Doubled-yet-Gauged, Semi-Covariance & Twofold Spin

Jeong-Hyuck Park

Sogang University, Seoul

Stringy Geometry, Mainz

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Three themes in this talk:

- **Doubled-yet-gauged coordinate system**
- **Semi-covariant formulation of DFT/SDFT**
- **Twofold spin and Standard Model**

To summarize collaborated works with Imtak Jeon (8), Kanghoon Lee (8), Yoonji Suh (3),

Chris Blair (1), Emanuel Malek (1), Wonyoung Cho (1), Jose Fernández-Melgarejo (1),

Kang-Sin Choi (1: *Phenomenologist*), Soo-Jong Rey (1), Woohyun Rim (1),

Yuho Sakatani (1), Sung Moon Ko (1), Charles Melby-Thompson (1), Rene Meyér (1).

- *Differential geometry with a projection: Application to double field theory* 1011.1324
- *Stringy differential geometry, beyond Riemann* 1105.6294
- *Incorporation of fermions into double field theory* 1109.2035
- *Ramond-Ramond Cohomology and O(D,D) T-duality* 1206.3478
- *Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity* 1112.0069
- *Stringy Unification of IIA and IIB Supergravities under* $\mathcal{N}=2$ $D=10$ *Supersymmetric Double Field Theory* 1210.5078
- *Supersymmetric gauged Double Field Theory: Systematic derivation by virtue of 'Twist'* 1505.01301
- *Comments on double field theory and diffeomorphisms* 1304.5946
- *Covariant action for a string in doubled yet gauged spacetime* 1307.8377
- *Double field formulation of Yang-Mills theory* $\Rightarrow$ *Standard Model Double Field Theory* 1102.0419/1506.05277
- $\mathbf{O}(D,D)$ *Covariant Noether Currents and Global Charges in Double Field Theory* 1507.07545
- *Dynamics of Perturbations in Double Field Theory & Non-Relativistic String Theory* 1508.01121
- *U-geometry: SL(5)* $\Rightarrow$ *U-gravity: SL(N)* 1302.1652/1402.5027
- *M-theory and F-theory from a Duality Manifest Action* 1311.5109

Notation 1/2

- Capital Latin letters denote the $\mathbf{O}(D,D)$ vector indices,

$$A, B, C, \cdots, L, M, N, \cdots = 1, 2, \cdots, D{+}D\,.$$

- They can be freely raised or lowered by the $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right).$$

Doubled-yet-gauged coordinate system

1304.5946/1307.8377

Section condition in DFT

- The section condition in DFT,

$$\partial_M \partial^M = 0\,,$$

implies that, arbitrary functions and their arbitrary derivatives, collectively Φ, are invariant under translations generated by a **derivative-index-valued vector,**

$$\Phi_0(x+\Delta) = \Phi_0(x)\,, \qquad \Delta^M = \Phi_1 \partial^M \Phi_2\,.$$

- In fact, the converse is also true. *c.f.* 1307.8377

Doubled-yet-gauged coordinate system

- Start with any D-dimensional coordinate system, x^{μ}, *e.g.* (t,x,y,z) *or* (t,r,θ,ϕ) *etc.*
- The doubled coordinates,

$$x^{M}=(\,\tilde{x}_{\mu}\,,\,x^{\nu}\,)\,,$$

  are then required **to be gauged**: they are subject to an **equivalence relation**,

$$x^{M}\ \sim\ x^{M}+\Phi_{1}\partial^{M}\Phi_{2}\,,$$

  which we call **coordinate gauge symmetry**.

- Each equivalence class, or gauge orbit, represents a single physical point.
- (Strongly constrained) DFT employs such a **doubled-yet-gagued coordinate system.**
- Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits', rather than 'points' in the doubled coordinate space:
  - Hohm-Zwiebach ansatz for finite transformations $\equiv$ $\mathbf{exp}(\hat{\mathcal{L}}_{X})$.

JHP, Berman-Cederwall-Perry, Hull

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$$\hat{\mathcal{L}}_{X}T_{A_{1}\cdots A_{n}}:=X^{B}\partial_{B}T_{A_{1}\cdots A_{n}}+\omega_{T}\,\partial_{B}X^{B}T_{A_{1}\cdots A_{n}}+\sum_{i=1}^{n}(\partial_{A_{i}}X_{B}-\partial_{B}X_{A_{i}})T_{A_{1}\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_{n}}\,,$$

  where ω_{T} denotes the weight.

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Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- The usual infinitesimal one-form, $\mathrm{d}x^M$, is NOT a covariant vector in DFT: it does not transform covariantly under DFT diffeomorphisms, obeying the way the 'generalized Lie derivative' would dictate.
- Hence, $\mathrm{d}x^M \mathrm{d}x^N \mathcal{H}_{MN}$ can NOT give a 'proper length' in DFT.
- Further, it is NOT coordinate gauge symmetry invariant,

$$\mathrm{d}x^M \quad \longrightarrow \quad \mathrm{d}(x^M + \Phi_1 \partial^M \Phi_2) \neq \mathrm{d}x^M \,.$$

- These can be all cured by introducing a **gauged infinitesimal one-form**,

$$Dx^M := \mathrm{d}x^M - \mathcal{A}^M \,,$$

where $\mathcal{A}^M$ is the 'coordinate gauge potential'. Being a derivative-index-valued vector, it satisfies $\mathcal{A}^M \partial_M = 0$, $\mathcal{A}_M \mathcal{A}^M = 0$, or suggestively the 'gauged section condition',

$$(\partial_M + \mathcal{A}_M)(\partial^M + \mathcal{A}^M) = 0 \,.$$

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Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- Under coordinate gauge symmetry, we have the invariance of Dx^M,

$$
\begin{array}{lcll}
x^M & \longrightarrow & x'^M = x^M + \Phi_1\partial^M\Phi_2\,, & \\
\mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^M + \mathrm{d}(\Phi_1\partial^M\Phi_2) & : \quad \mathcal{A}'^M\partial'_M \equiv 0\,, \\
Dx^M & \longrightarrow & D'x'^M = Dx^M = \mathrm{d}x^M - \mathcal{A}^M\,. &
\end{array}
$$

