Loops in exceptional field theory

Axel Kleinschmidt (Albert Einstein Institute, Potsdam)

Stringy Geometry, Mainz, September 14, 2015

Joint work (in progress) with Guillaume Bossard

Motivation and goal

Supergravity is the (ultra-)low energy effective action of M-theory. Certainly not the full story since theory contains many more states...

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Tools

- Hidden symmetries $E_d(\mathbb{R})$ and U-duality $E_d(\mathbb{Z})$
- Exceptional field theory structures
- Relation between field theory loops and BPS-protected corrections
- Automorphic forms

Local higher derivative corrections

Schematic form of effective action in $D = 11 - d$ on T^d

$$e^{-1}\mathcal{L} = \ell^{2-D}\left[R - \frac{1}{2}G_{IJ}(\Phi)\partial\Phi^I\partial\Phi^J + \ldots\right]$$
$$+ \ell^{8-D}\left[\mathcal{E}^D_{(0,0)}(\Phi)R^4 + \ldots\right] + \ell^{12-D}\left[\mathcal{E}^D_{(1,0)}(\Phi)\nabla^4 R^4 + \ldots\right]$$
$$+ \ell^{14-D}\left[\mathcal{E}^D_{(0,1)}(\Phi)\nabla^6 R^4 + \ldots\right] + \ldots$$

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The moduli fields Φ belong to quantum moduli space

$$E_d(\mathbb{Z})\backslash E_d(\mathbb{R})/K(E_d)$$

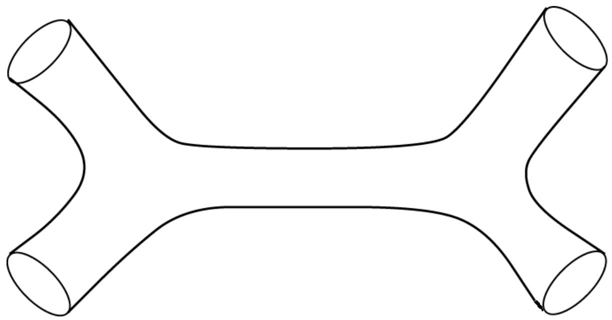
$K(E_d)$: max. compact subgroup of CJ symmetry $E_d(\mathbb{R})$ [Cremmer, Julia]

$E_d(\mathbb{Z})$: U-duality [Hull, Townsend]

Local higher derivative terms (II)

Functions $\mathcal{E}^{D}_{(p,q)}(\Phi)$

- contain perturbative and non-perturbative information on graviton scattering [Green, Gutperle]. String picture:



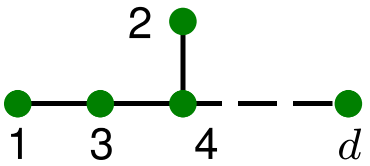
$\alpha' = \ell_s^2$ expansion

$$\mathcal{A}^{\text{tree}}(s,t,u) = g_s^{-2} \frac{4}{stu} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)} \mathcal{R}^4$$

- multiply full supersymmetric invariants

Higher derivative corrections

- satisfy $\mathcal{E}^{D}_{(p,q)}(\gamma\Phi k) = \mathcal{E}^{D}_{(p,q)}(\Phi)$ for $\gamma \in E_d(\mathbb{Z})$, $k \in K(E_d)$



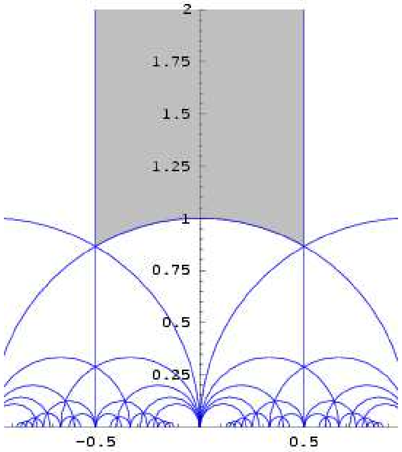
- A lot known for lowest $\mathcal{E}^{D}_{(p,q)}$ from supersymmetry and internal consistency [Green, Gutperle, Kiritsis, Miller, Obers, Pioline, Russo, Sethi, Vanhove,...]

