Asymptotic Conformal Symmetry and Gravity Localisation in Brane Worlds

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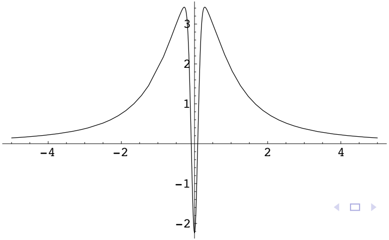
A. Salam & E. Sezgin, Phys.Lett. B147 (1984) 47
M. Cvetič, H. Lü & C.N. Pope, Nucl. Phys. B600 (2001) 103
M. Cvetič, G. Gibbons & C.N. Pope, Nucl. Phys. B677 (2004) 164
T. Pugh, E. Sezgin & K.S.S., JHEP 1102 (2011) 115
B. Crampton, C.N. Pope & K.S.S., JHEP 1412 (2014) 035; 1408.7072

The universe as a membrane

The idea of formulating the cosmology of our universe on a brane embedded in a higher-dimensional spacetime dates back, at least, to Rubakov and Shaposhnikov. Phys. Lett. B125 (1983), 136

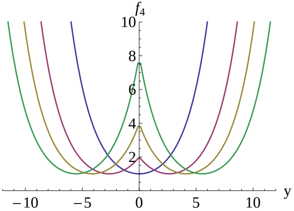
Attempts in a supergravity context to achieve a localization of gravity on a brane embedded in an infinite transverse space were made by Randall and Sundrum (RS II) Phys. Rev. Lett. 83 (1999) 4690 and by Karch and Randall JHEP 0105 (2001) 008 using patched-together sections of AdS_5 space with a delta-function source at the joining surface. This produced a "volcano potential" for the effective Schrödinger problem in the direction transverse to the brane, giving rise to a bound state concentrating gravity in the 4D directions.



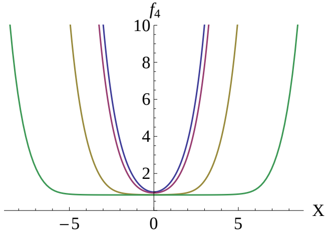
General problems with localization

Attempting to embed such models into a full supergravity/string-theory context have proved to be problematic, however. Splicing together sections of AdS_5 is clearly an artificial construction which does not make use of the natural D-brane or NS-brane objects of string or supergravity theory.

These difficulties were studied more generally by Bachas and Estes JHEP 1106 (2011) 005 , who traced the difficulty in obtaining localization within a string or supergravity context to the behavior of the warp factor for the 4D subspace. In the Karch-Randall spliced model, one obtains a "kink" in the warp factor at the junction:



The problem with string-theory attempts to localize gravity on a brane subspace as found by Bachas and Estes, *e.g.*, for a Janus discontinuous-dilaton solution, is that there is no similar “bump” in the warp factor for the 4D subgeometry:



In consequence, there is no concentration of gravity on the 4D subspace of such a model. Bachas and Estes raised the possibility that this difficulty could be generic for asymptotically maximally symmetric geometries of the embedding spacetime.

c.f. also Freedman, Gubser, Pilch & Warner, Adv. Theor. Math. Phys. 3 (1999) 363

One interpretation of the patched AdS constructions is in terms of an effective Schrödinger problem, in which the kink in the warp factor produces a bound state for the transverse part of the gravitational wavefunction. Trying to do this without an artificially generated kink runs into a key difficulty in attempts to obtain massless gravity in a lower-dimensional brane subspace when the transverse space is infinite. Here's a sketch:

Given an eigenvalue $-\lambda$ for a normalizable wavefunction ξ of the transverse wave operator $\frac{e^{-2A}}{\sqrt{\hat{g}}}(\partial_a\sqrt{\hat{g}}e^{4A}\hat{g}^{ab}\partial_b)$ (where e^{2A} is the warp factor of the 4d subspace), and *provided one may integrate by parts*, one may write

$$\lambda||\xi||^2 = -\int d^{d-4}y\xi(\partial_a\sqrt{\hat{g}}e^{4A}\hat{g}^{ab}\partial_b\xi) \to \int d^{d-4}y\sqrt{\hat{g}}e^{4A}|\partial\xi|^2$$

If one is looking for a transverse wavefunction ξ with $\lambda = 0$, corresponding to massless gravitational excitations in the 4d subspace, it would seem therefore that ξ has to be constant, which would be inconsistent with it being normalizable in an infinite transverse space.

