\sum_{j=1}^{i} x_j} \langle x_{i+1}, \mu_i(x_1, \dots x_i) \rangle$$

- (Homotopy) Maurer-Cartan element $a \in \mathfrak{g}$ $(a \in \mathfrak{L})$ from (h) MC-action:

$$f_a = da + \tfrac{1}{2}[a, a] = 0, \qquad S_{\mathrm{MC}} = \tfrac{1}{2}\langle a, da \rangle + \tfrac{1}{3!}\langle a, [a, a] \rangle$$

$$F_a = \sum_k \tfrac{1}{k!}\mu_k(a, a, \dots, a) = 0, \qquad S_{\mathrm{hMC}} = \sum_k \tfrac{1}{(k+1)!}\langle a, \mu_k(a, a, \dots, a) \rangle$$

- Covariant derivative, Bianchi identity and gauge transformations:

$$D_a x = \sum_k \frac{(-1)^k}{k!}\mu_{k+1}(x, a, \dots, a), \quad D_a F_a = 0, \quad \delta_c a = D_a c$$