

Supersymmetric gauged Double Field Theory: Systematic derivation by virtue of *Twist*

Stringy Geometry, MITP

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arXiv1505.01301, with Wonyong Cho, J.J. Fernandez-Melgarejo, Jeong-Hyuck Park

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Based on works

in collaboration with Jeong-Hyuck Park, Kanghoon Lee, Yoonji Suh*

- **Differential geometry with a projection: Application to double field theory**
JHEP 1104:014 (2011), arXiv:1011.1324
- **Double field formulation of Yang-Mills theory**
Phys. Lett. B 701:260 (2011), arXiv:1102.0419
- **Stringy differential geometry, beyond Riemann**
Phys. Rev. D 84:044022 (2011), arXiv:1105.6294
- **Incorporation of fermions into double field theory**
JHEP 1111:025 (2011), arXiv:1109.2035
- **Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity** Phys. Rev. D Rapid Comm. (2012), arXiv:1112.0069
- **Ramond-Ramond Cohomology and $O(D,D)$ T-duality**
JHEP 09 (2012) 079, arXiv:1206.3478
- **Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory***
PLB723 (2013) 245-250, arXiv:1210.5078

1.Introduction

- String theory says that $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ should be treated on an equal footing , because they form a **multiplet of T-duality**.
- This suggests that there should be an **unifying description of them**, beyond the Riemannian geometry.
- **Double Field Theory(DFT)** has been suggested as an unifying description of string effective theory, which captures the stringy effect by **manifesting the T-duality structure** [Siegel, Tseytlin, Duff, Hull, Zwiebach, Hohm]
cf. U-duality extension: Exceptional Field Theory(EFT)[Hohm, Samtleben, Berman, Cederwal...]

(Bosonic) Double Field Theory

“Double”: $x \longrightarrow (x, \tilde{x})$

- This originates from the string field theory.

The fields in SFT on a toroidal background have dependance on p_i and w^i ,

$$p_i \longleftrightarrow x^i \quad w^i \longleftrightarrow \tilde{x}_i$$

so have natural dependance on both x and \tilde{x} .: $\Phi(x^i, \tilde{x}_i)$ [Kugo, Zwiebach]

- In DFT, we focus on "massless sector".

Bosonic Double Field Theory

- DFT manifests the T-duality by using $\mathbf{O}(D, D)$ tensors as its dynamical variables.

e.g. NS-NS fields in DFT :

$$\begin{array}{ll} \text{dilaton ,} & \text{'generalized metric'} \\ \text{(scalar density)} & \text{(symmetric } \mathbf{O}(D, D) \text{ element)} \\ e^{-2d} = \sqrt{-g}e^{-2\phi} , & \mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \end{array}$$

- The Buscher's rule is realized as an subgroup of $\mathbf{O}(D, D)$ rotation, [Giveon, Rabinovici, Veneziano, Tseytlin, Siegel] :
 d is scalar and \mathcal{H}_{AB} is rank 2 tensor.
- Metric in DFT : $\mathbf{O}(D, D)$ metric,

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

freely raises or lowers the $(D + D)$ -dimensional vector indices, A, B .

Bosonic Double Field Theory

- DFT action for NS-NS sector: **Hull** and **Zwiebach**, later with **Hohm**

$$S_{\text{DFT}} = \int dy^{2D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d),$$

where

$$\begin{aligned} L_{\text{DFT}}(\mathcal{H}, d) = & \mathcal{H}^{AB} (4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD}) \\ & + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}. \end{aligned}$$

- $O(D, D)$ structure is manifest and background independent.
- All spacetime dimension is 'formally doubled', $y^A = (\tilde{x}_\mu, x^\nu)$,
 $A = 1, 2, \dots, D+D$.

Section condition

- DFT is a D -dimensional theory written in terms of $(D + D)$ -dimensional language, i.e. tensors.
- **Section condition (strong constraint)** :
The $\mathbf{O}(D, D)$ d'Alembert operator is trivial, acting on arbitrary fields or gauge parameters as well as their products:

$$\partial_A \partial^A = \mathcal{J}^{AB} \partial_A \partial_B = 2 \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} \sim 0$$

- equivalently understood as the *doubled-yet-gauged spacetime* [Lee, Park]
- DFT lives on a **D -dimensional null hyperplane** and $\mathbf{O}(D, D)$ rotates the null hyperplane.
- Up to $\mathbf{O}(D, D)$ rotation, we can choose a frame to set

$$\frac{\partial}{\partial \tilde{x}_\mu} \sim 0.$$

- DFT action is (locally) equivalent to the effective action:

$$S_{\text{DFT}} \implies S_{\text{eff.}} = \int dx^D \sqrt{-g} e^{-2\phi} \left(R_g + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right).$$

Section condition

- **Level matching condition** for the massless sector,

$$p \cdot w = N - \bar{N} \equiv 0 \iff \partial_A \partial^A = 2 \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} \equiv 0,$$

for all fields. (weak constraint)

So,

$$\partial_A \partial^A \Phi = 0, \quad \text{but } \partial_A \Phi_1 \partial^A \Phi_2 \neq 0$$

- Section condition (strong constraint) seems necessary to write a complete theory, because of **action invariance** and **closedness of symmetry algebra**.

Gauge symmetry: ‘DFT-diffeomorphism’

Unification of diffeomorphism and B -field gauge symmetry , expressed *via*

- ‘generalized Lie derivative’ [Siegel, Courant, Grana ...]

$$\hat{\mathcal{L}}_X T_{\omega_A} := X^B \partial_B T_{\omega_A} + \omega \partial_B X^B T_{\omega_A} + \partial_A X^B T_{\omega_B} - \partial^B X_A T_{\omega_B}.$$

- X^A is an unifying gauge parameter (B -field gauge symmetry + diffeomorphism),

$$X^A = (\Lambda_\mu, \delta x^\nu)$$

- \mathcal{H}_{AB} is a rank 2 DFT tensor, and e^{-2d} is a weight 1 DFT scalar,
- Finite version of transformation [Hohm, Zwiebach; Park]

Algebra of the gauge symmetry

- Commutator of the generalized Lie derivatives is closed, **up to the section condition**, by using **c**-bracket,

$$[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] \sim \hat{\mathcal{L}}_{[X, Y]_C},$$

where $[X, Y]_C$ denotes *C*-bracket

$$[X, Y]_C^A := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B,$$

known to be equivalent to the Courant bracket if dropping $\tilde{\partial}$.

- Understanding the level matching condition(weak constraint) in DFT is subtle and difficult.
- However, "relaxing" the section condition to some extent has been understood. [Aldazabal, Baron, Nunez, Grana, Marqués, Geissbühler]
The section condition is sufficient but not the necessary condition for the algebra closure and action invariance.
- The relaxation of the section condition is allowed when doing Sherk-Schwarz reduction in DFT and it gets gauged DFT
- A variety of the known gauged supergravities in lower dimensions can be reproduced, i.e. DFT provides the higher dimensional origin of a various gauged supereravities.
 - A systematic classification of all possible deformations as for gaugings is allowed by embedding tensor method [de Wit, Samtleben]
 - Only some of them can be obtained by a Sherk-Schwarz dimensional reduction of 11 or 10-dimensional supergravities.
- This is an indication that DFT goes beyond the ordinary supergravity or Generalized geometry .
- Particularly, one needs explicitly section condition breaking terms, which depend on both of x and \tilde{x} , to reproduce the complete classification of the gauged SUGRA[Geissbühler]

- Geometric understanding of DFT
 - Flux formulatin [Hohm, Kwak]
 - **Semi-covariant formulation** [IJ, Lee, Park]
 - Direct analogy of the Riemann geometry using Christofel connection,
 - Fully covariant with respect to all the symmetries in DFT,
 - Maximal and half maximal supersymmetric DFT is realized in full order of fermions, where Maximal supersymmetric DFT unifies the type II supergravities [IJ, Lee, Park, Suh]
- Geometric understanding of gauged DFT
 - in flux formulation (for bosonic DFT) [Geissbuhler, Marques, Aldazabal, Nunez, Berman, Perry, Malek]
 - in semi-covariant formulation, partially understood by torsionful deformation of gauged DFT, half-maximal supersymmetry was realized.[Berman, Lee]

Goal

- To have **systematic understanding** the low dimensional gauged SDFT in the semi-covariant formulation
 - We twist the semi-covariant formulation of the ungauged SDFT **without an ambiguity**.
 - By the formulation, all the symmetries in DFT are fully covariant.
 - Torsionful deformation of the gauged DFT is derived from twisting.
 - Definition of curvature includes the section condition breaking term
- To realize the **maximal** as well as **half maximal** supersymmetric gauged DFT in full order of fermions

Contents

1. Introduction

2. Semi-covariant formulation of ungauged $D = 10$ SDFT (section condition " \sim ")

: Semi-covariant derivative, connection and curvature, supersymmetry

3. Twisting the semi-covariant formulation to get gauged SDFT

twistability conditions for closure of the algebra

maximal and half maximal supersymmetry in full order of fermions

4. Summary

2. Semi-covariant formulation of DFT/SDFT

[1011.1324, 1105.6294, 1109.2035, 1112.0069, 1206.3478, 1210.5078]