$\mathcal{E}^{D}_{(0,0)}$	R^4 correction
$\mathcal{E}^{D}_{(1,0)}$	$\nabla^4 R^4$ correction
$\mathcal{E}^{D}_{(0,1)}$	$\nabla^6 R^4$ correction

Example: Type IIB

Hidden symmetry $SL(2,\mathbb{R})$; U-duality $SL(2,\mathbb{Z})$. Scalars $\Phi \equiv \tau \equiv \tau_1 + i\tau_2 = \chi + ie^{-\phi}$. Define

$$E_{[s]}(\tau) = \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq(0,0)}} \frac{\tau_2^s}{|c\tau+d|^{2s}} = 2\zeta(2s) \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} [\mathrm{Im}(\gamma\cdot\tau)]^s$$



Note: $\mathrm{Im}\,\tau = \tau_2 = e^{-\phi} = g_{\mathrm{s}}^{-1}$. Rewriting is sum over U-duality orbits.

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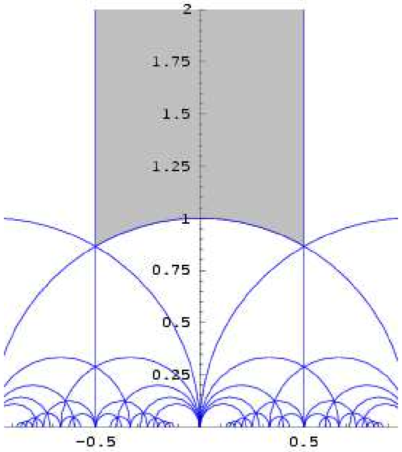

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$E_{[s]}(\tau)$ is a non-holomorphic Eisenstein series.

$\mathcal{E}^{10B}_{(0,0)} = E_{[3/2]}$ $\quad R^4$ correction [Green, Gutperle]

$\mathcal{E}^{10B}_{(1,0)} = E_{[5/2]}$ $\quad \nabla^4 R^4$ correction [Green, Vanhove]

$\mathcal{E}^{10B}_{(0,1)}$ $\quad \nabla^6 R^4$ correction. Not Eisenstein, explicit form by [Green, Miller, Vanhove]

Relation to field theory loops

Four-graviton process is very special. Low order corrections R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ enjoy SUSY protection.

$\Longrightarrow$ Only BPS states contribute; no other M-theory states visible at low energies

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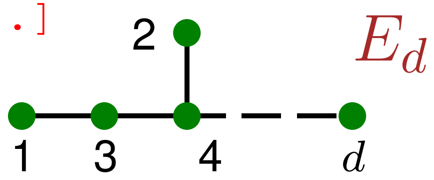
Aim: Investigate $\mathcal{E}^{D}_{(p,q)}$ for $D < 10$ by similar methods in manifestly U-duality covariant formalism

$\Longrightarrow$ Exceptional field theory loops

Exceptional field theory

[Hull; Waldram et al.; Hohm, Samtleben; West; ...]



Formalism to make hidden $E_d(\mathbb{R})$ (continuous!) manifest.

Consider extended space-time

$$\mathcal{M}^D \times \mathcal{M}^{d(\alpha_d)}$$

Coordinates x^μ, y^M with $\mu = 0, ..., D-1$ and $M = 1, ..., d(\alpha_d)$.

$d(\alpha_d) = \dim \mathbf{R}_{\alpha_d}$: hst. weight rep. on node α_d

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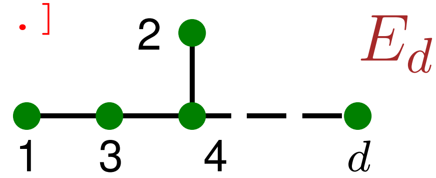

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$\mathbf{R}_{\alpha_d}$ decomposes under 'gravity line' $GL(d, \mathbb{R}) \subset E_d(\mathbb{R})$

$$y^M = (y^m, y_{[mn]}, y_{[m_1 \ldots m_5]}, \ldots) \qquad (m, n, ... = 1, ..., d)$$

KK momenta (→ y^m) M2 wrappings (→ $y_{[mn]}$)

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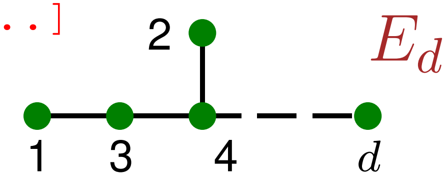