- Similarly, under (finite) DFT diffeomorphisms *à la* Hohm-Zwiebach

$$
L_M{}^N := \partial_M x'^N\,, \qquad \bar{L} := \mathcal{J}L^t\mathcal{J}^{-1}\,,
$$

$$
F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, \qquad \bar{F} := \mathcal{J}F^t\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,
$$

we have the covariance,

$$
\begin{array}{lcll}
x^M & \longrightarrow & x'^M(x)\,, & \\
\mathcal{H}_{MN}(x) & \longrightarrow & \mathcal{H}'_{MN}(x') = \bar{F}_M{}^K\bar{F}_N{}^L\mathcal{H}_{KL}(x)\,, & \\
\mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N(L-F)_N{}^M & : \quad \mathcal{A}'^M\partial'_M \equiv 0\,, \\
Dx^M & \longrightarrow & D'x'^M = Dx^N F_N{}^M\,. &
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Fixing the coordinate gauge symmetry : conventional choice of the section

- In DFT –unlike EFT or U-gravity– the solution of the section condition, *i.e.* the section is unique up to the duality rotations,

$$\frac{\partial}{\partial x^M} = \left(\frac{\partial}{\partial \tilde{x}_\mu}\,,\,\frac{\partial}{\partial x^\nu}\right) \equiv \left(0\,,\,\frac{\partial}{\partial x^\nu}\right) \qquad \text{'conventional' choice of the section}$$

- Then, the 'coordinate gauge symmetry' reads

$$(\,\tilde{x}_\mu\;,\;x^\nu\,)\;\sim\;(\,\tilde{x}_\mu+\Phi_1\partial_\mu\Phi_2\;,\;x^\nu\,)\,.$$

- The coordinate gauge potential and the gauged infinitesimal one-form become

$$\mathcal{A}^M = A_\lambda \partial^M x^\lambda = (\,A_\mu\;,\;0\,)\,, \qquad\qquad Dx^M = (\,\mathrm{d}\tilde{x}_\mu - A_\mu\;,\;\mathrm{d}x^\nu\,)\,.$$

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Newton mechanics with doubled-yet-gauged coordinate system

- The doubled-yet-gauged coordinates can be applied to any physical system, not exclusively to DFT.

- Newton mechanics can be formulated on the doubled-yet-gauged space, $x^I = (\tilde{x}_j, x^k)$,

$$\mathcal{L}_{\rm Newton} = \tfrac{1}{2} m\, D_t x^I D_t x^J\, \delta_{IJ} \,-\, V(x)\,,$$

where $I, J = 1, 2, \cdots, 6$ and the potential, $V(x)$, satisfies the section condition.

- With the conventional choice of the section, we get

$$\mathcal{L}_{\rm Newton} = \tfrac{1}{2} m\, \dot{x}^j \dot{x}^k\, \delta_{jk} \,-\, V(x) \,+\, \tfrac{1}{2} m \left(\dot{\tilde{x}}_j - A_j\right) \left(\dot{\tilde{x}}_k - A_k\right) \delta^{jk}\,.$$

Hence, after integrating out A_j, we recover the conventional formulation.

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String probes the doubled-yet-gauged spacetime

- **DFT string action** is with $D_iX^M = \partial_i X^M - \mathcal{A}_i^M$, JHP-Lee 2013

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\mathrm{string}}\,, \qquad \mathcal{L}_{\mathrm{string}} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^M D_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- The action is **fully symmetric**, essentially due to the auxiliary gauge field, $\mathcal{A}_i^M$, under
  - String worldsheet diffeomorphisms plus Weyl symmetry (as usual)
  - $\mathbf{O}(D,D)$ T-duality
  - Target spacetime DFT diffeomorphisms
  - The coordinate gauge symmetry

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

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- $\mathcal{H}_{AB}(x)$ is the "generalized metric" which can be defined as a symmetric $\mathbf{O}(D,D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}\,, \qquad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}\,,$$

and satisfies the section condition.

- There are **two types** of "generalized metric" : **Riemannian vs. non-Riemannian**.

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DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. the conventional choice of the section, $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$, **Riemannian generalized metric** assumes the well-known form,

$$\mathcal{H}_{AB} = \left(\begin{array}{cc} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{array}\right).$$

  Up to field redefinition (e.g. β-gravity Andriot-Betz) this is the most general form of a symmetric $\mathbf{O}(D,D)$ element **if the upper left $D \times D$ block is 'non-degenerate'.**

- The DFT sigma model then reduces to the standard string action,

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\rm string} \equiv \frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i \tilde{X}_\mu \partial_j X^\mu\right],$$

  with the bonus of the topological term introduced by Giveon-Rocek; Hull.

- The EOM of $\mathcal{A}_i^M$ implies **self-duality** on the full doubled spacetime,

$$\mathcal{H}^M{}_N D^i X^N + \frac{1}{\sqrt{-h}}\epsilon^{ij} D_j X^M = 0\,.$$

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DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$ again, the **non-Riemannian DFT background** is then characterized by the degenerate upper left $D \times D$ block, such that it does not admit any Riemannian interpretation even locally. For example, with the decomposition, $D = 10 = 2 + 8$,

$$\mathcal{H}_{MN} = \begin{pmatrix} 0 & 0 & e^{\alpha}{}_{\beta} & 0 \\ 0 & \delta^{ij} & 0 & 0 \\ -e_{\alpha}{}^{\beta} & 0 & f\eta_{\alpha\beta} & 0 \\ 0 & 0 & 0 & \delta_{ij} \end{pmatrix}, \qquad f = 1 + \frac{Q}{r^6}\,, \qquad r^2 = \textstyle\sum_{i=2}^{9}(x^i)^2\,.$$

  This is "doublly T-dual", $(t, x^1) \Leftrightarrow (\tilde{t}, \tilde{x}_1)$, DFT background to F1 *à la* Dabholkar-Gibbons-Harvey-Ruiz 1990, *c.f.* $2D$ null-wave *à la* Berkeley-Berman-Rudolf

- DFT as well as the DFT sigma model is well-defined even for such a non-Riemannian background.

- In particular, the Gomis-Ooguri 'non-relativistic' string theory can be identified precisely as the DFT sigma model on the above non-Riemannian background.