Another approach: Salam-Sezgin theory and its embedding

Abdus Salam and Ergin Sezgin constructed in 1984 a version of 6D minimal (chiral, *i.e.* (1,0)) supergravity coupled to a 6D 2-form tensor multiplet and a 6D super-Maxwell multiplet which gauges the U(1) R-symmetry of the theory. Phys.Lett. B147 (1984) 47 This Einstein-tensor-Maxwell system has the bosonic Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm SS} &= \tfrac{1}{2}R - \tfrac{1}{4g^2}e^{\sigma}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{6}e^{-2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - g^2e^{-\sigma} \\ G_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} + 3F_{[\mu\nu}A_{\rho]} \end{aligned}$$

Note the *positive* potential term for the scalar field σ. This is a key feature of all R-symmetry gauged models generalizing the Salam-Sezgin model, leading to models with noncompact symmetries. For example, upon coupling to yet more vector multiplets, the sigma-model target space can have a structure $SO(p,q)/(SO(p)\times SO(q))$.

The Salam-Sezgin theory does not admit a maximally symmetric 6D solution, but it does admit a $(\text{Minkowski})_4 \times S^2$ solution with the flux for a unit-strength $U(1)$ monopole turned on in the S^2 directions

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + a^2(d\theta^2 + \sin^2\theta d\phi^2) \\ A_m dy^m &= (n/2g)(\cos\theta \mp 1) d\phi \\ \sigma &= \sigma_0 = \text{const} , \qquad B_{\mu\nu} = 0 \\ g^2 &= \frac{e^{\sigma_0}}{2a^2} , \qquad n = \pm 1 \end{aligned}$$

This construction has been used in the SLED $\leftrightarrow$ Supersymmetry in Large Extra Dimensions proposal for dilution of the cosmological constant in the two extra S^2 dimensions, leaving a naturally small residue in the four x^μ dimensions.

Aghababaie, Burgess, Parameswaran & Quevedo, Nucl. Phys. B680 (2004) 389 et seq.

$\mathcal{H}^{(2,2)}$ embedding of the Salam-Sezgin theory

A way to obtain the Salam-Sezgin theory from M theory was given by Cvetič, Gibbons & Pope. Nucl. Phys. B677 (2004) 164 This employed a reduction from 10D type IIA supergravity on the space $\mathcal{H}^{(2,2)}$, or, equivalently, from 11D supergravity on $S^1 \times \mathcal{H}^{(2,2)}$. The $\mathcal{H}^{(2,2)}$ space is a cohomogeneity-one 3D hyperbolic space which can be obtained by embedding into $\mathbb{R}^4$ via the condition $\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1$. This embedding condition is SO(2, 2) invariant, but the embedding $\mathbb{R}^4$ space has SO(4) symmetry, so the isometries of this space are just $\mathrm{SO}(2,2) \cap \mathrm{SO}(4) = \mathrm{SO}(2) \times \mathrm{SO}(2)$. The cohomogeneity-one $\mathcal{H}^{(2,2)}$ metric is $ds_3^2 = \cosh 2\rho d\rho^2 + \cosh^2\rho d\alpha^2 + \sinh^2\rho d\beta^2$.

Since $\mathcal{H}^{(2,2)}$ admits a natural $SO(2,2)$ group action, the resulting 7D supergravity theory has maximal (32 supercharge) supersymmetry and a gauged $SO(2,2)$ symmetry, linearly realized on $SO(2) \times SO(2)$. Note how this fits neatly into the general scheme of extended Salam-Sezgin gauged models.