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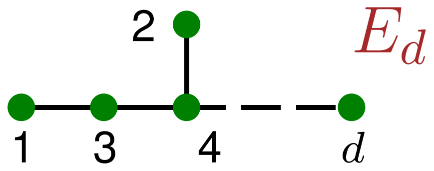
KK momenta M2 wrappings

Also in E_{11} [West]

Generalised coordinates $y^M \in \mathbf{R}_{\alpha_d}$

E_d	$\mathbf{R}_{\alpha_d}$	
$SO(5,5)$	$\mathbf{16}$	
E_6	$\mathbf{27}$	
E_7	$\mathbf{56}$	
E_8	$\mathbf{248}$	



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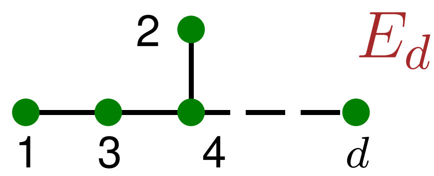
E_d	$\mathbf{R}_{\alpha_d}$	$\mathbf{R}_{\alpha_1}$
$SO(5,5)$	$\mathbf{16}$	$\mathbf{10}$
E_6	$\mathbf{27}$	$\overline{\mathbf{27}}$
E_7	$\mathbf{56}$	$\mathbf{133}$
E_8	$\mathbf{248}$	$\mathbf{3875} \oplus \mathbf{1}$



Generalised coordinates y^M have to obey section constraint

$$\left. \frac{\partial A}{\partial y^M} \frac{\partial B}{\partial y^N} \right|_{\mathbf{R}_{\alpha_1}} = 0$$

for any two fields $A(x^\mu, y^M)$, $B(x^\mu, y^M)$. LHS belongs to

$$\mathbf{R}_{\alpha_d} \otimes \mathbf{R}_{\alpha_d} = \mathbf{R}_{\alpha_1} \oplus \ldots$$

Section constraint

$$\left.\frac{\partial A}{\partial y^M}\frac{\partial B}{\partial y^N}\right|_{\mathbf{R}_{\alpha_1}} = 0$$

Possible solution: ‘M-theory’: $y^M = (y^m, \cancel{y_{mn}}, \cancel{y_{m_1 \ldots m_5}}, \ldots)$

Alternative: Type IIB [Blair, Malek, Park]. These are the only two vector space solutions [BK]

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Here: ‘Toroidal’ extended space for y^M. Conjugate momenta are quantised charges

$$\Gamma_M = (n_m, n^{m_1m_2}, n^{n_1\ldots n_5}, \ldots) \in \mathbb{Z}^{d(\alpha_d)}$$

Section constraint becomes $\frac{1}{2}$-BPS constraint on charges

$$\left.\Gamma \times \tilde{\Gamma}\right|_{\mathbf{R}_{\alpha_1}} = 0$$

Amplitudes in EFT (I)

Exceptional field theory is mainly a classical theory. QFT treatment complicated due to section constraint.

Consider 3-point vertex in EFT $\phi\,\partial\phi\,\partial\phi$

$$\int\limits_{\mathbb{R}^{11-d}} dx \int\limits_{\mathbb{R}^{d(\alpha_d)}} dy\phi(x,y)\left(\nabla\phi(x,y)\cdot\nabla\phi(x,y)\right)$$

y-Fourier expand $\phi(x,y) = \sum\limits_{\Gamma\in\mathbb{Z}^{d(\alpha_d)}} \phi_\Gamma(x) e^{i\ell^{-1}\Gamma\cdot y}$. Vertex

$$\sum_{\substack{\Gamma_1,\Gamma_2\in\mathbb{Z}^{d(\alpha_d)}\\ \Gamma_1\times\Gamma_2=0}} \int\limits_{\mathbb{R}^{11-d}} dx\,\phi_{-\Gamma_1-\Gamma_2}(x)\left[\partial_\mu\phi_{\Gamma_1}\partial^\mu\phi_{\Gamma_2} - \ell^{-2}\left\langle Z(\Gamma_1)|Z(\Gamma_2)\right\rangle\phi_{\Gamma_1}\phi_{\Gamma_2}\right]$$

Section constraint on y^M turned into constraint on charges

Amplitudes in EFT (II)