  Ko-Meyer-Melby-Thompson-JHP 2015

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  Ko-Meyer-Melby-Thompson-JHP 2015

Semi-covariant formulation of DFT/SDFT:

Gravity on doubled-yet-gauged spacetime

1011.1324/1105.6294/$\cdots$

- **Contrary to what it may sound like, the semi-covariant formalism is a completely covariant approach to DFT, as it manifests simultaneously**

  - **O**(D,D) **T-duality**

  - **DFT-diffeomorphisms (generalized Lie derivative)**

  - **A *pair* of local Lorentz symmetries, Spin**$(1,D{-}1)_L \times$ **Spin**$(D{-}1,1)_R$

In particular, it makes each term in $D=10$ Maximal SDFT completely covariant:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^q$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^p\mathcal{D}^\star_p\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^\star_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^q\mathcal{D}^\star_q\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^\star_{\bar{p}}\rho' + i\bar{\psi}'^p\mathcal{D}'^\star_p\rho' + i\tfrac{1}{2}\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\mathcal{D}'^\star_{\bar{q}}\psi'_p\Big]$$

Jeon-Lee-JHP-Suh 2012

- **It also works for SL**(N) **duality group,** $N\neq 4$ JHP-Suh 'U-gravity' 2014

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(D,D)$ & DFT-diffeom. vector	$\mathcal{J}_{AB}$
$p, q, \cdots$	$\mathbf{Spin}(1, D-1)_L$ vector	$\eta_{pq} = \text{diag}(- + + \cdots +)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D-1)_L$ spinor	$C_{+\alpha\beta}, \quad (\gamma^p)^T = C_+ \gamma^p C_+^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D-1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \text{diag}(+ - - \cdots -)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D-1, 1)_R$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}, \quad (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+ \bar{\gamma}^{\bar{p}} \bar{C}_+^{-1}$

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector**
    - **DFT-dilaton:** d
    - **DFT-vielbeins:** $V_{Ap}\,,\quad \bar{V}_{A\bar{p}}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (Majorana-Weyl)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,,\quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,,\quad \psi'^{\bar{\alpha}}_{p}$

R-R potential and Fermions carry NOT $(D+D)$-dimensional BUT undoubled D-dimensional indices.

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A priori, **O**(D, D) **rotates only the O**(D, D) **vector indices (capital Roman), and the R-R sector and all the fermions are O**(D, D) **T-duality singlet.**

The usual IIA $\Leftrightarrow$ IIB exchange will follow only after the diagonal gauge fixing of the twofold local Lorentz symmetries.

- The DFT-dilaton gives rise to a scalar density with weight one,

$$e^{-2d}\,.$$

- The DFT-vielbeins satisfy four 'defining' properties:

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}}=0\,,\qquad V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}\,.$$

- Naturally, they generate a pair of two-index 'projectors',

$$P_{AB}:=V_{A}{}^{p}V_{Bp}\,,\qquad P_{A}{}^{B}P_{B}{}^{C}=P_{A}{}^{C}\,,\qquad \bar{P}_{AB}:=\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B\bar{p}}\,,\qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C}=\bar{P}_{A}{}^{C}\,,$$

which are symmetric, orthogonal and complementary to each other,

$$P_{AB}=P_{BA}\,,\qquad \bar{P}_{AB}=\bar{P}_{BA}\,,\qquad P_{A}{}^{B}\bar{P}_{B}{}^{C}=0\,,\qquad P_{A}{}^{B}+\bar{P}_{A}{}^{B}=\delta_{A}{}^{B}\,.$$

- Some further projection properties follow

$$P_{A}{}^{B}V_{Bp}=V_{Ap}\,,\qquad \bar{P}_{A}{}^{B}\bar{V}_{B\bar{p}}=\bar{V}_{A\bar{p}}\,,\qquad \bar{P}_{A}{}^{B}V_{Bp}=0\,,\qquad P_{A}{}^{B}\bar{V}_{B\bar{p}}=0\,.$$

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- Note also $\mathcal{H}_{AB}=P_{AB}-\bar{P}_{AB}$. However, our emphasis lies on the 'projectors' rather than the "generalized metric".

- We continue to define a pair of six-index projectors,

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}\,, \qquad \mathcal{P}_{CAB}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI}\,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D}\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{CAB}{}^{GHI}\,,$$

which are symmetric and traceless,

$$\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}\,, \qquad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}\,,$$

$$\mathcal{P}^{A}{}_{ABDEF} = 0\,, \quad P^{AB}\mathcal{P}_{ABCDEF} = 0\,, \qquad \bar{\mathcal{P}}^{A}{}_{ABDEF} = 0\,, \quad \bar{P}^{AB}\bar{\mathcal{P}}_{ABCDEF} = 0\,.$$

- As we shall see shorlty, these projectors govern the DFT-diffeomorphic anomaly in the semi-covariant formalism, which can be then easily projected out.

Chirality of $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$:

$$\gamma^{(D+1)}\psi_{\bar{p}} = \mathbf{c}\,\psi_{\bar{p}}\,, \qquad \gamma^{(D+1)}\rho = -\mathbf{c}\,\rho\,,$$

$$\bar{\gamma}^{(D+1)}\psi'_{p} = \mathbf{c}'\psi'_{p}\,, \qquad \bar{\gamma}^{(D+1)}\rho' = -\mathbf{c}'\rho'\,,$$

$$\gamma^{(D+1)}\mathcal{C}\bar{\gamma}^{(D+1)} = \mathbf{c}\mathbf{c}'\,\mathcal{C}\,,$$

where $\mathbf{c}$ and $\mathbf{c}'$ are arbitrary independent two sign factors, $\mathbf{c}^2 = \mathbf{c}'^2 = 1$.

- *A priori,* all the possible four different sign choices are equivalent up to $\mathbf{Pin}(1,D{-}1)_L \times \mathbf{Pin}(D{-}1,1)_R$ rotations.
- That is to say, $D = 10$ maximal SDFT is chiral with respect to both $\mathbf{Pin}(1,D{-}1)_L$ and $\mathbf{Pin}(D{-}1,1)_R$, and the theory is unique, unlike IIA/IIB SUGRAs.
- Hence, without loss of generality, we may safely set

$$\mathbf{c} \equiv \mathbf{c}' \equiv +1\,.$$

- Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.

- **Having all the 'right' field-variables prepared, we now discuss their derivatives or what we call, 'semi-covariant derivative'.**

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Semi-covariant derivatives

- For each gauge symmetry we assign a corresponding connection,
  - Γ_A for the DFT-diffeomorphism (generalized Lie derivative),
  - Φ_A for the 'unbarred' local Lorentz symmetry, $\mathbf{Spin}(1, D{-}1)_L$,
  - $\bar{\Phi}_A$ for the 'barred' local Lorentz symmetry, $\mathbf{Spin}(D{-}1, 1)_R$.
- Combining all of them, we introduce **master 'semi-covariant' derivative,**

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A\,.$$

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A\,, \qquad\qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A\,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n}\,.$$

- And the latter is the covariant derivative for the twofold local Lorenz symmetries.