The Kaluza-Klein spectrum

Reduction on the non-compact $\mathcal{H}^{(2,2)}$ space from ten to seven dimensions, despite its mathematical consistency, does not provide a full physical basis for compactification to 4D, however. The chief problem is that the truncated Kaluza-Klein modes form a continuum instead of a discrete set with mass gaps. Moreover, the wavefunction of "reduced" 4D states when viewed from 10D or 11D includes a non-normalizable factor owing to the infinite $\mathcal{H}^{(2,2)}$ directions. This infinite transverse volume also has the consequence that the resulting 4D Newton constant vanishes. Accordingly, the higher-dimensional supergravity theory does not naturally localize gravity in the lower-dimensional subspace when handled by ordinary Kaluza-Klein methods.

Bound states and mass gaps Crampton, Pope & K.S.S., JHEP 1412 (2014) 035; 1408.7072

An approach to solving the non-localization problem of gravity on the 4D subspace of the ground-state Salam-Sezgin (SS) solution is to look for a *normalizable* transverse-space wavefunction with a mass gap before the onset of the continuous massive Kaluza-Klein spectrum. This could be viewed as analogous to an effective field theory for a system confined to a metal by a nonzero work function.

General study of the fluctuation spectra about brane solutions shows that the mass spectrum of the spin-two fluctuations about a brane background is given by the spectrum of the scalar Laplacian in the transverse embedding space of the brane

Csaki, Erlich, Hollowood & Shirman, Nucl.Phys. B581 (2000) 309; Bachas & Estes, JHEP 1106 (2011) 005

$$
\begin{aligned}
\Box_{(10)} F &= \frac{1}{\sqrt{-\det g_{(10)}}} \partial_M \left(\sqrt{-\det g_{(10)}}\, g_{(10)}^{MN} \partial_N F \right) \\
&= H_{\rm SS}^{\frac{1}{4}} (\Box_{(4)} + g^2 \triangle_{\theta,\phi,y,\psi,\chi} + g^2 \triangle_{\rm rad}) \\
H_{\rm SS} &= (\cosh 2\rho)^{-1} \text{ warp factor;} \quad \triangle_{\rm rad} = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\tanh(2\rho)} \frac{\partial}{\partial \rho}
\end{aligned}
$$

The directions θ, ϕ, y, ψ & χ are all compact, and one can employ ordinary Kaluza-Klein methods for reduction on them by truncating to the invariant sector for these coordinates, *i.e.* by making an S-wave reduction.

To handle the noncompact radial direction ρ, one needs to expand in eigenmodes of $\triangle_{\rm rad}$. The ansatz for 4D metric fluctuations simply replaces $\eta_{\mu\nu}$ in the 10D metric by $\eta_{\mu\nu} + h_{\mu\nu}(x, \rho)$, where one may take $\partial^\mu h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$

$$h_{\mu\nu}(x,\rho) = \sum_i h^{\lambda_i}_{\mu\nu}(x)\xi_{\lambda_i}(\rho) + \int_{\Lambda_{\rm edge}}^{\infty} d\lambda h^{\lambda}_{\mu\nu}(x)\xi_\lambda(\rho)$$

in which the ξ_{λ_i} are discrete eigenmodes and the ξ_λ are continuous Kaluza-Klein eigenmodes of the scalar Laplacian $\triangle_{\rm rad}$; their eigenvalues give the Kaluza-Klein masses $m^2 = g^2\lambda$ in 4D from $\Box_{(10)} h^\lambda_{\mu\nu} = 0$ using $\triangle_{\theta,\phi,y,\psi,\chi} h^\lambda_{\mu\nu}(x,\rho) = 0$:

$$\begin{aligned} \triangle_{\rm rad}\xi_\lambda(\rho) &= -\lambda\xi_\lambda(\rho) \\ \Box_{(4)} h^\lambda_{\mu\nu}(x) &= (g^2\lambda) h^\lambda_{\mu\nu}(x) \end{aligned}$$