$\langle Z(\Gamma)|Z(\Gamma)\rangle$ like BPS-mass. In M-theory frame

$$ds_{11}^2 = e^{\frac{9-d}{3}\phi} M_{mn} dy^m dy^n + e^{-\frac{d}{3}\phi} \eta_{\mu\nu} dx^\mu dx^\nu$$

ϕ now dilaton; M_{mn} uni-modular metric on T^d.

$$|Z(\Gamma)|^2 = e^{-3\phi} M^{mn} n_m n_n + \frac{1}{2} e^{(6-d)\phi} M_{m_1 n_1} M_{m_2 n_2} n^{m_1 m_2} n^{n_1 n_2} + \ldots$$

From form of vertex see that momenta in propagators are effectively shifted by Kaluza–Klein mass

$$p^2 \quad \longrightarrow \quad p^2 + \ell^{-2} |Z(\Gamma)|^2$$

and section constraint $\Gamma_i \times \Gamma_j = 0$ at every vertex.

Amplitudes in EFT (III)

General structure of four-point EFT amplitude at L loops

$$\mathcal{M}_{4,L}(1,2,3,4) = i^{L+1}\left(\frac{\kappa_{11-d}}{2}\right)^{2+2L} \sum_{\text{graphs}\, G} \frac{1}{S_G} \sum_{\substack{\Gamma_l \in \mathbb{Z}^{d(\alpha_d)L} \\ \Gamma_i \times \Gamma_j = 0}}$$

$$\times \int \prod_{l=1}^{L} \frac{d^{11-d}p_l}{(2\pi)^{11-d}} \frac{N_G\left(k_A, e_A, p_l, \Gamma_l\right)}{\prod_{I_G} p_{I_G}^2(k_A, p_l \oplus \Gamma_l)}$$

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external momenta and polarisations $A = 1, \ldots 4$

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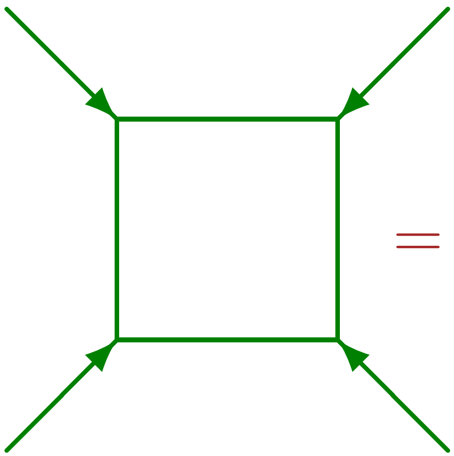
Next: Calculate $L = 1$ and $L = 2$ assuming reduction to scalar diagrams as in [Bern et al.; Green, Vanhove]

One-loop in EFT (I)

Four-graviton amplitude reduces to scalar box



$$= \underbrace{\left[\frac{i\kappa^2}{2} t_8 t_8 \prod_{A=1}^{4} k_A R(k_A, e_A)\right]}_{\mathcal{R}^4} A^{\text{1-loop}}(k_1, k_2, k_3, k_4)$$

Pull out kinematic part

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = \kappa^2 \int \frac{d^{11-d}p}{(2\pi)^{11-d}} \sum_{\substack{\Gamma\in\mathbb{Z}^{d(\alpha_d)} \\ \Gamma\times\Gamma=0}} \frac{1}{(p^2+\ell^{-2}|Z|^2)}$$

$$\times \frac{1}{((p-k_1)^2+\ell^{-2}|Z|^2)((p-k_1-k_2)^2+\ell^{-2}|Z|^2)((p+k_4)^2+\ell^{-2}|Z|^2)}$$

$$+ \text{perms.}$$

One-loop in EFT (II)

$\Gamma = 0$ term corresponds to SUGRA in $D = 11 - d$; usual log threshold contribution $\Rightarrow$ remove for analytic eff. action

Treat loop integral over $d^{11-d}p$ with usual Schwinger and Feynman techniques:

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = 4\pi\ell^{9-d} \sum_{\substack{\Gamma \in \mathbb{Z}^{d(\alpha_d)} \\ \Gamma \times \Gamma = 0}} \int_0^\infty \frac{dv}{v^{\frac{d-1}{2}}} \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3$$

$$\times \exp\left[\frac{\pi}{v}\left((1-x_1)(x_2-x_3)s + x_3(x_1-x_2)t - \ell^{-2}|Z|^2\right)\right] + \text{perms.}$$