Symmetries of SDFT

- **$O(D, D)$ T-duality:**
- **DFT-diffeomorphism (generalized Lie derivative)**
 - **Diffeomorphism**
 - **B -field gauge symmetry**
- **A pair of local Lorentz symmetries, $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$**
- **$D = 10$ Local SUSY**
- **Ordinary SUGRA \equiv diagonal gauge-fixed SDFT,**

Field contents of $D = 10$ Maximal SFT

- Bosons**

- NS-NS sector** $\left\{ \begin{array}{ll} \text{DFT-dilaton:} & d \\ \text{Double-vielbeins:} & V_{Ap}, \quad \bar{V}_{A\bar{p}} \end{array} \right.$
 - R-R potential:** $C^{\alpha}_{\bar{\alpha}}$

- Fermions**

- DFT-dilatinos:** $\rho^{\alpha}, \quad \rho'^{\bar{\alpha}}$
 - Gravitinos:** $\psi_{\bar{p}}^{\alpha}, \quad \psi_{\bar{p}}'^{\bar{\alpha}}$

Index	Representation	Metric (raising/lowering indices)
A, B, \dots	$\mathbf{O}(D, D)$ vector	\mathcal{J}_{AB}
p, q, \dots	$\text{Spin}(1, D-1)_{\mathbb{L}}$ vector	$\eta_{pq} = \text{diag}(- + + \dots +)$
α, β, \dots	$\text{Spin}(1, D-1)_{\mathbb{L}}$ spinor	$C_{+\alpha\beta}, \quad (\gamma^p)^T = C_+ \gamma^p C_+^{-1}$
\bar{p}, \bar{q}, \dots	$\text{Spin}(D-1, 1)_{\mathbb{R}}$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \text{diag}(+ - - \dots -)$
$\bar{\alpha}, \bar{\beta}, \dots$	$\text{Spin}(D-1, 1)_{\mathbb{R}}$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}, \quad (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+ \bar{\gamma}^{\bar{p}} \bar{C}_+^{-1}$

Field contents of $D = 10$ Maximal SFT

- **Bosons**

- **NS-NS sector** $\left\{ \begin{array}{l} \text{DFT-dilaton:} \\ \text{Double-vielbeins:} \end{array} \right. \quad \begin{array}{l} d \\ V_{Ap}, \quad \bar{V}_{A\bar{p}} \end{array}$
- **R-R potential:** $C^{\alpha}_{\bar{\alpha}}$

- **Fermions**

- **DFT-dilatinos:** $\rho^{\alpha}, \quad \rho'^{\bar{\alpha}}$
- **Gravitinos:** $\psi_{\bar{p}}^{\alpha}, \quad \psi'_p{}^{\bar{\alpha}}$

All NS-NS fields, $d, V_{Ap}, \bar{V}_{A\bar{p}}$, will be equally treated as basic geometric objects.

Field contents of $D = 10$ Maximal SFT

- **Bosons**

- NS-NS sector $\left\{ \begin{array}{l} \text{DFT-dilaton:} \\ \text{Double-vielbeins:} \end{array} \right. \begin{array}{l} d \\ V_{Ap}, \quad \bar{V}_{A\bar{p}} \end{array}$
- **R-R potential:** $C^{\alpha}_{\bar{\alpha}}$

- **Fermions**

- DFT-dilatinos: $\rho^{\alpha}, \quad \rho'^{\bar{\alpha}}$
- Gravitinos: $\psi_{\bar{p}}^{\alpha}, \quad \psi'_p{}^{\bar{\alpha}}$

**R-R potential is bi-fundamental spinor representation
as a democratic description.**

cf. $\mathbf{O}(D, D)$ spinor representation **Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach**

Field contents of $D = 10$ Maximal SFT

- **Bosons**

- NS-NS sector $\left\{ \begin{array}{l} \text{DFT-dilaton:} \\ \text{Double-vielbeins:} \end{array} \right. \quad \begin{array}{l} d \\ V_{Ap}, \quad \bar{V}_{A\bar{p}} \end{array}$
- R-R potential: $C^{\alpha}_{\bar{\alpha}}$

- **Fermions**

- **DFT-dilatinos:** $\rho^{\alpha}, \quad \rho'^{\bar{\alpha}}$
- **Gravitinos:** $\psi_p^{\alpha}, \quad \psi_p'^{\bar{\alpha}}$

Double-vielbein 1105.6294, 1109.2035

- **Double-vielbein** simultaneously diagonalizes \mathcal{J}_{AB} and \mathcal{H}_{AB} ,

$$\mathcal{J} = (V \ \bar{V}) \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \bar{\eta} \end{pmatrix} (V \ \bar{V})^T, \quad \mathcal{H} = (V \ \bar{V}) \begin{pmatrix} \eta^{-1} & 0 \\ 0 & -\bar{\eta} \end{pmatrix} (V \ \bar{V})^T.$$

- It follows the defining properties

$$V_{Ap} V^A_q = \eta_{pq}, \quad V_{Ap} \bar{V}^A_{\bar{p}} = 0, \quad V_A^p V_{Bp} = P_{AB},$$

$$\bar{V}_{A\bar{p}} \bar{V}^A_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}, \quad \bar{V}_A^{\bar{p}} \bar{V}_{B\bar{p}} = \bar{P}_{AB},$$

P_{AB}, \bar{P}_{AB} are projection matrices ('left and right'),

$$P_A^B P_B^C = P_A^C, \quad \bar{P}_A^B \bar{P}_B^C = \bar{P}_A^C, \quad P_A^B \bar{P}_B^C = 0$$

which are related to \mathcal{H} and \mathcal{J} ,

$$P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}, \quad P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}$$

- **Projection is the characteristic property of DFT geometry.**
- The basic geometric objects, which should be treated equally, are

$$(d, V_{Ap}, \bar{V}_{A\bar{p}}), \quad \text{or} \quad (d, P_{AB}, \bar{P}_{AB}).$$

- We further define a pair of six-index projectors,

$$\begin{aligned} \mathcal{P}_{CAB}^{DEF} &:= P_C^D P_{[A}^{[E} P_{B]}^{F]} + \frac{2}{D-1} P_{C[A} P_{B]}^{[E} P^{F]D}, & \mathcal{P}_{CAB}^{DEF} \mathcal{P}_{DEF}^{GHI} &= \mathcal{P}_{CAB}^{GHI}, \\ \bar{\mathcal{P}}_{CAB}^{DEF} &:= \bar{P}_C^D \bar{P}_{[A}^{[E} \bar{P}_{B]}^{F]} + \frac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}^{[E} \bar{P}^{F]D}, & \bar{\mathcal{P}}_{CAB}^{DEF} \bar{\mathcal{P}}_{DEF}^{GHI} &= \bar{\mathcal{P}}_{CAB}^{GHI}, \end{aligned}$$

which satisfy the following properties, symmetric and traceless,

$$\begin{aligned} \mathcal{P}_{CABDEF} &= \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}, & \bar{\mathcal{P}}_{CABDEF} &= \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}, \\ \mathcal{P}^A_{ABDEF} &= 0, \quad P^{AB} \mathcal{P}_{ABCDEF} = 0, & \bar{\mathcal{P}}^A_{ABDEF} &= 0, \quad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0. \end{aligned}$$

- These projectors will govern the DFT-diffeomorphic anomaly in the semi-covariant formalism, which can be easily projected out.

Semi-covariant derivatives

- We introduce master ‘semi-covariant’ derivative

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A .$$

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A ,$$

- The ‘semi-covariant’ derivative for the DFT-diffeomorphism is

$$\nabla_C T_{\omega_{A_1 A_2 \dots A_n}} := \partial_C T_{\omega_{A_1 A_2 \dots A_n}} - \omega \Gamma^B{}_{BC} T_{\omega_{A_1 A_2 \dots A_n}} + \sum_{i=1}^n \Gamma_{CA_i}{}^B T_{\omega_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}} .$$

- compatibility for the whole NS-NS sector

$$\mathcal{D}_A d = 0, \quad \mathcal{D}_A V_{Bp} = 0, \quad \mathcal{D}_A \bar{V}_{B\bar{p}} = 0. \quad (\text{cf. } \mathcal{D}_\mu e_\nu^a = 0)$$

together with

$$\mathcal{D}_A \eta_{pq} = \mathcal{D}_A \bar{\eta}_{\bar{p}\bar{q}} = \mathcal{D}_A (\gamma^p)^\alpha_\beta = \mathcal{D}_A (\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}_{\bar{\beta}} = \mathcal{D}_A C_{+\alpha\beta} = \mathcal{D}_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = 0.$$

It follows that

$$\nabla_A d = 0, \quad \nabla_A P_{BC} = 0, \quad \nabla_A \bar{P}_{BC} = 0, \quad (\text{cf. } \nabla_\mu g_{\nu\lambda} = 0)$$

- Spin connections

$$\Phi_{Apq} = V^B_p \nabla_A V_{Bq}, \quad \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^B_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}},$$

- **Torsion free conection** is uniquely determined in terms of basic geometrical variables, [IJ, Lee, Park '11]

$$\Gamma_{CAB}^0 = 2(P\partial_C P\bar{P})_{[AB]} + 2(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E)\partial_D P_{EC} \\ - \frac{4}{D-1}(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D)(\partial_D d + (P\partial^E P\bar{P})_{[ED]}) ,$$

satisfying the torsion free condition,

$$\Gamma_{[ABC]}^0 = 0, \quad (\Leftrightarrow \quad \hat{\mathcal{L}}_X^\partial = \hat{\mathcal{L}}_X^\nabla)$$

and further satisfying

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma_{DEF}^0 = 0, \quad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma_{DEF}^0 = 0.$$

Stringy differential geometry 1105.6294 (1011.1324)

- Under $\delta_X \mathcal{H}_{AB} = \hat{\mathcal{L}}_X \mathcal{H}_{AB}$, $\delta_X d = \hat{\mathcal{L}}_X d$, namely DFT-diffeomorphism (= diffeomorphism + B -field gauge symmetry), the variation of $\nabla_C T_A$ contains an anomalous non-covariant part,

$$(\delta_X - \hat{\mathcal{L}}_X) \nabla_C T_A \sim 2(\mathcal{P} + \bar{\mathcal{P}})_{CA}{}^{BFDE} \partial_F \partial_{[D} X_{E]} T_B.$$

- Hence, these are not DFT-diffeomorphism covariant,
- However, the anomalous term are controlled by the rank six projectors, so they can be projected out by combining the projection matrices P_{AB} and \bar{P}_{AB} .