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- And the latter is the covariant derivative for the twofold local Lorenz symmetries.

- **By definition, the master derivative annihilates all the 'constants',**

$$\mathcal{D}_A\mathcal{J}_{BC}=\nabla_A\mathcal{J}_{BC}=\Gamma_{AB}{}^D\mathcal{J}_{DC}+\Gamma_{AC}{}^D\mathcal{J}_{BD}=0\,,$$

$$\mathcal{D}_A\eta_{pq}=D_A\eta_{pq}=\Phi_{Ap}{}^r\eta_{rq}+\Phi_{Aq}{}^r\eta_{pr}=0\,,$$

$$\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=D_A\bar{\eta}_{\bar{p}\bar{q}}=\bar{\Phi}_{A\bar{p}}{}^{\bar{r}}\bar{\eta}_{\bar{r}\bar{q}}+\bar{\Phi}_{A\bar{q}}{}^{\bar{r}}\bar{\eta}_{\bar{p}\bar{r}}=0\,,$$

$$\mathcal{D}_A C_{+\alpha\beta}=D_A C_{+\alpha\beta}=\Phi_{A\alpha}{}^{\delta}C_{+\delta\beta}+\Phi_{A\beta}{}^{\delta}C_{+\alpha\delta}=0\,,$$

$$\mathcal{D}_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=D_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=\bar{\Phi}_{A\bar{\alpha}}{}^{\bar{\delta}}\bar{C}_{+\bar{\delta}\bar{\beta}}+\bar{\Phi}_{A\bar{\beta}}{}^{\bar{\delta}}\bar{C}_{+\bar{\alpha}\bar{\delta}}=0\,,$$

including the gamma matrices,

$$\mathcal{D}_A(\gamma^p)^{\alpha}{}_{\beta}=D_A(\gamma^p)^{\alpha}{}_{\beta}=\Phi_A{}^p{}_q(\gamma^q)^{\alpha}{}_{\beta}+\Phi_A{}^{\alpha}{}_{\delta}(\gamma^p)^{\delta}{}_{\beta}-(\gamma^p)^{\alpha}{}_{\delta}\Phi_A{}^{\delta}{}_{\beta}=0\,,$$

$$\mathcal{D}_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=D_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=\bar{\Phi}_A{}^{\bar{p}}{}_{\bar{q}}(\bar{\gamma}^{\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}+\bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\delta}}(\bar{\gamma}^{\bar{p}})^{\bar{\delta}}{}_{\bar{\beta}}-(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\delta}}\bar{\Phi}_A{}^{\bar{\delta}}{}_{\bar{\beta}}=0\,.$$

- It follows then that the connections are all anti-symmetric,

$$\Gamma_{ABC} = -\Gamma_{ACB}\,,$$

$$\Phi_{Apq} = -\Phi_{Aqp}\,, \qquad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = -\bar{\Phi}_{A\bar{q}\bar{p}}\,, \qquad \bar{\Phi}_{A\bar{\alpha}\bar{\beta}} = -\bar{\Phi}_{A\bar{\beta}\bar{\alpha}}\,,$$

and as usual,

$$\Phi_{A}{}^{\alpha}{}_{\beta} = \tfrac{1}{4}\Phi_{Apq}(\gamma^{pq})^{\alpha}{}_{\beta}\,, \qquad \bar{\Phi}_{A}{}^{\bar{\alpha}}{}_{\bar{\beta}} = \tfrac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}(\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}\,.$$

- **Further, the master derivative is compatible with the whole NS-NS sector,**

$$\mathcal{D}_A d = \nabla_A d := -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA} = 0\,,$$

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0\,,$$

$$\mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

- It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$

and the connections are related to each other,

$$\Gamma_{ABC} = V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}\,,$$

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- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \,,$$

$$\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq} \,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} \,.$$

Here

$$\Gamma^0_{CAB} = 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \,,$$

Jeon-Lee-JHP 2011

and, with the corresponding derivative, $\nabla^0_A = \partial_A + \Gamma^0_A$,

$$\Phi^0_{Apq} = V^B{}_p \nabla^0_A V_{Bq} = V^B{}_p \partial_A V_{Bq} + \Gamma^0_{ABC} V^B{}_p V^C{}_q \,,$$

$$\bar{\Phi}^0_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla^0_A \bar{V}_{B\bar{q}} = \bar{V}^B{}_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma^0_{ABC} \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}} \,.$$

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- The extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the torsion of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_A d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 \,, \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 \,.$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_A \,, \qquad \bar{\psi}_{\bar{p}}\gamma_A\psi_{\bar{q}} \,, \qquad \bar{\rho}\gamma_{Apq}\rho \,, \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} \,,$$

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$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} .$$

- The extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_A d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 , \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 .$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_A , \qquad \bar{\psi}_{\bar{p}}\gamma_A\psi_{\bar{q}} , \qquad \bar{\rho}\gamma_{Apq}\rho , \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} ,$$

  where we set $\psi_A = \bar{V}_A{}^{\bar{p}}\psi_{\bar{p}}$, $\gamma_A = V_A{}^p\gamma_p$.

- The 'torsionless' connection,

$$\Gamma^{0}_{CAB} = \quad 2\left(P\partial_{C}P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_{D}P_{EC}$$
$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_{D}d + (P\partial^{E}P\bar{P})_{[ED]}\right)\,,$$

further obeys

$$\Gamma^{0}_{ABC} + \Gamma^{0}_{BCA} + \Gamma^{0}_{CAB} = 0\,,$$

and

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma^{0}_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma^{0}_{DEF} = 0\,.$$

- **In fact, the torsionless connection,**

$$\Gamma^{0}_{CAB} = 2\left(P\partial_{C}P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_{D}P_{EC} - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_{D}d + (P\partial^{E}P\bar{P})_{[ED]}\right),$$

is the unique solution to the following constraints:

$$\Gamma_{CAB} + \Gamma_{CBA} = 0 \qquad \Longrightarrow \qquad \nabla_{A}\mathcal{J}_{BC} = 0\,,$$

$$\nabla_{A}P_{BC} = \nabla_{A}\bar{P}_{BC} = 0\,,$$

$$\nabla_{A}d = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0 \quad \Longrightarrow \quad \hat{\mathcal{L}}(\partial) = \hat{\mathcal{L}}(\nabla)\,,$$

$$(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{DEF}\Gamma_{DEF} = 0\,.$$

- In this way, Γ^{0}_{ABC} is the **DFT analogy of the Christoffel connection.**
  However, unlike Christoffel symbol, the DFT-diffeomorphism cannot transform it to vanish point-wise. This can be viewed as the failure of the Equivalence Principle applied to an extended object, *i.e.* string.