Low energy from expanding in Mandelstam variables

$$s = -\ell^2(k_1+k_2)^2, \quad t = -\ell^2(k_1+k_4)^2, \quad u = -\ell^2(k_1+k_3)^2.$$

Low energy correction terms

For lowest two orders

$$A^{\text{1-loop}}(s,t,u) = 4\pi\ell^6\Bigg(\xi(d-3)E_{\alpha_d,\frac{d-3}{2}} + \frac{\pi^2\ell^4(s^2+t^2+u^2)}{720}\xi(d+1)E_{\alpha_d,\frac{d+1}{2}} + \dots\Bigg)$$

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R^4 correction

$\nabla^4 R^4$ correction

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$\nabla^4 R^4$ correction

Notation

- $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ [completed Riemann zeta]
- $E_{\alpha_d,s} = \frac{1}{2\zeta(2s)} \sum_{\substack{\Gamma\neq 0\\ \Gamma\times\Gamma=0}} |Z(\Gamma)|^{-2s}$ [Eisenstein series]

  Restricted lattice sum rewritable as single U-duality orbit

Interpretation

Expressions converge for $\nabla^{2k}R^4$ term on T^d when $k > \frac{3-d}{2}$

- For $k = 0$ (R^4) and $d > 3$ ($D < 8$) one after using Langlands' functional relation the correct correction function $\mathcal{E}^D_{(0,0)}$ (including numerical coefficient).
  For $d = 3$ one has to regularise; related to known one-loop R^4 divergence in SUGRA.
- For $k = 2$ ($\nabla^4 R^4$) expressions converge. For $d \leq 5$ one obtains only one supersymmetric invariant of [Bossard, Verschinin]; for $7 \leq d < 5$ full (unique) invariant with correct coefficient. For $d = 8$ ancestor of 3-loop divergence [BK].

Expressions also ok for $d > 8$; Kac–Moody case [Fleig, AK]

Two loops in EFT (I)

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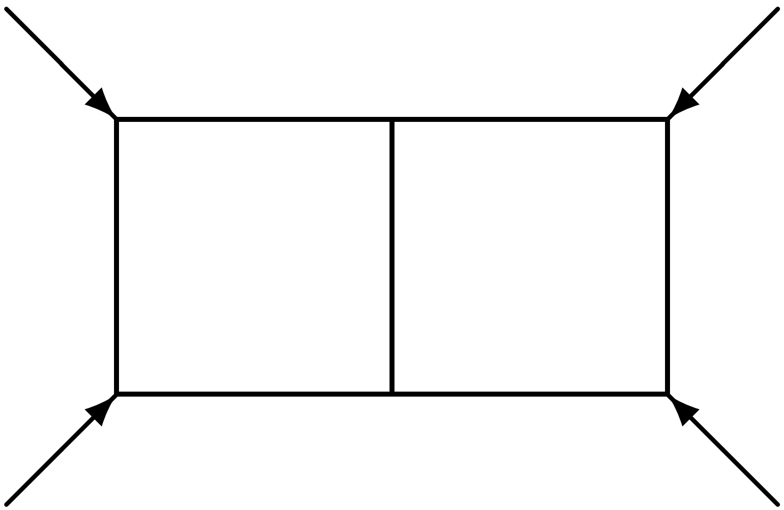
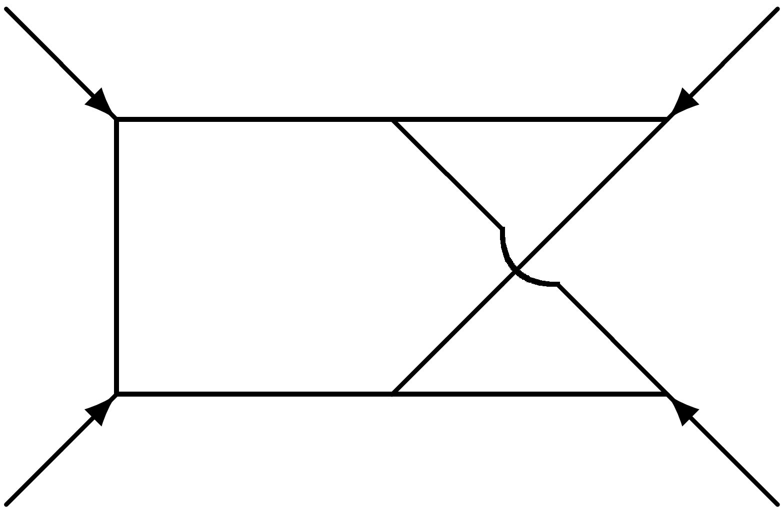
[Bern et al.]: combination of planar and non-planar scalar diagram at $L = 2$