Projection-aided covariant derivatives

“semi-covariant derivative” :

combined with the projections , we can get various covariant quantities:

Examples:

- For $O(D, D)$ tensors:

$$P_C{}^D \bar{P}_A{}^B \nabla_D T_B,$$

$$\bar{P}_C{}^D P_A{}^B \nabla_D T_B,$$

$$P^{AB} \nabla_A T_B,$$

$$\bar{P}^{AB} \nabla_A T_B,$$

Divergences ,

$$P^{AB} \bar{P}_C{}^D \nabla_A \nabla_B T_D,$$

$$\bar{P}^{AB} P_C{}^D \nabla_A \nabla_B T_D.$$

Laplacians

- Rule: need opposite chirality or contraction

Projection-aided covariant derivatives

- For $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ tensors:

$$\mathcal{D}_p T_{\bar{q}}, \quad \mathcal{D}_{\bar{p}} T_q,$$

$$\mathcal{D}^p T_p, \quad \mathcal{D}^{\bar{p}} T_{\bar{p}},$$

$$\mathcal{D}_p \mathcal{D}^p T_{\bar{q}}, \quad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_q,$$

where we set

$$\mathcal{D}_p := V^A_p \mathcal{D}_A, \quad \mathcal{D}_{\bar{p}} := \bar{V}^A_{\bar{p}} \mathcal{D}_A.$$

These are the **pull-back** of the previous results using the double-vielbeins.

Projection-aided covariant derivatives

- **Dirac operators for fermions, $\rho^\alpha, \psi_{\bar{p}}^\alpha, \rho'^{\bar{\alpha}}, \psi'_p{}^{\bar{\alpha}}$: [IJ, Lee, Park '11]**

$$\gamma^p \mathcal{D}_p \rho = \gamma^A \mathcal{D}_A \rho, \quad \gamma^p \mathcal{D}_p \psi_{\bar{p}} = \gamma^A \mathcal{D}_A \psi_{\bar{p}},$$

$$\mathcal{D}_{\bar{p}} \rho, \quad \mathcal{D}_{\bar{p}} \psi^{\bar{p}} = \mathcal{D}_A \psi^A,$$

$$\bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \rho' = \bar{\gamma}^A \mathcal{D}_A \rho', \quad \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \psi'_p = \bar{\gamma}^A \mathcal{D}_A \psi'_p,$$

$$\mathcal{D}_p \rho', \quad \mathcal{D}_p \psi'^p = \mathcal{D}_A \psi'^A,$$

Projection-aided covariant derivatives

- For $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ bi-fundamental spinors, $\mathcal{C}^\alpha_{\bar{\beta}}$:
[IJ, Lee, Park '12]

$$\gamma^A \mathcal{D}_A \mathcal{C}, \quad \mathcal{D}_A \mathcal{C} \bar{\gamma}^A.$$

- Further define

$$\mathcal{D}_+ \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} + \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A,$$

$$\mathcal{D}_- \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} - \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent up to the section condition**

$$(\mathcal{D}_+^0)^2 \mathcal{C} \sim 0, \quad (\mathcal{D}_-^0)^2 \mathcal{C} \sim 0,$$

- The field strength of the R-R potential, $\mathcal{C}^\alpha_{\bar{\alpha}}$, is then defined by

$$\mathcal{F} := \mathcal{D}_+^0 \mathcal{C}.$$

Curvatures I [1105.6294]

- From the DFT-diffeomorphism field strength defined by

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED},$$

which is NOT covariant,

- we define *semi-covariant four-index curvature*, as for a **key quantity** in our formalism, cf. [Siegel; Waldram; Hohm, Zwiebach]

$$S_{ABCD} := \frac{1}{2} (R_{ABCD} + R_{CDAB} - \Gamma_{AB}{}^E \Gamma_{ECD}).$$

- It satisfies
 - just like the Riemann curvature,

$$S_{ABCD} = \frac{1}{2} (S_{[AB][CD]} + S_{[CD][AB]}),$$

$$S_{A[BCD]} = 0 \quad : \quad \text{Bianchi identity},$$

- and with projectors,

$$(P^{AB} P^{CD} + \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} \sim 0,$$

$$P_I^A P_J^B \bar{P}_K^C \bar{P}_L^D S_{ABCD} \sim 0,$$

$$P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} \sim 0, \text{ etc.}$$

Curvatures I [1105.6294]

- This is still Not covariant tensor, but contracting with projection operators, we can obtain covariant quantities.

- Rank two-tensor:

$$P_I^A \bar{P}_J^B S_{AB}, \quad \text{where } S_{AB} := S^C{}_{ACB}$$

- **Scalar curvature:** defines the Lagrangian for NS-NS sector

$$(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD}$$

- The above scalar curvature exactly reproduces the bosonic Lagrangian by Hull, Zwiebach, Hohm.
- There is no covariant rank 4 tensor.

Curvatures II

- Alternative way to define the curvature is using the field strength for the local Lorentz group, *c.f.* YM-gauge field strength

$$\mathcal{F}_{ABpq} := \nabla_A \Phi_{Bpq} - \nabla_B \Phi_{Apq} + \Phi_{Ap}{}^r \Phi_{Brq} - \Phi_{Bp}{}^r \Phi_{Arq},$$

$$\bar{\mathcal{F}}_{AB\bar{p}\bar{q}} := \nabla_A \bar{\Phi}_{B\bar{p}\bar{q}} - \nabla_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{r}} \bar{\Phi}_{B\bar{r}\bar{q}} - \bar{\Phi}_{B\bar{p}}{}^{\bar{r}} \bar{\Phi}_{A\bar{r}\bar{q}},$$

We define *Semi-covariant four-index curvature of the spin connections*,

$$\mathcal{G}_{ABCD} := \frac{1}{2} [(\mathcal{F} + \bar{\mathcal{F}})_{ABCD} + (\mathcal{F} + \bar{\mathcal{F}})_{CDAB} + (\Phi + \bar{\Phi})^E{}_{AB} (\Phi + \bar{\Phi})_{ECD}] ,$$

where

$$\mathcal{F}_{ABCD} = \mathcal{F}_{ABpq} V_C{}^p V_D{}^q, \quad \bar{\mathcal{F}}_{ABCD} = \bar{\mathcal{F}}_{AB\bar{p}\bar{q}} \bar{V}_C{}^{\bar{p}} \bar{V}_D{}^{\bar{q}}.$$

Curvatures II

- These two four-index curvatures are closely related to each other,

$$\mathcal{G}_{ABCD} = S_{ABCD} + \frac{1}{2}(V_A{}^P \partial_E V_{Bp} + \bar{V}_A{}^{\bar{P}} \partial_E \bar{V}_{B\bar{P}})(V_C{}^q \partial^E V_{Dq} + \bar{V}_C{}^{\bar{q}} \partial^E \bar{V}_{D\bar{q}}),$$

such that upon the section condition they are equivalent.

- Later, \mathcal{G}_{ABCD} will be the proper curvature when we relax the section condition.
- The later terms will correspond to what Geissbuler introduced in order to reproduce the the low dimensional gauged SUGRA.

D = 10 Maximal SDFT [1210.5078]

- **Lagrangian** (full order of fermions):

$$\mathcal{L}_{\text{Type II}} = e^{-2d} \left[\frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \bar{\mathcal{F}}) - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi}_{\bar{p}} \gamma_q \mathcal{F} \bar{\gamma}^{\bar{p}} \psi'^q \right. \\ \left. + i \frac{1}{2} \bar{\rho} \gamma^p \mathcal{D}_p^* \rho - i \bar{\psi}^{\bar{p}} \mathcal{D}_{\bar{p}}^* \rho - i \frac{1}{2} \bar{\psi}^{\bar{p}} \gamma^q \mathcal{D}_q^* \psi_{\bar{p}} - i \frac{1}{2} \bar{\rho}' \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}'^* \rho' + i \bar{\psi}'^p \mathcal{D}_p'^* \rho' + i \frac{1}{2} \bar{\psi}'^p \bar{\gamma}^{\bar{q}} \mathcal{D}_{\bar{q}}'^* \psi'_p \right]$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_+^{-1} \mathcal{F}^T C_+$.