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Semi-covariant Riemann curvature

- The usual curvatures for the three connections,

$$R_{CDAB}=\partial_{A}\Gamma_{BCD}-\partial_{B}\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}\,,$$

$$F_{ABpq}=\partial_{A}\Phi_{Bpq}-\partial_{B}\Phi_{Apq}+\Phi_{Apr}\Phi_{B}{}^{r}{}_{q}-\Phi_{Bpr}\Phi_{A}{}^{r}{}_{q}\,,$$

$$\bar{F}_{AB\bar{p}\bar{q}}=\partial_{A}\bar{\Phi}_{B\bar{p}\bar{q}}-\partial_{B}\bar{\Phi}_{A\bar{p}\bar{q}}+\bar{\Phi}_{A\bar{p}\bar{r}}\bar{\Phi}_{B}{}^{\bar{r}}{}_{\bar{q}}-\bar{\Phi}_{B\bar{p}\bar{r}}\bar{\Phi}_{A}{}^{\bar{r}}{}_{\bar{q}}\,,$$

are, from $[\mathcal{D}_{A},\mathcal{D}_{B}]V_{Cp}=0$ and $[\mathcal{D}_{A},\mathcal{D}_{B}]\bar{V}_{C\bar{p}}=0$, related to each other,

$$R_{ABCD}=F_{CDpq}V_{A}{}^{p}V_{B}{}^{q}+\bar{F}_{CD\bar{p}\bar{q}}\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B}{}^{\bar{q}}\,.$$

- **However, the crucial object in DFT turns out to be**

$$S_{ABCD}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^{E}{}_{AB}\Gamma_{ECD}\right),$$

which we name semi-covariant Riemann curvature.

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which we name semi-covariant Riemann curvature.

Properties of the semi-covariant curvature

- Under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S^{0}_{ABCD}=\mathcal{D}_{[A}\delta\Gamma^{0}_{B]CD}+\mathcal{D}_{[C}\delta\Gamma^{0}_{D]AB}\,,$$

$$\delta S_{ABCD}=\mathcal{D}_{[A}\delta\Gamma_{B]CD}+\mathcal{D}_{[C}\delta\Gamma_{D]AB}-\tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^{E}{}_{CD}-\tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^{E}{}_{AB}\,.$$

- It also satisfies precisely the same symmetric property as the ordinary Riemann curvature,

$$S_{ABCD}=\tfrac{1}{2}\left(S_{[AB][CD]}+S_{[CD][AB]}\right)\,,$$

$$S^{0}_{[ABC]D}=0\,,$$

as well as projection property,

$$S_{p\bar{p}q\bar{q}}=S_{ABCD}V^{A}{}_{p}\bar{V}^{B}{}_{\bar{p}}V^{C}{}_{q}\bar{V}^{D}{}_{\bar{q}}=0\,.$$

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$$S_{ABCD} = \tfrac{1}{2}\left(S_{[AB][CD]} + S_{[CD][AB]}\right)\,,$$

$$S^{0}_{[ABC]D} = 0\,,$$

as well as projection property,

$$S_{p\bar{p}q\bar{q}} = S_{ABCD}\,V^{A}{}_{p}\bar{V}^{B}{}_{\bar{p}}V^{C}{}_{q}\bar{V}^{D}{}_{\bar{q}} = 0\,.$$

- Generically, under DFT-diffeomorphisms, the variation of the semi-covariant derivative carries anomalous terms which are dictated by the six-index projectors,

$$\delta_X \left(\nabla_C T_{A_1 \cdots A_n}\right) \equiv \hat{\mathcal{L}}_X \left(\nabla_C T_{A_1 \cdots A_n}\right) + \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE} \partial_F \partial_{[D} X_{E]} T_{\cdots B \cdots} \,.$$

- Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{\mathcal{L}}_X \,.$$

- However, the characteristic property of our 'semi-covariant' derivative/curvature is that, the anomaly can be easily projected out, and can thus produce completely covariant derivatives/curvatures.

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- However, the characteristic property of our 'semi-covariant' derivative/curvature is that, the anomaly can be easily projected out, and can thus produce completely covariant derivatives/curvatures.

Completely covariant derivatives

- **For $\mathbf{O}(D,D)$ tensors:**

$$P_C{}^D\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_D T_{B_1B_2\cdots B_n}\,,\qquad \bar{P}_C{}^D P_{A_1}{}^{B_1}P_{A_2}{}^{B_2}\cdots P_{A_n}{}^{B_n}\nabla_D T_{B_1B_2\cdots B_n}\,,$$

$$\left.\begin{array}{l} P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A T_{BD_1D_2\cdots D_n}\,,\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_A T_{BD_1D_2\cdots D_n}\end{array}\right\}\quad\text{Divergences}\,,$$

$$\left.\begin{array}{l} P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1D_2\cdots D_n}\,,\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1D_2\cdots D_n}\end{array}\right\}\quad\text{Laplacians}\,,$$

and

$$\mathfrak{D}_A{}^C\bar{P}_{B_1}{}^{D_1}\cdots\bar{P}_{B_n}{}^{D_n}T_{CD_1\cdots D_n}\,,\qquad \bar{\mathfrak{D}}_A{}^C P_{B_1}{}^{D_1}\cdots P_{B_n}{}^{D_n}T_{CD_1\cdots D_n}\,,$$

where we set a pair of semi-covariant second order differential operators,

$$\mathfrak{D}_A{}^B := (P_A{}^BP^{CD}-2P_A{}^DP^{BC})(\nabla_C\nabla_D-S_{CD})\,,\qquad \bar{\mathfrak{D}}_A{}^B := (\bar{P}_A{}^B\bar{P}^{CD}-2\bar{P}_A{}^D\bar{P}^{BC})(\nabla_C\nabla_D-S_{CD})\,,$$

which are relevant to the DFT fluctuation analysis Ko-Meyer-Melby-Thompson-JHP

Completely covariant derivatives

- **For local Lorentz tensors, Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$:**

$$\begin{array}{ll}
\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, & \mathcal{D}_{\bar{p}} T_{q_1q_2\cdots q_n}\,,\\
\mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, & \mathcal{D}^{\bar{p}} T_{\bar{p}q_1q_2\cdots q_n}\,,\\
\mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, & \mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}} T_{q_1q_2\cdots q_n}\,,\\
\mathfrak{D}_p{}^q T_{q\bar{p}_1\bar{p}_2\cdots\bar{p}_n}\,, & \bar{\mathfrak{D}}_{\bar{p}}{}^{\bar{q}} T_{\bar{q}p_1p_2\cdots p_n}\,.
\end{array}$$