The Schrödinger problem

One can rewrite the $\triangle_{\text{rad}}$ eigenvalue problem as a Schrödinger equation by making the substitution

$$\Psi_\lambda = \sqrt{\sinh(2\rho)}\xi_\lambda$$

after which the first derivative term is eliminated and the eigenfunction equation takes the Schrödinger equation form

$$-\frac{d^2\Psi_\lambda}{d\rho^2} + V(\rho)\Psi_\lambda = \lambda\Psi_\lambda$$

where the potential is

$$V(\rho) = 2 - \frac{1}{\tanh^2(2\rho)}$$

The SS Schrödinger equation potential $V(\rho)$ asymptotes to the value 1 for large ρ. In this large-ρ limit, the Schrödinger equation becomes

$$\frac{d^2\Psi_\lambda}{d\rho^2} + 4e^{-4\rho}\Psi_\lambda + (\lambda - 1)\Psi_\lambda = 0$$

giving scattering-state solutions for $\lambda > 1$:

$$\Psi_\lambda(\rho) \sim \left(A_\lambda e^{i\rho\sqrt{\lambda-1}} + B_\lambda e^{-i\rho\sqrt{\lambda-1}}\right) \qquad \text{for large } \rho$$

while for $\lambda < 1$, one can have L^2 normalizable candidate bound states. Recalling the ρ dependence of the measure $\sqrt{-g_{(10)}} \sim (\cosh(2\rho))^{\frac{1}{4}} \sinh(2\rho)$, one finds for large ρ the normalizability requirement

$$\int_{\rho_1 \gg 1}^{\infty} |\Psi_\lambda(\rho)|^2 d\rho < \infty \Rightarrow \Psi_\lambda \sim B_\lambda e^{-\rho\sqrt{1-\lambda}} \text{ for } \lambda < 1$$

So for $\lambda < 1$ we can have candidate bound states.

Asymptotic conformal invariance and its puzzles

The limit as $\rho \to 0$ of the potential $V(\rho) = 2 - 1/\tanh^2(2\rho)$ is just $V(\rho) = -1/(4\rho^2)$. The associated Schrödinger problem has a long history as one of the most puzzling cases in one-dimensional quantum mechanics. It has been studied and commented upon by Von Neumann; Pauli; Case; Landau & Lifshitz; de Alfaro, Fubini & Furlan, and many others.

A key feature of this 1D problem is its $SO(1,2)$ *conformal invariance*. This symmetry has the consequence that, at the classical level, there is no way to form a definite scale for the transverse Laplacian eigenvalue of an L^2 normalizable ground state. (Except for the value zero, which is what will happen, as we shall see.)

Discussions of the corresponding quantum theory require a regularization that breaks this 1D conformal symmetry and gives rise to the choice of a *self-adjoint extension* for the domain of the Laplacian in order to determine the ground state. The $-1/4$ coefficient is also key: for coefficients $\alpha > -1/4$, there is no L^2 normalizable ground state, while for $\alpha < -1/4$, an infinity of L^2 normalizable discrete bound states appear.

For the precise coefficient $\alpha = -1/4$, a regularized treatment shows the existence of a *single* L^2 normalizable bound state separated by a mass gap and lying below the continuum of scattering states. A.M. Essin & D.J. Griffiths, Am.J. Phys. 74, 109 (2006) The precise eigenvalue of this ground state, however, is not fixed by normalizability considerations and hence remains, so far, a free parameter of the quantum theory.

The zero-mode bound state

The SS Schrödinger potential $V(\rho) = 2 - \coth^2(2\rho)$ diverges as $\rho \to 0$; this is a regular singular point of the Schrödinger equation. Near $\rho = 0$, solutions have a structure given by a Frobenius expansion

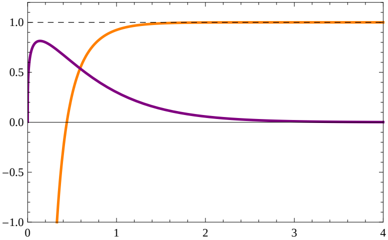
$$\Psi_\lambda \sim \sqrt{\rho}(a_\lambda + b_\lambda \log \rho)$$

This behavior at the origin does not affect L^2 normalizability, but it does indicate that we have a family of candidate bound states characterized by $\theta = \arctan(\frac{a_\lambda}{b_\lambda})$. Numerical study shows that there is a $1 \leftrightarrow 1$ relationship between θ and the eigenvalue λ. Moreover, the limit of a candidate wavefunction $\xi_\lambda \sim a_\lambda + b_\lambda \log \rho$ is singular as $\rho \to 0$, in contrast to the smooth character of the underlying Salam-Sezgin spacetime.