- \mathcal{D}_A in S_{ACBD} , \mathcal{D}_A^* and $\mathcal{D}_A'^*$ are defined by their own **torsionful connection**,
- The torsions are determined to satisfy usual **1.5 formalism**,

$$\delta \mathcal{L}_{\text{SDFT}} = \delta \Gamma_{ABC} \times 0.$$

- The Lagrangian is **pseudo : self-duality** of the R-R field strength needs to be imposed by hand, just like the ‘democratic’ type II SUGRA **Bergshoeff, et al.**

$$\left(1 - \gamma^{(D+1)} \right) \left(\mathcal{F} - i \frac{1}{2} \rho \bar{\rho}' + i \frac{1}{2} \gamma^p \psi_{\bar{q}} \psi'_{\bar{p}} \bar{\gamma}^{\bar{q}} \right) \sim 0.$$

3. Twisted supersymmetric Double field theory

- Twisting ansatz (Scherk-Schwarz reduction ansatz)
- Twistability condition ("relaxing" the section condition) by closure of the algebra.
- Obtain low dimensional gauged SDFT with maximal and half maximal supersymmetry

U-twisted double field theory

- For the twisting we use the two twisting datas:
a scalar $\lambda(x)$ and $U(x)_A^{\dot{A}} \in \mathbf{O}(D, D)$,

$$U\dot{\mathcal{J}}U^t = \mathcal{J}, \quad \dot{\mathcal{J}}_{\dot{M}\dot{N}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

using which we set the ansatz for *U-twist*

$$T_{A_1 \dots A_n} = e^{-2\omega\lambda} U_{A_1}^{\dot{A}_1} \dots U_{A_n}^{\dot{A}_n} \dot{T}_{\dot{A}_1 \dots \dot{A}_n}.$$

- The $\lambda(x)$ and $U(x)_A^{\dot{A}}$ do not satisfy the section condition, but shall be require to satisfy the consistency conditions, i.e. *twistability condition*.
- The twisted field is denoted by dot with dotted indices,
- and $U(x)_A^{\dot{A}}$ carries one undated index and other dotted index, such that the additional $\mathbf{O}(D, D)$ metric, $\dot{\mathcal{J}}_{\dot{M}\dot{N}}$ is introduced.
- While the twisted metric $\dot{\mathcal{J}}_{\dot{M}\dot{N}}$ may coincide numerically with the untwisted metric \mathcal{J}_{MN} , we deliberately distinguish them as the two different kinds of indices will never be contracted.

U-twisted double field theory

- For the twisting we use the two twisting datas:
a scalar $\lambda(x)$ and $U(x)_{A\dot{A}} \in \mathbf{O}(D, D)$,

$$U\dot{\mathcal{J}}U^t = \mathcal{J}, \quad \dot{\mathcal{J}}_{\dot{M}\dot{N}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

using which we set the ansatz for *U-twist*

$$T_{A_1 \dots A_n} = e^{-2\omega\lambda} U_{A_1 \dot{A}_1} \dots U_{A_n \dot{A}_n} \dot{T}_{\dot{A}_1 \dots \dot{A}_n}.$$

- If we assume the U matrix to be in a block diagonal form,

$$U = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix},$$

and split all the internal coordinate dependency into the U matrix, then this twisting ansatz is nothing but the Sherk-Schwarz reduction ansatz.

- But all the forthcoming analyses do not necessarily demand this ansatz, so we will use above general twisting ansatz.

U-twisted double field theory

- For the twisting we use the two twisting datas:
a scalar $\lambda(x)$ and $U(x)_A^{\dot{A}} \in \mathbf{O}(D, D)$,

$$U \dot{\mathcal{J}} U^t = \mathcal{J}, \quad \dot{\mathcal{J}}_{\dot{M}\dot{N}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

using which we set the ansatz for U -twist

$$T_{A_1 \dots A_n} = e^{-2\omega\lambda} U_{A_1}^{\dot{A}_1} \dots U_{A_n}^{\dot{A}_n} \dot{T}_{\dot{A}_1 \dots \dot{A}_n}.$$

- *The only field variables to be twisted* are

$$e^{-2d} = e^{-2\lambda} e^{-2\dot{d}}, \quad V_{Ap} = U_A^{\dot{A}} \dot{V}_{\dot{A}p}, \quad \bar{V}_{A\bar{p}} = U_A^{\dot{A}} \dot{\bar{V}}_{\dot{A}\bar{p}}.$$

Other fields (fermions and the R-R potential) are weightless and $\mathbf{O}(D, D)$ singlet.

- The twist of the $\mathcal{N} = 1$ or the $\mathcal{N} = 2, D = 10$ SDFT simply amounts to inserting the above expressions for the dilaton and the vielbeins into the untwisted Lagrangian.

U-twisted double field theory

- The derivatives of the untwisted fields then assume a generic form,

$$\partial_C T_{A_1 \dots A_n} = e^{-2\omega\lambda} U_C^{\dot{C}} U_{A_1}^{\dot{A}_1} \dots U_{A_n}^{\dot{A}_n} \dot{D}_{\dot{C}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n},$$

- U-derivative, $\dot{D}_{\dot{C}}$, is defined to act on a twisted field by

$$\dot{D}_{\dot{C}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} := \dot{\partial}_{\dot{C}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} - 2\omega \dot{\partial}_{\dot{C}} \lambda \dot{T}_{\dot{A}_1 \dots \dot{A}_n} + \sum_{i=1}^n \Omega_{\dot{C}\dot{A}_i}^{\dot{B}} \dot{T}_{\dot{A}_1 \dots \dot{B} \dots \dot{A}_n}.$$

With the pull-back of the naked derivative,

$$\dot{\partial}_{\dot{C}} = U^{-1}{}_{\dot{C}}{}^C \partial_C,$$

and a pure gauge “connection”,

$$\Omega_{\dot{C}\dot{A}}^{\dot{B}} := \left(U^{-1} \dot{\partial}_{\dot{C}} U \right)_{\dot{A}}^{\dot{B}},$$

- The U-derivatives are all commutative,

$$[D_A, D_B] = 0, \quad [D_A, \dot{D}_{\dot{B}}] = 0, \quad [\dot{D}_{\dot{A}}, \dot{D}_{\dot{B}}] = 0.$$

U-twisted double field theory

- Those replacement leads to twisted SDFT Lagrangian,

$$\begin{aligned}\mathcal{L}_{D=10}^{\mathcal{N}=1}(\mathcal{J}_{AB}, \partial_A, d, V_{Ap}, \bar{V}_{A\bar{p}}, \rho, \psi_{\bar{p}}) &= e^{-2\lambda} \dot{\mathcal{L}}_{\text{Twisted SDFT}}^{\text{Half-maximal}}(\dot{\mathcal{J}}_{\dot{A}\dot{B}}, \dot{D}_{\dot{A}}, \dot{d}, \dot{V}_{\dot{A}p}, \dot{\bar{V}}_{\dot{A}\bar{p}}, \rho, \psi_{\bar{p}}), \\ \mathcal{L}_{D=10}^{\mathcal{N}=2}(\mathcal{J}_{AB}, \partial_A, d, V_{Ap}, \bar{V}_{A\bar{p}}, \mathcal{C}, \rho, \psi_{\bar{p}}, \rho', \psi'_p) & \\ &= e^{-2\lambda} \dot{\mathcal{L}}_{\text{Twisted SDFT}}^{\text{Maximal}}(\dot{\mathcal{J}}_{\dot{A}\dot{B}}, \dot{D}_{\dot{A}}, \dot{d}, \dot{V}_{\dot{A}p}, \dot{\bar{V}}_{\dot{A}\bar{p}}, \mathcal{C}, \rho, \psi_{\bar{p}}, \rho', \psi'_p).\end{aligned}$$

- The section condition

$$\dot{D}_{\dot{A}} \dot{D}^{\dot{A}} \sim 0.$$

- If we impose this, it is nothing but the field redefinition of the untwisted SDFT.

We want to find an alternative conditions i.e. *Twistability condition* by imposing the closure of the algebra.