These are the 'pull-back' of the previous page using the DFT-vielbeins, such as

$$\mathcal{D}_p := V^A{}_p\mathcal{D}_A\,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}}\mathcal{D}_A\,.$$

Completely covariant derivatives

- Following the aforementioned general prescription, **completely covariant Yang-Mills field strength** is given by two opposite projections, or

$$\mathcal{F}_{p\bar{q}} = V^{M}{}_{p}\bar{V}^{N}{}_{\bar{q}}\mathcal{F}_{MN}\,,$$

where $\mathcal{F}_{MN}$ is the semi-covariant field strength of a YM potential, $\mathcal{V}_M$,

$$\mathcal{F}_{MN} := \nabla_M\mathcal{V}_N - \nabla_N\mathcal{V}_M - i\,[\mathcal{V}_M,\mathcal{V}_N]\ .$$

Unlike the Riemannian case, the Γ-connections are not canceled out.

- Further, we may freely impose "gauged" section condition to halve the off-shell degrees:

$$(\partial_M - i\,\mathcal{V}_M)(\partial^M - i\,\mathcal{V}^M) = 0\,,$$

which implies $\mathcal{V}^M\partial_M = 0, \partial_M\mathcal{V}^M = 0, \mathcal{V}_M\mathcal{V}^M = 0$, like the coordinate gauge potential.

For consistency, the above condition is preserved under all the symmetry transformations: $\mathbf{O}(D,D)$ rotations, diffeomorphisms, and the Yang-Mills gauge symmetry,

$$[\mathbf{g}\mathcal{V}^M\mathbf{g}^{-1} - i(\partial^M\mathbf{g})\mathbf{g}^{-1}]\partial_M = 0\,,\quad (\hat{\mathcal{L}}_X\mathcal{V}^M)\partial_M = [X^N\partial_N\mathcal{V}^M + (\partial^M X_N - \partial_N X^M)\mathcal{V}^N]\partial_M = 0\,.$$

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Completely covariant derivatives

- $\mathbf{O}(D,D)$ covariant **Killing equations** in DFT:

$$\hat{\mathcal{L}}_X\mathcal{H}_{MN}=0\qquad\Longleftrightarrow\qquad(P\nabla)_M(\bar{P}X)_N-(\bar{P}\nabla)_N(PX)_M=0\,,$$

$$\hat{\mathcal{L}}_X d=0\qquad\Longleftrightarrow\qquad\nabla_M X^M=0\,.$$

JHP-Rey-Rim-Sakatani 2015

Chris Blair 2015

- **Dirac operators for fermions,** $\rho^{\alpha},\ \psi^{\alpha}_{\bar{p}},\ \rho'^{\bar{\alpha}},\ \psi'^{\bar{\alpha}}_{p}$:

$$\gamma^{p}\mathcal{D}_{p}\rho=\gamma^{A}\mathcal{D}_{A}\rho\,,\qquad\qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}}=\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,,\qquad\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}=\mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\psi}^{A}\gamma_{p}(\mathcal{D}_{A}\psi_{\bar{q}}-\tfrac{1}{2}\mathcal{D}_{\bar{q}}\psi_{A})\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'=\bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p}=\bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,,\qquad\qquad \mathcal{D}_{p}\psi'^{p}=\mathcal{D}_{A}\psi'^{A}\,,$$

$$\bar{\psi}'^{A}\bar{\gamma}_{\bar{p}}(\mathcal{D}_{A}\psi'_{q}-\tfrac{1}{2}\mathcal{D}_{q}\psi'_{A})\,.$$

Incorporation of fermions into DFT 1109.2035

Completely covariant derivatives

- **For R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\beta}}$:**

$$\mathcal{D}_{+}\mathcal{C} := \gamma^{A}\mathcal{D}_{A}\mathcal{C} + \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{C}\bar{\gamma}^{A}\,,$$

$$\mathcal{D}_{-}\mathcal{C} := \gamma^{A}\mathcal{D}_{A}\mathcal{C} - \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{C}\bar{\gamma}^{A}\,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent,**

$$(\mathcal{D}^{0}_{+})^{2}\mathcal{C} = 0\,, \qquad (\mathcal{D}^{0}_{-})^{2}\mathcal{C} = 0\,,$$

and hence, they define **$\mathbf{O}(D,D)$ covariant cohomology.**

- **The field strength of the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, is then defined by**

$$\mathcal{F} := \mathcal{D}^{0}_{+}\mathcal{C}\,.$$

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized

$$\delta\mathcal{C} = \mathcal{D}^{0}_{+}\Delta \qquad \Longrightarrow \qquad \delta\mathcal{F} = \mathcal{D}^{0}_{+}(\delta\mathcal{C}) = (\mathcal{D}^{0}_{+})^{2}\Delta = 0\,.$$

Completely covariant derivatives

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Completely covariant curvatures

- **Scalar curvature:**

$$\mathcal{S} := (P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}$$

$$c.f.\quad S_{AB}{}^{AB} = 0\,.$$

- **"Ricci" curvature:**

$$S^0_{p\bar{q}} = V^A{}_p\bar{V}^B{}_{\bar{q}}S^0_{AB}$$

where we set $S^0_{AB} = S^0_{ACB}{}^C$.