We need some way to select a specific ground state, hopefully corresponding to massless 4D gravitons, and at the same time to justify the $\rho \to 0$ behavior.

For $\lambda = 0$ the Schrödinger equation luckily can be solved in terms of simple functions. The exact result, corresponding to $\theta = 0$ (*i.e.* to a Ψ wavefunction that is asymptotically pure $\sqrt{\rho}\log\rho$ as $\rho \to 0$) is

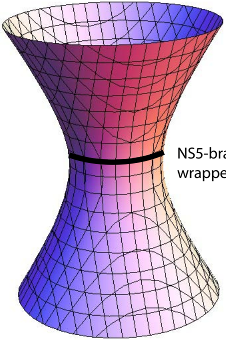
$$\Psi_0(\rho) = \sqrt{\sinh(2\rho)}\xi_0(\rho) = \frac{2\sqrt{3}}{\pi}\sqrt{\sinh(2\rho)}\log(\tanh\rho)$$



$\mathcal{H}^{(2,2)}$ Schrödinger equation potential (orange) and zero-mode Ψ_0 (purple)

The Salam-Sezgin background with an NS5-brane inclusion

Justifying the singularity of the $\xi(\rho)$ bound state as $\rho \to 0$ requires introduction of some other element into the solution. It turns out that what can be included nicely is an NS5-brane. Güven 1992



$\mathcal{H}^{(2,2)}$ space with an NS5-brane source wrapped around its 'waist' and smeared on a transverse S^2

In the Einstein frame, the 10D nonsingular SS solution has the metric (with $\mu = 0, 1, 2, 3$ corresponding to the 4D subspace)

$$\begin{aligned} d\hat{s}_{10}^2 &= (\cosh 2\rho)^{\frac{1}{4}} \Big(dx^\mu dx_\mu + dy^2 \\ &\quad + \frac{1}{4g^2} \Big\{ 4d\rho^2 + \big(d\psi + \mathrm{sech} 2\rho \,(d\chi + \cos\theta d\varphi) \big)^2 \\ &\quad + \tanh^2 2\rho \,(d\chi + \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta \, d\varphi^2 \Big\} \Big) \end{aligned}$$

accompanied by flux from the 2-form gauge field

$$\hat{A}_2 = \frac{1}{4g^2} \Big[d\chi + \mathrm{sech} 2\rho \, d\psi \Big] \wedge (d\chi + \cos\theta \, d\varphi)$$

and the dilaton, asymptotically linear as $\rho \to \infty$,

$$e^{-2\hat{\phi}} = \cosh 2\rho$$

The SS solution has 8 unbroken supersymmetries arising as solutions of the 10D Killing spinor equations (written in string frame)

$$\begin{aligned}\delta\psi_M &= \nabla_M \epsilon - \tfrac{1}{8} F_{MNP}\, \Gamma^{NP}\, \Gamma_{11}\, \epsilon = 0 \\ \delta\lambda &= \Gamma^M \partial_M \phi\, \epsilon - \tfrac{1}{12} F_{MNP}\, \Gamma^{MNP}\, \Gamma_{11}\, \epsilon = 0\,.\end{aligned}$$

These Killing spinor equations have solutions

$$\epsilon = e^{-\frac{1}{2}\chi\, \Gamma_{89}}\, \eta$$

where the constant spinor η satisfies the projection conditions

$$\Gamma_{11}\, \eta = -\eta\,, \qquad \Gamma_{67}\, \eta = \Gamma_{89}\, \eta\,.$$

The key to generalizing the SS solution by the inclusion of an NS5-brane Güven 1992 is first to dimensionally reduce it to 9D on the 'waist' coordinate $z = \frac{1}{2g}\psi$ and then to recognize its structure as a "brane resolved through transgression".