Twistability condition

- Define key quantities out of the twisting data are *c.f.* [Grana, Marques]

$$f_{\dot{A}} := \Omega^{\dot{B}}_{\dot{B}\dot{A}} - 2\dot{\partial}_{\dot{A}}\lambda = \partial_C U^C_{\dot{A}} - 2\dot{\partial}_{\dot{A}}\lambda,$$

and the ‘structure constant’,

$$f_{\dot{A}\dot{B}\dot{C}} := \Omega_{\dot{A}\dot{B}\dot{C}} + \Omega_{\dot{B}\dot{C}\dot{A}} + \Omega_{\dot{C}\dot{A}\dot{B}} = f_{[\dot{A}\dot{B}\dot{C}]}.$$

Twistability condition

- Consider the diffeomorphism, which also be twisted and generated by the *U-twisted generalized Lie derivative*,

$$\dot{\mathcal{L}}_{\dot{X}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} := \dot{X}^{\dot{B}} \dot{D}_{\dot{B}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} + \omega \dot{D}_{\dot{B}} \dot{X}^{\dot{B}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} + \sum_{i=1}^n (\dot{D}_{\dot{A}_i} \dot{X}_{\dot{B}} - \dot{D}_{\dot{B}} \dot{X}_{\dot{A}_i}) \dot{T}_{\dot{A}_1 \dots \dot{A}_{i-1} \dot{A}_{i+1} \dots \dot{A}_n}.$$

- Closure of the diffeomorphism

$$\begin{aligned} & \left([\dot{\mathcal{L}}_{\dot{X}}, \dot{\mathcal{L}}_{\dot{Y}}] - \dot{\mathcal{L}}_{[\dot{X}, \dot{Y}]_{\dot{C}}} \right) \dot{T}_{\dot{A}_1 \dots \dot{A}_n} \\ &= \frac{1}{2} (\dot{X}^{\dot{N}} \dot{D}^{\dot{M}} \dot{Y}_{\dot{N}} - \dot{Y}^{\dot{N}} \dot{D}^{\dot{M}} \dot{X}_{\dot{N}}) \dot{D}_{\dot{M}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} + \frac{1}{2} \omega (\dot{X}^{\dot{N}} \dot{D}_{\dot{M}} \dot{D}^{\dot{M}} \dot{Y}_{\dot{N}} - \dot{Y}^{\dot{N}} \dot{D}_{\dot{M}} \dot{D}^{\dot{M}} \dot{X}_{\dot{N}}) \dot{T}_{\dot{A}_1 \dots \dot{A}_n} \\ &+ \sum_{i=1}^n (\dot{D}_{\dot{M}} \dot{Y}_{\dot{A}_i} \dot{D}^{\dot{M}} \dot{X}_{\dot{B}} - \dot{D}_{\dot{M}} \dot{X}_{\dot{A}_i} \dot{D}^{\dot{M}} \dot{Y}_{\dot{B}}) \dot{T}_{\dot{A}_1 \dots \dot{A}_{i-1} \dot{A}_{i+1} \dots \dot{A}_n}, \end{aligned}$$

where $[\dot{X}, \dot{Y}]_{\dot{C}}$ denotes the U-twisted C-bracket,

$$[\dot{X}, \dot{Y}]_{\dot{C}}^{\dot{A}} := \dot{X}^{\dot{B}} \dot{D}_{\dot{B}} \dot{Y}^{\dot{A}} - \dot{Y}^{\dot{B}} \dot{D}_{\dot{B}} \dot{X}^{\dot{A}} + \frac{1}{2} \dot{Y}^{\dot{B}} \dot{D}^{\dot{A}} \dot{X}_{\dot{B}} - \frac{1}{2} \dot{X}^{\dot{B}} \dot{D}^{\dot{A}} \dot{Y}_{\dot{B}}.$$

Twistability condition

- Closure of the diffeomorphism

$$\begin{aligned}
 & \left([\dot{\mathcal{L}}_{\dot{X}}, \dot{\mathcal{L}}_{\dot{Y}}] - \dot{\mathcal{L}}_{[\dot{X}, \dot{Y}]_C} \right) \dot{T}_{\dot{A}_1 \dots \dot{A}_n} \\
 &= \left(\frac{1}{2} \dot{X}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{Y}_{\dot{N}} - \frac{1}{2} \dot{Y}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{X}_{\dot{N}} + \Omega^{\dot{M}}_{\dot{N}\dot{G}} \dot{X}^{\dot{N}} \dot{Y}^{\dot{G}} \right) \dot{\partial}_{\dot{M}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} \\
 & \quad + \frac{1}{2} \omega \left[\dot{X}^{\dot{N}} \dot{\partial}_{\dot{M}} \dot{\partial}^{\dot{M}} \dot{Y}_{\dot{N}} - \dot{Y}^{\dot{N}} \dot{\partial}_{\dot{M}} \dot{\partial}^{\dot{M}} \dot{X}_{\dot{N}} + 2 \dot{X}^{\dot{N}} \Omega^{\dot{M}}_{\dot{N}\dot{G}} \dot{\partial}_{\dot{M}} \dot{Y}^{\dot{G}} - 2 \dot{Y}^{\dot{N}} \Omega^{\dot{M}}_{\dot{N}\dot{G}} \dot{\partial}_{\dot{M}} \dot{X}^{\dot{G}} \right. \\
 & \quad \left. + 2 \dot{X}^{\dot{N}} \dot{Y}^{\dot{G}} \left(\dot{\partial}^{\dot{M}} f_{\dot{M}\dot{N}\dot{G}} + f^{\dot{M}} f_{\dot{M}\dot{N}\dot{G}} + 2 \dot{\partial}_{[\dot{N}} f_{\dot{G}]} \right) + f_{\dot{M}} \left(\dot{X}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{Y}_{\dot{N}} - \dot{Y}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{X}_{\dot{N}} \right) \right] \dot{T}_{\dot{A}_1 \dots \dot{A}_n} \\
 & \quad + \sum_{i=1}^n \left[\dot{\partial}_{\dot{M}} \dot{Y}_{\dot{A}_i} \dot{\partial}^{\dot{M}} \dot{X}_{\dot{B}} - \dot{\partial}_{\dot{M}} \dot{X}_{\dot{A}_i} \dot{\partial}^{\dot{M}} \dot{Y}_{\dot{B}} - \frac{1}{2} \Omega_{\dot{M}\dot{A}_i\dot{B}} \left(\dot{X}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{Y}_{\dot{N}} - \dot{Y}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{X}_{\dot{N}} \right) \right. \\
 & \quad \left. + 3 \Omega_{\dot{M}[\dot{A}_i\dot{B}} \dot{X}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{Y}_{\dot{N}}] - 3 \Omega_{\dot{M}[\dot{A}_i\dot{B}} \dot{Y}^{\dot{N}} \dot{\partial}^{\dot{M}} \dot{X}_{\dot{N}}] \right. \\
 & \quad \left. + \dot{X}^{\dot{N}} \dot{Y}^{\dot{G}} \left(\dot{\partial}_{\dot{A}_i} f_{\dot{B}\dot{N}\dot{G}} - 3 f_{\dot{M}[\dot{B}\dot{N}} f^{\dot{M}}_{\dot{G}]\dot{A}_i} - 3 \dot{\partial}_{[\dot{B}} f_{\dot{N}\dot{G}]\dot{A}_i} \right) \right] \dot{T}_{\dot{A}_1 \dots \dot{A}_{i-1} \dot{A}_{i+1} \dots \dot{A}_n}^{\dot{B}}.
 \end{aligned}$$

Twistability condition

Sufficient conditions for the closure

1. The section condition for all the dotted twisted fields,

$$\dot{\partial}_M \dot{\partial}^M \equiv 0.$$

2. The orthogonality between the connection and the derivatives of the dotted twisted fields,

$$\Omega^{\dot{M}}_{\dot{F}\dot{G}} \dot{\partial}_M \equiv 0.$$

3. The Jacobi identity for $f_{\dot{A}\dot{B}\dot{C}} = f_{[\dot{A}\dot{B}\dot{C}]}$,

$$f_{[\dot{A}\dot{B}}^{\dot{E}} f_{\dot{C}]\dot{D}\dot{E}} \equiv 0.$$

4. The constancy of the structure constant, $f_{\dot{A}\dot{B}\dot{C}}$,

$$\dot{\partial}_{\dot{E}} f_{\dot{A}\dot{B}\dot{C}} \equiv 0.$$

5. The triviality of $f_{\dot{A}}$,

$$f_{\dot{A}} = \Omega^{\dot{C}}_{\dot{C}\dot{A}} - 2\dot{\partial}_{\dot{A}} \lambda = \partial_C U^C_{\dot{A}} - 2\dot{\partial}_{\dot{A}} \lambda \equiv 0.$$

It might be interesting to investigate the general compatibility condition if any.