After a few pages of calculation

$$A^{\text{2-loop}}(s,t,u) \sim \ell^6 \sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_i \times \Gamma_j = 0}} \int_0^\infty \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

$$\times \left[\frac{\ell^4(s^2+t^2+u^2)}{6} + \frac{\ell^6(s^3+t^3+u^3)}{72}\Phi_{(0,1)}(\Omega) + \ldots \right]$$

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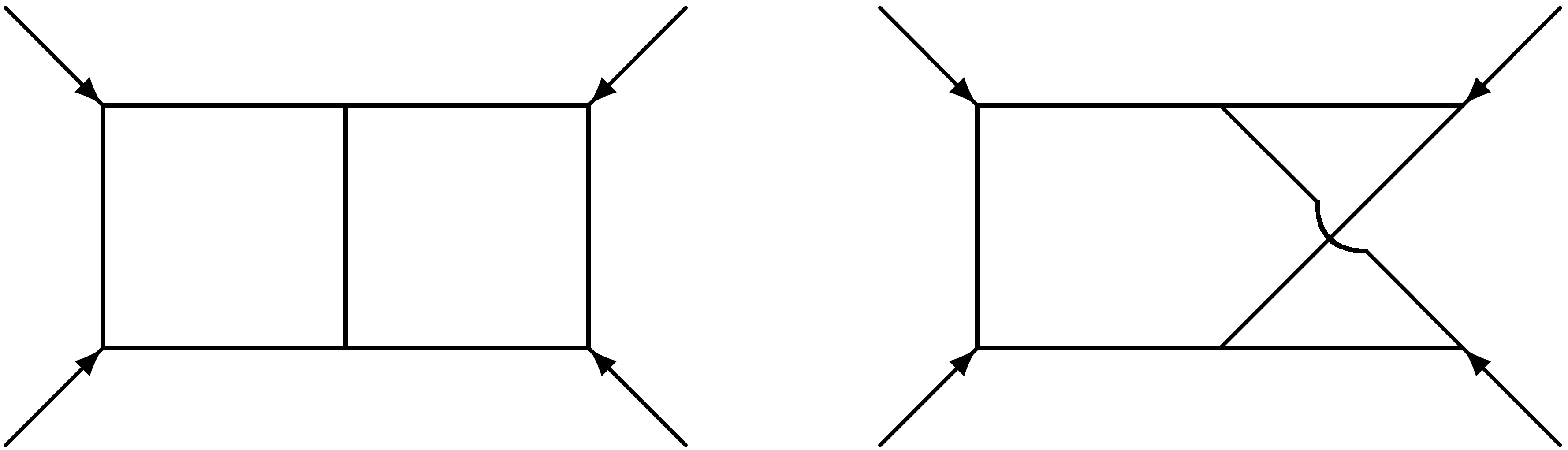

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$\nabla^4 R^4$ correction $\qquad\qquad$ $\nabla^6 R^4$

$$\times \left[\frac{\ell^4(s^2+t^2+u^2)}{6} + \frac{\ell^6(s^3+t^3+u^3)}{72} \Phi_{(0,1)}(\Omega) + \dots \right]$$

Two loops in EFT (II)

Focus on $\nabla^4 R^4$ contribution. Need to understand

$$\sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_i\times\Gamma_j=0}} \int_0^\infty \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

where $\Omega^{ij} = \Omega = \begin{pmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{pmatrix}$

Two loops in EFT (II)

Focus on $\nabla^4 R^4$ contribution. Need to understand

$$\sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_i\times\Gamma_j=0}} \int_0^\infty \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

where $\Omega^{ij} = \Omega = \begin{pmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{pmatrix}$

Sum is restricted to pairs of charges Γ_1, Γ_2 satisfying

$$\Gamma_i \times \Gamma_j \big|_{\mathbf{R}_{\alpha_1}} = 0$$

Solutions can be parametrised by suitable parabolic decompositions [BK]. Look at E_7 example.