Cvetič, Lü & Pope, Nucl. Phys. B600 (2001) 103

It is convenient to work first in 10D string frame, $d\hat{s}^2_{10\,\mathrm{str}} = e^{\frac{1}{2}\hat{\phi}}\, d\hat{s}^2_{10\,\mathrm{ein}}$, after which the reduction ansatz takes the simple form $d\hat{s}^2_{10\,\mathrm{str}} = ds^2_{9\,\mathrm{str}} + e^{\sqrt{2}\,\phi_2}\,(dz + \mathcal{A}_{(1)})^2$ and the 10D dilaton is given by $\hat{\phi} = -\sqrt{\frac{7}{8}}\,\phi_1 + \frac{1}{\sqrt{8}}\,\phi_2$.

One then recognizes the SS solution as a special 9D case of a 4-brane solution

$$ds^2_{9\,\mathrm{str}} = dX^{\tilde{\mu}} dX_{\tilde{\mu}} + g^{-2} H_{\mathrm{SS}}\, d\bar{s}^2_4\,, \qquad e^{-\sqrt{\frac{7}{2}}\,\phi_1} = H_{\mathrm{SS}}$$

where $\tilde{\mu} = \mu\ (= 0, 1, 2, 3)$; y are 5D coordinates and

$$\begin{aligned} d\bar{s}^2_4 &= \big(\cosh 2\rho\, d\rho^2 + \tfrac{1}{4}\cosh 2\rho (d\theta^2 + \sin\theta\, d\phi^2) \\ &\qquad + \tfrac{1}{4}\sinh 2\rho \tanh 2\rho (d\chi + \cos\theta\, d\phi)^2\big) \\ H_{\mathrm{SS}} &= (\cosh 2\rho)^{-1}\,. \end{aligned}$$

Changing the radial coordinate according to $\cosh 2\rho = r^2$, the underlying 4D transverse metric becomes

$$d\bar{s}_4^2 = \left(1 - \frac{1}{r^4}\right)^{-1} dr^2 + \tfrac{1}{4} r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) + \tfrac{1}{4}\left(1 - \frac{1}{r^4}\right) r^2 \, (d\chi + \cos\theta\, d\varphi)^2$$

which one recognizes as the unit-scale self-dual Ricci-flat Eguchi-Hanson metric.

This solution fits into the system of "brane resolution through transgression" because the 3-form field strength for the 9D 2-form reduced from A_2 obeys the Bianchi identity $dF_{(3)} = -F_{(2)} \wedge \mathcal{F}_{(2)}$ where $F_{(2)}$ and $\mathcal{F}_{(2)}$ are self-dual in the $d\bar{s}_4^2$ metric. The 4-brane ansatz

$$e^{\sqrt{\frac{7}{2}}\,\phi_1} * F_{(3)} = dA_{(5)} \ , \qquad A_{(5)} = H^{-1}\, d^5 X \, , \qquad e^{\sqrt{\frac{7}{2}}\phi_1} = H^{-1}$$

then yields then a solution provided H satisfies

$$\triangle_{\mathrm{EH}(4)}\, H = \tfrac{g^2}{2} F^{ij}\, \mathcal{F}_{ij} \, ,$$

where $\triangle_{\mathrm{EH}(4)} = \mathrm{sech} 2\rho\, \triangle_{\mathrm{rad}}$ is the radial Eguchi-Hanson Laplacian.

In the present case, one has the self-dual 2-forms

$$F_{(2)} = -\mathcal{F}_{(2)} = \frac{-2}{g \cosh^2 2\rho} \left(\bar{e}^6 \wedge \bar{e}^7 - \bar{e}^8 \wedge \bar{e}^9\right)$$

so for H one requires

$$\triangle_{\mathrm{EH}(4)} H = \tfrac{g^2}{2} F^{ij} \mathcal{F}_{ij} = -\frac{8}{\cosh^4 2\rho} \,.$$