Twistability condition

Sufficient conditions for the closure

1. The section condition for all the dotted twisted fields,

$$\dot{\partial}_M \dot{\partial}^M \equiv 0.$$

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5. The triviality of $f_{\dot{A}}$,

$$f_{\dot{A}} = \Omega^{\dot{C}}_{\dot{C}\dot{A}} - 2\dot{\partial}_{\dot{A}} \lambda = \partial_C U^C_{\dot{A}} - 2\dot{\partial}_{\dot{A}} \lambda \equiv 0.$$

For Sherk-Shwarz ansatz, 3-5th conditions are genuine consistency conditions same as [Grana, Marques]

Twistability condition

- The U-twisted generalized Lie derivative reduces, upon the twistability conditions, to

$$\hat{\mathcal{L}}_{\dot{X}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} \equiv \dot{X}^{\dot{B}} \dot{\partial}_{\dot{B}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} + \omega \dot{\partial}_{\dot{B}} \dot{X}^{\dot{B}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n} + \sum_{i=1}^n \left(2 \dot{\partial}_{[\dot{A}_i} \dot{X}_{\dot{B}]} + f_{\dot{A}_i \dot{B} \dot{C}} \dot{X}^{\dot{C}} \right) \dot{T}_{\dot{A}_1 \dots \dot{A}_{i-1} \dot{A}_{i+1} \dots \dot{A}_n}$$

This is the gauge transformation of gauged DFT

Twisted semi-covariant formalism

- *U-twisted master semi-covariant derivative* is

$$\dot{D}_A = \dot{\nabla}_A + \dot{\Phi}_A + \dot{\check{\Phi}}_A,$$

of which the twisted semi-covariant derivative and the twisted spin connections are given by

$$\dot{\nabla}_A = \dot{D}_A + \dot{\Gamma}_A = \dot{\partial}_A + \Omega_A + \dot{\Gamma}_A, \quad \dot{\Phi}_{Apq} = \dot{V}^B{}_p \dot{\nabla}_A \dot{V}_{Bq}, \quad \dot{\check{\Phi}}_{A\bar{p}\bar{q}} = \dot{V}^{\dot{B}}{}_{\bar{p}} \dot{\nabla}_A \dot{V}_{\dot{B}\bar{q}},$$

- the twisted torsionless connection reads

$$\begin{aligned} \dot{\Gamma}_{\dot{C}\dot{A}\dot{B}} = & 2(\dot{P}\dot{D}_C\dot{P}\dot{P})_{[\dot{A}\dot{B}]} + 2(\dot{P}_{[\dot{A}}^{\dot{D}}\dot{P}_{\dot{B}]}^{\dot{E}} - \dot{P}_{[\dot{A}}^{\dot{D}}\dot{P}_{\dot{B}]}^{\dot{E}})\dot{D}_D\dot{P}_{\dot{E}\dot{C}} \\ & - \frac{4}{D-1}(\dot{P}_{\dot{C}[\dot{A}}\dot{P}_{\dot{B}]}^{\dot{D}} + \dot{P}_{\dot{C}[\dot{A}}\dot{P}_{\dot{B}]}^{\dot{D}}) \left(\dot{D}_D\dot{d} + (\dot{P}\dot{D}^{\dot{E}}\dot{P}\dot{P})_{[\dot{E}\dot{D}]} \right), \end{aligned}$$

- “effective connection” reads explicitly,

$$\begin{aligned} \Omega_{\dot{C}\dot{A}\dot{B}} + \dot{\Gamma}_{\dot{C}\dot{A}\dot{B}} \equiv & 2(\dot{P}\dot{\partial}_C\dot{P}\dot{P})_{[\dot{A}\dot{B}]} + 2(\dot{P}_{[\dot{A}}^{\dot{D}}\dot{P}_{\dot{B}]}^{\dot{E}} - \dot{P}_{[\dot{A}}^{\dot{D}}\dot{P}_{\dot{B}]}^{\dot{E}})\dot{\partial}_D\dot{P}_{\dot{E}\dot{C}} \\ & - \frac{4}{D-1}(\dot{P}_{\dot{C}[\dot{A}}\dot{P}_{\dot{B}]}^{\dot{D}} + \dot{P}_{\dot{C}[\dot{A}}\dot{P}_{\dot{B}]}^{\dot{D}}) \left(\dot{\partial}_D\dot{d} + (\dot{P}\dot{\partial}^{\dot{E}}\dot{P}\dot{P})_{[\dot{E}\dot{D}]} \right) \\ & + (\dot{P}_C^{\dot{D}}\dot{P}_A^{\dot{E}}\dot{P}_B^{\dot{F}} + \dot{P}_C^{\dot{D}}\dot{P}_A^{\dot{E}}\dot{P}_B^{\dot{F}})f_{\dot{D}\dot{E}\dot{F}} \\ & + (\dot{P} + \dot{\check{P}})_{\dot{C}\dot{A}\dot{B}}\dot{D}^{\dot{E}\dot{F}}\Omega_{\dot{D}\dot{E}\dot{F}}. \end{aligned}$$

The torsionful deformation [Berman, Lee] is derived as the effective torsion.

Twisted semi-covariant formalism

- The semi-covariant formulation also works for the twisted semi-covariant derivative.

Upon all the twistability conditions, we obtain

$$(\delta_{\dot{X}} - \hat{\mathcal{L}}_{\dot{X}})(\dot{\nabla}_{\dot{C}} \dot{T}_{\dot{A}_1 \dots \dot{A}_n}) \equiv \sum_{i=1}^n (\mathcal{P} + \bar{\mathcal{P}})_{\dot{C}\dot{A}_i} \dot{B} \dot{T}_{\dot{A}_1 \dots \dot{A}_{i-1} \dot{B} \dot{A}_{i+1} \dots \dot{A}_n},$$

once again *the anomalies are all controlled by the index-six projection operators*. Namely, they are still semi-covariant.

- We get various covariant derivatives in the same way,

$$\dot{P}_{\dot{C}}^{\dot{D}} \dot{P}_{\dot{A}_1}^{\dot{B}_1} \dots \dot{P}_{\dot{A}_n}^{\dot{B}_n} \dot{\nabla}_{\dot{D}} \dot{T}_{\dot{B}_1 \dots \dot{B}_n}, \dots$$

$$\dot{D}_p T_{\bar{q}_1 \dots \bar{q}_n}, \quad \dot{D}_p T^p_{\bar{q}_1 \dots \bar{q}_n}, \dots$$

$$\gamma^p \dot{D}_p \rho, \quad \gamma^p \dot{D}_p \psi_{\bar{p}}, \dots, \text{etc.}$$

Curvature

- Compare the two possible semi-covariant curvatures upon the twistability conditions, upon the twistability conditions,

$$\dot{\mathcal{G}}_{\dot{A}\dot{B}\dot{C}\dot{D}} \equiv \dot{S}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \frac{1}{2}\Omega_{\dot{E}\dot{A}\dot{B}}\Omega^{\dot{E}}_{\dot{C}\dot{D}}.$$

- $\mathcal{G}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ differs from $\dot{S}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ after the twist.
- In the twisted SDFT to be constructed below, we shall employ $\mathcal{G}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ only. It turns out to be semi-covariant, while the other is not.
- The completely covariant index-two (“Ricci”) and index-zero (scalar) twisted curvatures are as untwisted cases,

$$\dot{\mathcal{G}}_{p\bar{q}}{}^r, \quad \dot{\mathcal{G}}_{p\bar{q}}{}^{\bar{r}}, \quad \dot{\mathcal{G}}_{pq}{}^{pq}, \quad \dot{\mathcal{G}}_{p\bar{q}}{}^{\bar{p}\bar{q}}.$$

Their covariance is guaranteed as they are related to the completely twisted covariant derivatives,

$$\begin{aligned} \frac{1}{2}[\dot{\mathcal{D}}_p, \dot{\mathcal{D}}_{\bar{q}}]T^p &\equiv \dot{\mathcal{G}}_{p\bar{q}}{}^r T^p, & \frac{1}{2}[\dot{\mathcal{D}}_p, \dot{\mathcal{D}}_{\bar{q}}]T^{\bar{q}} &\equiv -\dot{\mathcal{G}}_{p\bar{q}}{}^{\bar{r}} T^{\bar{q}}, \\ [\gamma^p \dot{\mathcal{D}}_p, \dot{\mathcal{D}}_{\bar{q}}]\varepsilon &\equiv \dot{\mathcal{G}}_{p\bar{q}}{}^r \gamma^p \varepsilon, & [\mathcal{D}_p, \bar{\gamma}^{\bar{q}} \dot{\mathcal{D}}_{\bar{q}}]\varepsilon' &\equiv -\dot{\mathcal{G}}_{p\bar{q}}{}^{\bar{r}} \gamma^{\bar{q}} \varepsilon', \\ (\gamma^p \dot{\mathcal{D}}_p)^2 \varepsilon + \dot{\mathcal{D}}_{\bar{p}} \dot{\mathcal{D}}^{\bar{p}} \varepsilon &\equiv -\frac{1}{4} \dot{\mathcal{G}}_{pq}{}^{pq} \varepsilon, & (\bar{\gamma}^{\bar{p}} \dot{\mathcal{D}}_{\bar{p}})^2 \varepsilon' + \dot{\mathcal{D}}_p \dot{\mathcal{D}}^p \varepsilon' &\equiv -\frac{1}{4} \dot{\mathcal{G}}_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}} \varepsilon'. \end{aligned}$$

- Using $\mathcal{G}_{\dot{A}\dot{B}\dot{C}\dot{D}}$, the supersymmetric completion will be possible.

Twisted semi-covariant formalism

- We replace $\dot{S}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ by $\dot{\mathcal{G}}_{\dot{A}\dot{B}\dot{C}\dot{D}}$.
- Almost all the properties of the four-index curvature still hold after the twist, up to the twistability conditions,

$$\dot{\mathcal{G}}_{\dot{A}[\dot{B}\dot{C}\dot{D}]} \equiv 0. \quad \text{etc.}$$

The only exception is

$$\dot{\mathcal{G}}_{pq}{}^{pq} + \dot{\mathcal{G}}_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}} \equiv \frac{1}{6} f_{\dot{A}\dot{B}\dot{C}} f^{\dot{A}\dot{B}\dot{C}}.$$

- It follows the modification of the Ramond-Ramond cohomology,

$$(\dot{\mathcal{D}}_{\pm})^2 \mathcal{T} \equiv -\frac{1}{24} f_{\dot{A}\dot{B}\dot{C}} f^{\dot{A}\dot{B}\dot{C}} \mathcal{T}.$$

We should separately impose

$$f_{\dot{A}\dot{B}\dot{C}} f^{\dot{A}\dot{B}\dot{C}} \equiv 0.$$

- To incorporate RR field, we have to impose this extra consistency condition.
- This will also be necessary **for the maximal supersymmetry**.