- **Further, we have conserved "Einstein" curvature,**

$$G^{AB} = 2(P^{AC}\bar{P}^{BD} - \bar{P}^{AC}P^{BD})S_{CD} - \tfrac{1}{2}\mathcal{J}^{AB}\mathcal{S}\,, \qquad \nabla_A G^{AB} = 0\,.$$

Combining all the results above, we are now ready to spell

- $D = 10$ **Maximally Supersymmetric Double Field Theory**

- **Lagrangian**:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^q$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^p\mathcal{D}^\star_p\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^\star_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^q\mathcal{D}^\star_q\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^\star_{\bar{p}}\rho' + i\bar{\psi}'^p\mathcal{D}'^\star_p\rho' + i\tfrac{1}{2}\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\mathcal{D}'^\star_{\bar{q}}\psi'_p\Big]\,.$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_+^{-1}\mathcal{F}^T C_+$.

- As they are contracted with the DFT-vielbeins properly, **every term in the Lagrangian is completey covariant.**

c.f. Democratic SUGRA *à la* Bergshoeff-Kallosh-Ortin-Roest-Van Proeyen & Generalized Geometry *à la* Coimbra-Strickland-Constable-Waldram

- **Lagrangian**:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
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$D = 10$ Maximal SDFT [1210.5078]

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$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^q$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^p\mathcal{D}^\star_p\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^\star_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^q\mathcal{D}^\star_q\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^\star_{\bar{p}}\rho' + i\bar{\psi}'^p\mathcal{D}'^\star_p\rho' + i\tfrac{1}{2}\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\mathcal{D}'^\star_{\bar{q}}\psi'_p\Big]\,.$$

- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\Gamma_{ABC} = \Gamma^0_{ABC} + i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho - 2i\bar{\rho}\gamma_{BC}\psi_A - i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} + 4i\bar{\psi}_B\gamma_A\psi_C$$
$$+ i\tfrac{1}{3}\bar{\rho}'\bar{\gamma}_{ABC}\rho' - 2i\bar{\rho}'\bar{\gamma}_{BC}\psi'_A - i\tfrac{1}{3}\bar{\psi}'^p\bar{\gamma}_{ABC}\psi'_p + 4i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,,$$

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold: $\mathcal{D}^\star_A$ for the unprimed fermions and $\mathcal{D}'^\star_A$ for the primed fermions, set by

$$\Gamma^\star_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 2i\bar{\psi}_B\gamma_A\psi_C + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A\,,$$

$$\Gamma'^\star_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{4}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + i\tfrac{5}{24}\bar{\psi}'^p\bar{\gamma}_{ABC}\psi'_p - 2i\bar{\psi}'_B\bar{\gamma}_A\psi'_C + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A\,.$$

- **Lagrangian**:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
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- **Maximal supersymmetry transformation rules are also completely covariant,**

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,$$

$$\delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,$$

$$\delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^{q}(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon \mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon \rho = -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,$$

$$\delta_\varepsilon \rho' = -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,$$

$$\delta_\varepsilon \psi_{\bar{p}} = \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,$$

$$\delta_\varepsilon \psi'_p = \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,,$$

where

$$\hat{\Gamma}_{ABC} = \Gamma_{ABC} - i\tfrac{17}{48}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{1}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 3i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,,$$

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- **Lagrangian**:

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
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- The Lagrangian is **pseudo:** It is necessary to impose a **self-duality** of the R-R field strength by hand,

$$\tilde{\mathcal{F}}_{-}:=\left(1-\gamma^{(D+1)}\right)\left(\mathcal{F}-i\tfrac{1}{2}\rho\bar{\rho}'+i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{q}}\right)\equiv 0\,.$$

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$D = 10$ Maximal SDFT [1210.5078]

- Under the $\mathcal{N} = 2$ SUSY transformation rule, the Lagrangian transforms as

$$\delta_{\varepsilon}\mathcal{L}_{\rm Type\,II} = -\tfrac{1}{8}e^{-2d}\,\bar{V}^{A}{}_{\bar{q}}\delta_{\varepsilon}V_{Ap}\mathrm{Tr}\left(\gamma^{p}\tilde{\mathcal{F}}_{-}\bar{\gamma}^{\bar{q}}\overline{\tilde{\mathcal{F}}_{-}}\right) \;+\; \text{total derivatives}\,,$$

where the precise self-duality relation appears quadratically,

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This verifies, to the full order in fermions, **the supersymmetric invariance of the action, modulo the self-duality.**

- For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_{\varepsilon}\tilde{\mathcal{F}}_{-} = -i\left(\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^{p}\tilde{\mathcal{D}}_{p}\psi_{\bar{p}} - \gamma^{p}\mathcal{F}\bar{\gamma}_{\bar{p}}\psi'{}_{p}\right)\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} - i\gamma^{p}\varepsilon\left(\tilde{\mathcal{D}}'_{p}\bar{\rho}' + \tilde{\mathcal{D}}'_{\bar{p}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{p}} - \bar{\psi}_{\bar{p}}\gamma_{p}\mathcal{F}\bar{\gamma}^{\bar{p}}\right)\,.$$

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$$\delta_\varepsilon \mathcal{L}_{\text{Type II}} = -\tfrac{1}{8} e^{-2d}\,\bar{V}^{A}{}_{\bar{q}}\delta_\varepsilon V_{Ap}\mathrm{Tr}\left(\gamma^{p}\tilde{\mathcal{F}}_{-}\bar{\gamma}^{\bar{q}}\overline{\tilde{\mathcal{F}}_{-}}\right) \ + \ \text{total derivatives}\,,$$

where the precise self-duality relation appears quadratically,

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Equations of Motion

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_{p}\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_{p}\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_{p}\rho-i\bar{\psi}^{\bar{p}}\gamma_{p}\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}^{\prime}\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_{p}\rho^{\prime}+2i\bar{\psi}^{\prime}{}_{p}\tilde{\mathcal{D}}_{\bar{q}}\rho^{\prime}-i\bar{\psi}^{\prime q}\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_{p}\psi^{\prime}_{q}=0.$$

  This is **DFT-generalization of Einstein equation.**

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

  Namely, **the on-shell Lagrangian vanishes.**

- R-R potential:

$$\mathcal{D}^{0}_{-}\left(\mathcal{F}-i\rho\bar{\rho}^{\prime}+i\gamma^{r}\psi_{\bar{s}}\bar{\psi}^{\prime}_{r}\bar{\gamma}^{\bar{s}}\right)=0\,,$$