The SS "vacuum" solution to this equation has $H_{\mathrm{SS}} = \mathrm{sech}2\rho$, but one can now straightforwardly generalize this by inclusion of a homogeneous $\tilde{H}$ solution: $H = \tilde{H} + H_{\mathrm{SS}}$, where

$$\tilde{H} = c_1 + c_2 \log \tanh \rho$$

in which c_1 and c_2 are integration constants. Then, returning to Einstein frame in 10D, one has the generalized SS + NS-5 solution

$$\begin{aligned} d\hat{s}^2_{10} &= H^{-\frac{1}{4}}(dx^\mu dx_\mu + dy^2 + \tfrac{1}{4g^2}\left[d\psi + \mathrm{sech}2\rho\,(d\chi + \cos\theta\, d\varphi)\right]^2) + H^{\frac{3}{4}}\, d\bar{s}^2 \\ e^{\hat{\phi}} &= H^{\frac{1}{2}}\,, \quad \hat{A}_2 = \tfrac{1}{4g^2}\left[(1-c_2)\, d\chi + \mathrm{sech}2\rho\, d\psi\right] \wedge (d\chi + \cos\theta\, d\varphi)\,. \end{aligned}$$

Reconsidering the fluctuation problem about the deformed SS + NS5-brane metric, one now finds that the transverse wavefunction ξ_λ with eigenvalue λ must satisfy

$$\triangle_{\mathsf{EH(4)}}\xi_\lambda + \lambda H\xi = 0 \quad \text{(up to NS-5 source terms)}$$

in which $\triangle_{\mathsf{EH(4)}} = (\sinh 2\rho \cosh 2\rho)^{-1}\frac{\partial}{\partial\rho}(\sinh 2\rho \frac{\partial}{\partial\rho})$ is, as above, the radial part of the Eguchi-Hanson Laplacian. Demanding L^2 normalizability of eigenmodes in the generalized metric requires choosing $c_1 = 0$. Note then that $-\log(\tanh\rho)$ and the original H_{SS} function sech2ρ in H have the same $2e^{-2\rho}$ asymptotic behavior as $\rho \to \infty$. Consequently, the $\rho \to \infty$ asymptotic form of the Schrödinger problem remains unchanged with respect to the undeformed SS system. Letting $c_2 = -k$ with $k > 0$, all that happens asymptotically is that the eigenvalue $\lambda = g^{-2}m^2$ effectively gets replaced by $\tilde{\lambda} = \lambda(1+k)$.

Since the modified function H has factorized out, the zero mode ξ_0 turns out to be exactly the *same* as in the original SS ground-state solution prior to the NS5-brane inclusion:

$$\xi_0 = \log(\tanh\rho)\,.$$

The NS5-brane source and boundary conditions on $\xi(\rho)$

The source action for an NS5-brane smeared over a transverse S^2 is

$$I_s = \frac{-T}{\Omega_2} \int d^2\Omega \int d^6\zeta \left(-\det\left(\partial_i x^M \partial_j x^N g_{MN}(x(\zeta)) \right) \right)^{\frac{1}{2}} e^{-\phi/2}$$

With the inclusion of this source, the relevant part of the Einstein equation for the static SS + NS5 background plus the transverse part of the 4D gravity fluctuation is:

$$g^2 \eta_{\mu\nu} \triangle_{EH} \tilde{H} - H^2 \Box_{(4)} h_{\mu\nu} \xi - g^2 H h_{\mu\nu} \triangle_{EH} \xi = \\ - T \frac{g^4}{\sqrt{g_{EH}}} (\eta_{\mu\nu} - h_{\mu\nu} \xi(\rho)) \delta^2(z)$$

Integrating this system over a disc around the origin out to radius ϵ yields, consistently for the static background and for the fluctuation term, a relation between the source tension T and the integration constant k in $\tilde{H}$: $k = \frac{2Tg^2}{\pi}$.

In order to determine fully the boundary condition on the transverse wavefunction ξ, it turns out to be necessary to expand the delta-function source slightly and then take a limit. Accordingly, one replaces the pointlike delta-function by a ring delta-function $d^2z\delta^2(z) = \frac{1}{2\pi}d\rho d\chi\delta(\rho - \epsilon)$.

The indicial equation for ξ shows that the asymptotic structure of ξ for any candidate eigenvalue λ is $\xi(\rho) = a + b\log\rho$. From the NS5-sourced field equation, one then obtains the relation $a = b(\frac{\pi k}{2Tg^2} - 1)\log\epsilon$. At the same time, the relation between k and T is modified to give $\frac{\pi k}{2Tg^2} - 1 = \frac{2}{3}\epsilon^2 + \mathcal{O}(\epsilon^4)$.