Supersymmetric gauged double field theory

- The half-maximal supersymmetric gauged double field theory Lagrangian

$$\dot{\mathcal{L}}_{\text{Twisted SDFT}}^{\text{Half-maximal}} = e^{-2\dot{d}} \left[\frac{1}{4} \dot{\mathcal{G}}_{pq}{}^{pq} + i \frac{1}{2} \bar{\rho} \gamma^p \dot{\mathcal{D}}_p \rho - i \bar{\psi}^{\bar{p}} \dot{\mathcal{D}}_{\bar{p}} \rho - i \frac{1}{2} \bar{\psi}^{\bar{p}} \gamma^q \dot{\mathcal{D}}_q \psi_{\bar{p}} \right].$$

The leading order half-maximal twisted supersymmetry transformation rules

$$\delta_\varepsilon \dot{d} = -i \frac{1}{2} \bar{\varepsilon} \rho, \quad \delta_\varepsilon \dot{V}_{Ap} = -i \dot{V}_A{}^{\bar{q}} \bar{\varepsilon} \gamma_p \psi_{\bar{q}}, \quad \delta_\varepsilon \dot{V}_{A\bar{p}} = +i \dot{V}_A{}^q \bar{\varepsilon} \gamma_q \psi_{\bar{p}},$$

$$\delta_\varepsilon \rho = -\gamma^p \dot{\mathcal{D}}_p \varepsilon, \quad \delta_\varepsilon \psi_{\bar{p}} = \dot{\mathcal{D}}_{\bar{p}} \varepsilon.$$

- Higher order of fermionic terms are same as the terms in untwisted SDFT.

Supersymmetric gauged double field theory

- From the Z_2 symmetry which exchanges the two spin groups, $\mathbf{Spin}(1, 9) \leftrightarrow \mathbf{Spin}(9, 1)$, there is a parallel formulation of the half-maximal SDFT,

$$\overline{\dot{\mathcal{L}}_{\text{Twisted SDFT}}^{\text{Half-maximal}}} = e^{-2\dot{d}} \left[-\frac{1}{4} \dot{\mathcal{G}}_{\bar{p}\bar{q}}^{\bar{p}\bar{q}} - i\frac{1}{2} \bar{\rho}' \bar{\gamma}^{\bar{p}} \dot{\mathcal{D}}_{\bar{p}} \rho' + i\bar{\psi}'^p \dot{\mathcal{D}}_p \rho' + i\frac{1}{2} \bar{\psi}'^p \bar{\gamma}^{\bar{q}} \dot{\mathcal{D}}_{\bar{q}} \psi'_p \right].$$

The supersymmetry is realized by

$$\begin{aligned} \delta_\varepsilon \dot{d} &= -i\frac{1}{2} \bar{\varepsilon}' \rho', & \delta_\varepsilon \dot{V}_{\dot{A}p} &= +i\bar{\varepsilon}' \bar{\gamma}_{\dot{A}} \psi'_p, & \delta_\varepsilon \dot{V}_{\dot{A}\bar{p}} &= -i\bar{\varepsilon}' \bar{\gamma}_{\bar{p}} \psi'_{\dot{A}}, \\ \delta_\varepsilon \rho' &= -\bar{\gamma}^{\bar{p}} \dot{\mathcal{D}}_{\bar{p}} \varepsilon', & \delta_\varepsilon \psi'_p &= \dot{\mathcal{D}}_p \varepsilon'. \end{aligned}$$

Supersymmetric gauged double field theory

- Maximal supersymmetric gauged double field theory Lagrangian,

$$\begin{aligned} \dot{\mathcal{L}}_{\text{Twisted SDFT}}^{\text{Maximal}} = e^{-2\dot{d}} & \left[\frac{1}{8} (\dot{\mathcal{G}}_{pq}{}^{pq} - \dot{\mathcal{G}}_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}}) + \frac{1}{2} \text{Tr}(\dot{\mathcal{F}}\bar{\mathcal{F}}) - i\bar{\rho}\dot{\mathcal{F}}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\dot{\mathcal{F}}\bar{\gamma}^{\bar{p}}\psi'^q \right. \\ & \left. + i\frac{1}{2}\bar{\rho}\gamma^p\dot{\mathcal{D}}_p\rho - i\bar{\psi}^{\bar{p}}\dot{\mathcal{D}}_{\bar{p}}\rho - i\frac{1}{2}\bar{\psi}^{\bar{p}}\gamma^q\dot{\mathcal{D}}_q\psi_{\bar{p}} - i\frac{1}{2}\rho'\bar{\gamma}^{\bar{p}}\dot{\mathcal{D}}_{\bar{p}}\rho' + i\bar{\psi}'^p\dot{\mathcal{D}}_p\rho' + i\frac{1}{2}\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\dot{\mathcal{D}}_{\bar{q}}\psi'_p \right] \end{aligned}$$

$$\delta_\varepsilon \dot{d} = -i\frac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho'),$$

$$\delta_\varepsilon \dot{V}_{\dot{A}p} = i\dot{V}_{\dot{A}}{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}}), \quad \delta_\varepsilon \dot{V}_{\dot{A}\bar{p}} = i\dot{V}_{\dot{A}}{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q),$$

$$\delta_\varepsilon \mathcal{C} = i\frac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon \dot{d} - \frac{1}{2}(\dot{V}_{\dot{A}}{}^{\bar{q}}\delta_\varepsilon \dot{V}_{\dot{A}p})\gamma^{(11)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}},$$

$$\delta_\varepsilon \rho = -\gamma^p\dot{\mathcal{D}}_p\varepsilon, \quad \delta_\varepsilon \rho' = -\bar{\gamma}^{\bar{p}}\dot{\mathcal{D}}_{\bar{p}}\varepsilon',$$

$$\delta_\varepsilon \psi_{\bar{p}} = \dot{\mathcal{D}}_{\bar{p}}\varepsilon + \dot{\mathcal{F}}\bar{\gamma}_{\bar{p}}\varepsilon', \quad \delta_\varepsilon \psi'_p = \dot{\mathcal{D}}_p\varepsilon' + \bar{\mathcal{F}}\gamma_p\varepsilon.$$

Supersymmetric gauged double field theory

Under the supersymmetry transformation

$$\delta_\varepsilon \dot{\mathcal{L}}_{\text{Twisted SDFT}}^{\text{Maximal}} \equiv i \frac{1}{48} e^{-2d} (\bar{\rho}\varepsilon - \bar{\rho}'\varepsilon' + \bar{\varepsilon}\mathcal{C}\rho' + \bar{\varepsilon}\gamma^p\mathcal{C}\psi'_p + \bar{\rho}\mathcal{C}\varepsilon' + \bar{\psi}_p\mathcal{C}\bar{\gamma}^{\bar{p}}\varepsilon') \times f_{\dot{A}\dot{B}\dot{C}} f^{\dot{A}\dot{B}\dot{C}} \\ + i \frac{1}{8} e^{-2d} (\bar{\varepsilon}\gamma_p\psi_{\bar{q}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p) \text{Tr} \left(\gamma^p \dot{\mathcal{F}}_- \bar{\gamma}^{\bar{q}} \overline{\dot{\mathcal{F}}_-} \right).$$

- Thus, requiring the extra condition

$$f_{\dot{A}\dot{B}\dot{C}} f^{\dot{A}\dot{B}\dot{C}} \equiv 0,$$

the action is supersymmetric invariant modulo the self-duality.

Comparison with the untwisted case

- Look at the NS-NS sector of two half-maximal Lagrangian.