Two loops in EFT (III)

$\mathbf{R}_{\alpha_d} = \mathbf{56}$ representation for charges Γ_i

$\mathbf{R}_{\alpha_1} = \mathbf{133} = \mathfrak{e}_7$ representation for section constraint

Decompose under $E_6 \subset E_7$

$$\mathfrak{e}_7 \cong \overline{\mathbf{27}}^{(-2)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{e}_6)^{(0)} \oplus \mathbf{27}^{(2)}$$

$$\mathbf{56} \cong \mathbf{1}^{(-3)} \oplus \mathbf{27}^{(-1)} \oplus \overline{\mathbf{27}}^{(1)} \oplus \mathbf{1}^{(3)}$$

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- Use $E_7(\mathbb{Z})$ to bring rank one Γ_1 into $\mathbf{1}^{(3)}$. Solves constraint $\Gamma_1 \times \Gamma_1 = 0$.

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- Use $E_7(\mathbb{Z})$ to bring rank one Γ_1 into $\mathbf{1}^{(3)}$. Solves constraint $\Gamma_1 \times \Gamma_1 = 0$.
- Any Γ_2 with $\Gamma_1 \times \Gamma_2 = 0$ then has to be in $\overline{\mathbf{27}}^{(1)} \oplus \mathbf{1}^{(3)}$.
- Γ_2 has to also satisfy $\Gamma_2 \times \Gamma_2 = 0 \Rightarrow \overline{\mathbf{27}}^{(1)}$ component has to satisfy section constraint of E_6 stabiliser. Bring into same canonical form as in step one; breaks to $SO(5,5)$

Two loops in EFT (IV)

Upshot: Any pair of Γ_i subject to section constraint written as a pair of elements in $\mathbf{2}^{(4)}$ in $SO(5,5)$ decomp. of E_7

$$\mathfrak{e}_7 \cong \mathbf{10}^{(-4)} \oplus (\mathbf{2}, \mathbf{16})^{(-2)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{so}_{5,5})^{(0)} \oplus (\mathbf{2}, \overline{\mathbf{16}})^{(2)} \oplus \mathbf{10}^{(4)}$$

$$\mathbf{56} \cong \mathbf{2}^{(-4)} \oplus \overline{\mathbf{16}}^{(-2)} \oplus (\mathbf{2} \otimes \mathbf{10})^{(0)} \oplus \mathbf{16}^{(2)} \oplus \mathbf{2}^{(4)}$$

Two loops in EFT (IV)

Upshot: Any pair of Γ_i subject to section constraint written as a pair of elements in $\mathbf{2}^{(4)}$ in $SO(5,5)$ decomp. of E_7

$$\mathfrak{e}_7 \cong \mathbf{10}^{(-4)} \oplus (\mathbf{2},\mathbf{16})^{(-2)} \oplus \overbrace{(\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{so}_{5,5})^{(0)} \oplus (\mathbf{2},\overline{\mathbf{16}})^{(2)} \oplus \mathbf{10}^{(4)}}^{P_{d-1}}$$

$$\mathbf{56} \cong \mathbf{2}^{(-4)} \oplus \overline{\mathbf{16}}^{(-2)} \oplus (\mathbf{2}\otimes\mathbf{10})^{(0)} \oplus \mathbf{16}^{(2)} \oplus \mathbf{2}^{(4)}$$

Pair of elements in $\mathbf{2}^{(4)}$ $\Leftrightarrow$ 2×2 matrix M

$$\sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_i\times\Gamma_j=0}} f(\Gamma_i) = \sum_{\gamma\in E_d/P_{d-1}} \sum_M f(\gamma M)$$

Three different orbits on RHS from $\mathrm{rank}(M) = 0,1,2$. Most interesting is non-degenerate one.