Putting these together, one learns $a = \frac{2}{3}b\epsilon^2\log\epsilon + \mathcal{O}(\epsilon^4)$, so upon taking $\epsilon \to 0$ one learns $a/b \to 0$, *i.e.* $\theta = 0$.

Numerical study of the Schrödinger eigenvalue problem shows that $\theta = \arctan(a/b)$ is a monotonic function of the eigenvalue λ. Since the zero-mode $\xi_0 = \log\tanh\rho$ becomes pure $\log\rho$ as $\rho \to 0$, this must be the *only* bound state consistent with the boundary conditions imposed by the NS5-brane source.

The above results establish a Kaluza-Klein spectrum with a *mass gap* between the massless 4D graviton states supported by the transverse zero-mode $\xi_0 = \log\tanh\rho$ and the continuum of massive states supported by the transverse scattering states, with m^2 values beginning at the continuum edge $g^2(1+k)$. Although this braneworld spectrum does not constitute a fully KK consistent truncation to 4D of the 10D theory, the mass gap establishes a band of low energies at which the theory becomes effectively four-dimensional: gravity is localized on the 4D subspace.

Another aspect of this SS+NS5 system that remains unchanged is supersymmetry: the modified solution has 8 unbroken 4D supersymmetries, just like the original SS solution on which it was based. This may be further broken down to 4D $N=1$ supersymmetry by incorporating a Hořava-Witten mechanism on the y coordinate; gauge anomaly cancellation may also be achieved this way. Pugh, Sezgin & K.S.S., JHEP 1102 (2011) 115 Moreover, the reduction to 4D may also be arranged so as to preserve chirality in the reduced 4D theory. Pugh, Pope & K.S.S., JHEP 1202 (2012) 098

The braneworld Newton constant

Reducing to 4D on the NS-5 modified SS solution, gravity has an effective action

$$\frac{g^3}{16\pi G_{(10)}} V_{(5)} \int d\rho \sqrt{g_{\mathrm{EH}}} H \int d^4x (\partial_\mu h_{\sigma\tau}(x)\partial^\mu h^{\sigma\tau}(x)|\xi(\rho)|^2 + \dots)$$

where $V_{(5)} = g^{-4}\pi^2 \ell_y$ is the volume of the 5 compact directions.

For conventional Kaluza-Klein reduction with $\xi(\rho) = \mathrm{const}$, the ρ integral diverges and one finds $G_{(4)} = 0$ for the 4D Newton constant. For the $\xi_0(\rho) = \log\tanh\rho$ bound state in the SS + NS5 geometry, however, the integral now converges and one obtains a finite 4D Newton constant. The corresponding gravitational coupling constant $\kappa_{(4)} = \sqrt{32\pi G_{(4)}}$ is

$$\begin{aligned} \kappa_{(4)} &= \sqrt{32\pi g}\sqrt{\frac{G_{(10)}}{V_{(5)}}} \frac{\int d\rho \sinh 2\rho (1 - k\cosh 2\rho \log\tanh\rho)\xi^3}{(\int d\rho \sinh 2\rho (1 - k\cosh 2\rho \log\tanh\rho)\xi^2)^{\frac{3}{2}}} \\ &= 144\sqrt{6}\zeta(3)\left(\frac{G_{(10)} g^5}{\pi^7 \ell_y}\right)^{\frac{1}{2}} \frac{(1+2k)}{(2+3k)^{\frac{3}{2}}} . \end{aligned}$$

Conclusions and further questions

• Inclusion of an NS5-brane on a Salam-Sezgin hyperbolic 10D spacetime solution of type IIA supergravity successfully localizes massless gravity near the NS5-brane subsurface. This is in contrast to situations previously considered, *e.g.* with asymptotically maximally symmetric spacetimes, where localization fails and was thought to be impossible when attempted with natural string or M-theory constructions.

• Incorporation of this structure into a string theory construction remains an important topic for investigation. The linear dilaton background is a familiar enough string theory background. As one approaches $\rho \to 0$, the Güven NS-5 brane dominates. There may be a relation, *e.g.* to the Compère & Marolf boundary conditions for AdS/CFT that retain the boundary metric degrees of freedom.