$$\begin{aligned}
 +\dot{\mathcal{G}}_{pq}{}^{pq} &\equiv \frac{1}{16}\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{A}}\dot{\mathcal{H}}_{\dot{C}\dot{D}}\dot{\partial}_{\dot{B}}\dot{\mathcal{H}}^{\dot{C}\dot{D}} + \frac{1}{4}\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}^{\dot{C}}\dot{\mathcal{H}}_{\dot{A}\dot{D}}\dot{\partial}^{\dot{D}}\dot{\mathcal{H}}_{\dot{B}\dot{C}} - \frac{1}{2}\dot{\partial}_{\dot{A}}\dot{\partial}_{\dot{B}}\dot{\mathcal{H}}^{\dot{A}\dot{B}} \\
 &\quad - 2\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{A}}\dot{d}\dot{\partial}_{\dot{B}}\dot{d} + 2\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{A}}\dot{\partial}_{\dot{B}}\dot{d} + 2\dot{\partial}_{\dot{A}}\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{B}}\dot{d} \\
 &\quad + \frac{1}{8}f_{\dot{A}\dot{B}\dot{C}}f^{\dot{A}\dot{B}}{}_{\dot{D}}\dot{\mathcal{H}}^{\dot{C}\dot{D}} - \frac{1}{24}f_{\dot{A}\dot{B}\dot{C}}f_{\dot{D}\dot{E}\dot{F}}\dot{\mathcal{H}}^{\dot{A}\dot{D}}\dot{\mathcal{H}}^{\dot{B}\dot{E}}\dot{\mathcal{H}}^{\dot{C}\dot{F}} - \frac{1}{4}f_{\dot{A}\dot{B}\dot{C}}\dot{\mathcal{H}}^{\dot{B}\dot{D}}\dot{\mathcal{H}}^{\dot{C}\dot{E}}\dot{\partial}_{\dot{D}}\dot{\mathcal{H}}_{\dot{E}}{}^{\dot{A}} \\
 &\quad + \frac{1}{12}f_{\dot{A}\dot{B}\dot{C}}f^{\dot{A}\dot{B}\dot{C}}, \\
 -\dot{\mathcal{G}}_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}} &\equiv \frac{1}{16}\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{A}}\dot{\mathcal{H}}_{\dot{C}\dot{D}}\dot{\partial}_{\dot{B}}\dot{\mathcal{H}}^{\dot{C}\dot{D}} + \frac{1}{4}\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}^{\dot{C}}\dot{\mathcal{H}}_{\dot{A}\dot{D}}\dot{\partial}^{\dot{D}}\dot{\mathcal{H}}_{\dot{B}\dot{C}} - \frac{1}{2}\dot{\partial}_{\dot{A}}\dot{\partial}_{\dot{B}}\dot{\mathcal{H}}^{\dot{A}\dot{B}} \\
 &\quad - 2\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{A}}\dot{d}\dot{\partial}_{\dot{B}}\dot{d} + 2\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{A}}\dot{\partial}_{\dot{B}}\dot{d} + 2\dot{\partial}_{\dot{A}}\dot{\mathcal{H}}^{\dot{A}\dot{B}}\dot{\partial}_{\dot{B}}\dot{d} \\
 &\quad + \frac{1}{8}f_{\dot{A}\dot{B}\dot{C}}f^{\dot{A}\dot{B}}{}_{\dot{D}}\dot{\mathcal{H}}^{\dot{C}\dot{D}} - \frac{1}{24}f_{\dot{A}\dot{B}\dot{C}}f_{\dot{D}\dot{E}\dot{F}}\dot{\mathcal{H}}^{\dot{A}\dot{D}}\dot{\mathcal{H}}^{\dot{B}\dot{E}}\dot{\mathcal{H}}^{\dot{C}\dot{F}} - \frac{1}{4}f_{\dot{A}\dot{B}\dot{C}}\dot{\mathcal{H}}^{\dot{B}\dot{D}}\dot{\mathcal{H}}^{\dot{C}\dot{E}}\dot{\partial}_{\dot{D}}\dot{\mathcal{H}}_{\dot{E}}{}^{\dot{A}} \\
 &\quad - \frac{1}{12}f_{\dot{A}\dot{B}\dot{C}}f^{\dot{A}\dot{B}\dot{C}}.
 \end{aligned}$$

- It reproduces the previous result [Geissbuler,Aldazabal,Grana, Marques] , and it matches with the $N = 4 D = 4$ gauged SUGRA.
- The fourth line is the cosmological constant which is the special feature of half-maximal supersymmetric DFT.

Comparison with the untwisted case

- NS-NS sector of maximal gauged SDFT.

$$\begin{aligned}
 & + \dot{\mathcal{G}}_{pq}{}^{pq} - \dot{\mathcal{G}}_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}} \\
 & \equiv \frac{1}{8} \dot{\mathcal{H}}^{\dot{A}\dot{B}} \dot{\partial}_A \dot{\mathcal{H}}_{\dot{C}\dot{D}} \dot{\partial}_B \dot{\mathcal{H}}^{\dot{C}\dot{D}} + \frac{1}{2} \dot{\mathcal{H}}^{\dot{A}\dot{B}} \dot{\partial}^{\dot{C}} \dot{\mathcal{H}}_{\dot{A}\dot{D}} \dot{\partial}^{\dot{D}} \dot{\mathcal{H}}_{\dot{B}\dot{C}} - \dot{\partial}_A \dot{\partial}_B \dot{\mathcal{H}}^{\dot{A}\dot{B}} \\
 & \quad - 4 \dot{\mathcal{H}}^{\dot{A}\dot{B}} \dot{\partial}_A \dot{d} \dot{\partial}_B \dot{d} + 4 \dot{\mathcal{H}}^{\dot{A}\dot{B}} \dot{\partial}_A \dot{\partial}_B \dot{d} + 4 \dot{\partial}_A \dot{\mathcal{H}}^{\dot{A}\dot{B}} \dot{\partial}_B \dot{d} \\
 & \quad + \frac{1}{4} f_{\dot{A}\dot{B}\dot{C}} f^{\dot{A}\dot{B}}{}_{\dot{D}} \dot{\mathcal{H}}^{\dot{C}\dot{D}} - \frac{1}{12} f_{\dot{A}\dot{B}\dot{C}} f_{\dot{D}\dot{E}\dot{F}} \dot{\mathcal{H}}^{\dot{A}\dot{D}} \dot{\mathcal{H}}^{\dot{B}\dot{E}} \dot{\mathcal{H}}^{\dot{C}\dot{F}} - \frac{1}{2} f_{\dot{A}\dot{B}\dot{C}} \dot{\mathcal{H}}^{\dot{B}\dot{D}} \dot{\mathcal{H}}^{\dot{C}\dot{E}} \dot{\partial}_D \dot{\mathcal{H}}_E{}^{\dot{A}}
 \end{aligned}$$

Comparison with the untwisted case

- For fermions,

$$\begin{aligned}\gamma^p \dot{D}_p \rho &\equiv \gamma^p \dot{D}_p \rho \Big|_{\dot{\theta}} + \frac{1}{12} f_{pqr} \gamma^{pqr} \rho, \\ \dot{D}_{\bar{p}} \rho &\equiv \dot{D}_{\bar{p}} \rho \Big|_{\dot{\theta}} + \frac{1}{4} f_{\bar{p}qr} \gamma^{qr} \rho, \\ \gamma^q \dot{D}_q \psi_{\bar{p}} &\equiv \gamma^q \dot{D}_q \psi_{\bar{p}} \Big|_{\dot{\theta}} + \frac{1}{12} f_{qrs} \gamma^{qrs} \psi_{\bar{p}} + f_{r\bar{p}q} \gamma^r \psi^{\bar{q}},\end{aligned}$$

and

$$\begin{aligned}\bar{\gamma}^{\bar{p}} \dot{D}_{\bar{p}} \rho' &\equiv \bar{\gamma}^{\bar{p}} \dot{D}_{\bar{p}} \rho' \Big|_{\dot{\theta}} + \frac{1}{12} f_{\bar{p}q\bar{r}} \bar{\gamma}^{\bar{p}q\bar{r}} \rho', \\ \dot{D}_p \rho' &\equiv \dot{D}_p \rho' \Big|_{\dot{\theta}} + \frac{1}{4} f_{p\bar{q}\bar{r}} \bar{\gamma}^{\bar{q}\bar{r}} \rho', \\ \bar{\gamma}^{\bar{q}} \dot{D}_{\bar{q}} \psi'_p &\equiv \bar{\gamma}^{\bar{q}} \dot{D}_{\bar{q}} \psi'_p \Big|_{\dot{\theta}} + \frac{1}{12} f_{\bar{q}rs} \bar{\gamma}^{\bar{q}rs} \psi'_p + f_{r\bar{p}q} \bar{\gamma}^{\bar{r}} \psi'^q.\end{aligned}$$

agree with Berman and Lee.

- For R-R as our new result

$$\dot{\mathcal{F}} = \dot{D}_+ \mathcal{C} \equiv \dot{D}_+ \mathcal{C} \Big|_{\dot{\theta}} + \frac{1}{12} f_{pqr} \gamma^{pqr} \mathcal{C} - \frac{1}{4} f_{p\bar{q}\bar{r}} \gamma^p \mathcal{C} \bar{\gamma}^{\bar{q}\bar{r}} - \frac{1}{12} f_{\bar{p}q\bar{r}} \gamma^{(11)} \mathcal{C} \bar{\gamma}^{\bar{p}q\bar{r}} + \frac{1}{4} f_{p\bar{q}\bar{r}} \gamma^{(11)} \gamma^{pq} \mathcal{C} \bar{\gamma}^{\bar{r}},$$

The nilpotency of this twisted R-R cohomology implies the Bianchi identity for the twisted R-R flux, which is expected to produce the ‘tensor hierarchy’

[Bergshoeff, *et. al*; Fernandez-Melgarejo *et. al*]

Summary and comments

- We successfully twisted the semi-covariant formulations of the $\mathcal{N} = 2$ and the $\mathcal{N} = 1, D = 10$ SDFT.
- Imposing the twistability conditions, it **systematically derives** the gauged **maximal** and **half-maximal** supersymmetric double field theories,
- R-R potential \mathcal{C} is not twisted. Only the field strength $\dot{\mathcal{F}} = \dot{D}_+\mathcal{C}$ is influenced by twisting though the twisted nilpotent operator. We expect that this will change when U-duality is twisted in \mathcal{M} -theory setup.

Conclusion

Thank you.