Two loops in EFT (V)

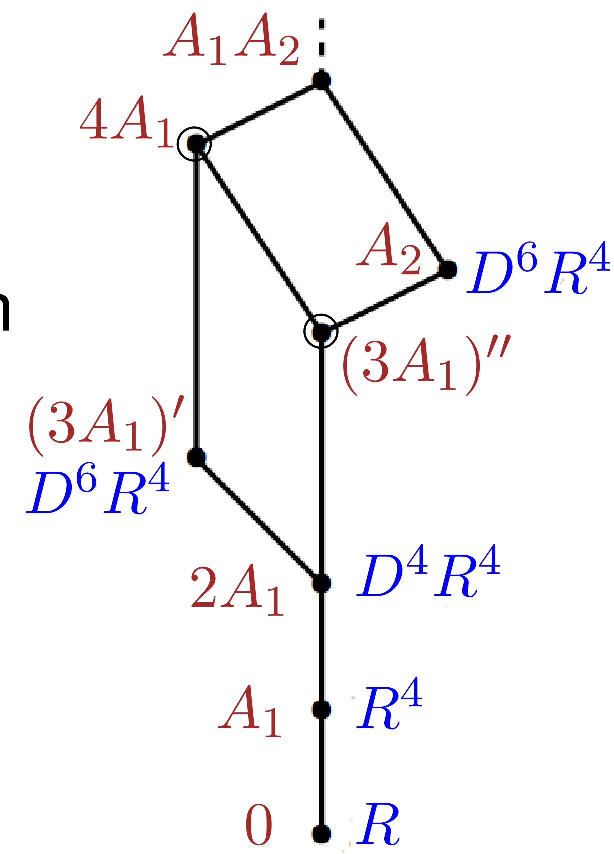
Putting everything together

$$A^{\text{2-loop},\nabla^4R^4}(s,t,u) = 8\pi\ell^{10}\xi(d-4)\xi(d-5)E_{\alpha_{d-1},\frac{d-4}{2}}$$

- This gives the correct function and coefficient for $3 \le d \le 8$ with the right coefficient. Case $d=5$ ($D=6$) trickier due to IR divergences.
- Certain doubling of contributions from one loop and two loops. Corrected if one-loop result renormalised.
- Other orbits of M subdominant at low energies except $d=5$.

Summary and outlook

- Explicitly evaluated loop amplitudes in EFT associated with BPS-protected higher derivative c M-theory
- Reproduced known $\mathcal{E}_{(p,q)}$ in manifestly U-duality covariant form
- Useful tools for dealing with section constraint
- Analysis of differential equation for higher order corrections and their wavefront sets



Hasse diagram for $E_{7(7)}$

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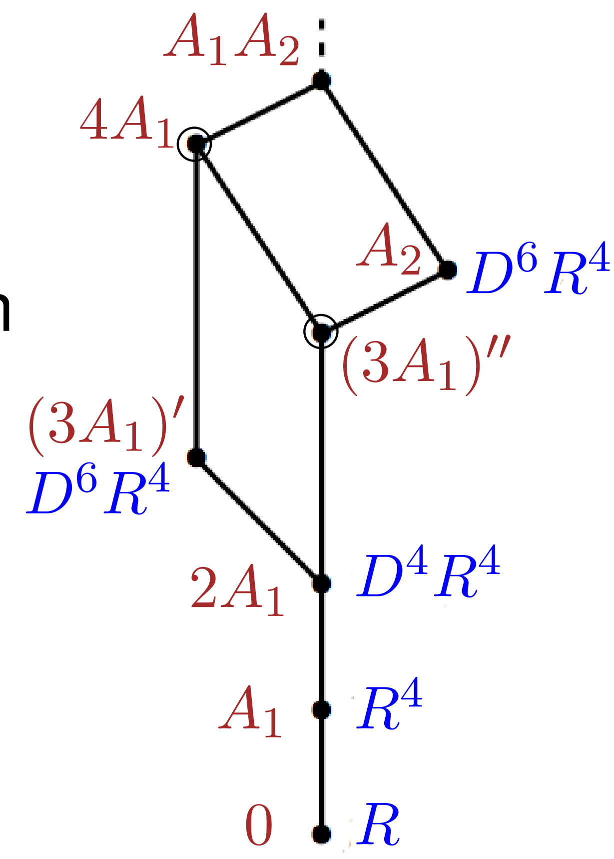
Thank you for your attention!



Hasse diagram for $E_{7(7)}$