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- Turning off the primed fermions and the R-R sector truncates the $\mathcal{N}=2$ $D=10$ SDFT to $\mathcal{N}=1$ $D=10$ SDFT,

$$\mathcal{L}_{\mathcal{N}=1} = e^{-2d}\Big[\tfrac{1}{8}\left(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD}\right)S_{ACBD} + i\tfrac{1}{2}\bar{\rho}\gamma^{A}\mathcal{D}^{\star}_{A}\rho - i\bar{\psi}^{A}\mathcal{D}^{\star}_{A}\rho - i\tfrac{1}{2}\bar{\psi}^{B}\gamma^{A}\mathcal{D}^{\star}_{A}\psi_{B}\Big]\,.$$

- $\mathcal{N}=1$ **Local SUSY:**

$$\begin{aligned}
\delta_{\varepsilon}d &= -i\tfrac{1}{2}\bar{\varepsilon}\rho\,,\\
\delta_{\varepsilon}V_{Ap} &= -i\bar{\varepsilon}\gamma_{p}\psi_{A}\,,\\
\delta_{\varepsilon}\bar{V}_{A\bar{p}} &= i\bar{\varepsilon}\gamma_{A}\psi_{\bar{p}}\,,\\
\delta_{\varepsilon}\rho &= -\gamma^{A}\hat{\mathcal{D}}_{A}\varepsilon\,,\\
\delta_{\varepsilon}\psi_{\bar{p}} &= \bar{V}^{A}{}_{\bar{p}}\hat{\mathcal{D}}_{A}\varepsilon - i\tfrac{1}{4}(\bar{\rho}\psi_{\bar{p}})\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\rho)\psi_{\bar{p}}\,.
\end{aligned}$$

- Commutator of supersymmetry reads

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] \equiv \hat{\mathcal{L}}_{X_3} + \delta_{\varepsilon_3} + \delta_{\mathbf{so}(1,9)_L} + \delta_{\mathbf{so}(9,1)_R} + \delta_{\rm trivial}\,.$$

where

$$X_3^A = i\bar{\varepsilon}_1\gamma^A\varepsilon_2\,, \qquad \varepsilon_3 = i\tfrac{1}{2}\left[(\bar{\varepsilon}_1\gamma^p\varepsilon_2)\gamma_p\rho + (\bar{\rho}\varepsilon_2)\varepsilon_1 - (\bar{\rho}\varepsilon_1)\varepsilon_2\right]\,, \qquad \text{etc.}$$

and $\delta_{\rm trivial}$ corresponds to the fermionic equations of motion.

Reduction to SUGRA

Now we are going to

- **parametrize the DFT-field-variables in terms of Riemannian variables,**
- **discuss the 'unification' of IIA and IIB,**
- **choose a diagonal gauge of Spin**$(1,D{-}1)_L\times$ **Spin**$(D{-}1,1)_R$**,**
- **and reduce SDFT to SUGRAs.**

Parametrization: Reduction to Generalized Geometry

- As stressed before, one of the characteristic features in our construction of $D = 10$ maximal SDFT is the usage of the $\mathbf{O}(D, D)$ covariant, genuine DFT-field-variables.
- However, the relation to an ordinary supergravity can be established only after we solve the defining algebraic relations of the DFT-vielbeins and parametrize the solution in terms of Riemannian variables, *i.e.* zehnbeins and B-field.
- Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^{\mu} \\ (B + e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\ (B + \bar{e})_{\nu\bar{p}} \end{pmatrix}.$$

Here $e_{\mu}{}^{p}$ and $\bar{e}_{\nu}{}^{\bar{p}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

$$e_{\mu}{}^{p} e_{\nu}{}^{q} \eta_{pq} = -\bar{e}_{\mu}{}^{\bar{p}} \bar{e}_{\nu}{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} = g_{\mu\nu} \,,$$

and further we set $B_{\mu p} = B_{\mu\nu}(e^{-1})_p{}^{\nu}$, $B_{\mu\bar{p}} = B_{\mu\nu}(\bar{e}^{-1})_{\bar{p}}{}^{\nu}$.

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Here $e_\mu{}^p$ and $\bar{e}_\nu{}^{\bar{p}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

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and further we set $B_{\mu p} = B_{\mu\nu}(e^{-1})_p{}^{\nu}$, $B_{\mu\bar{p}} = B_{\mu\nu}(\bar{e}^{-1})_{\bar{p}}{}^{\nu}$.

Parametrization: Reduction to Generalized Geometry

- Instead, we may choose an alternative parametrization,

$$V_A{}^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^p{}_\nu \end{pmatrix} , \qquad \bar{V}_A{}^{\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{\tilde{e}})^{\mu p} \\ (\bar{\tilde{e}}^{-1})^p{}_\nu \end{pmatrix} ,$$

  where $\beta^{\mu p} = \beta^{\mu\nu}(\tilde{e}^{-1})^p{}_\nu$, $\beta^{\mu\bar{p}} = \beta^{\mu\nu}(\bar{\tilde{e}}^{-1})^p{}_\nu$, and $\tilde{e}^\mu{}_p$, $\bar{\tilde{e}}^\mu{}_{\bar{p}}$ correspond to a pair of T-dual vielbeins for winding modes,

$$\tilde{e}^\mu{}_p \tilde{e}^\nu{}_q \eta^{pq} = -\bar{\tilde{e}}^\mu{}_{\bar{p}} \bar{\tilde{e}}^\nu{}_{\bar{q}} \eta^{\bar{p}\bar{q}} = (g - Bg^{-1}B)^{-1\,\mu\nu} .$$

- Note that in the above T-dual winding mode sector, the D-dimensional curved spacetime indices are all upside-down: $\tilde{x}_\mu$, $\tilde{e}^\mu{}_p$, $\bar{\tilde{e}}^\mu{}_{\bar{p}}$, $\beta^{\mu\nu}$ (*cf.* x^μ, $e_\mu{}^p$, $\bar{e}_\mu{}^{\bar{p}}$, $B_{\mu\nu}$).

Parametrization: Reduction to Generalized Geometry

- Two parametrizations:

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^{\mu} \\ (B+e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}$$

versus

$$V_A{}^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta+\tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^p{}_{\nu} \end{pmatrix}, \qquad \bar{V}_A{}^{\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta+\bar{\tilde{e}})^{\mu p} \\ (\bar{\tilde{e}}^{-1})^p{}_{\nu} \end{pmatrix}.$$

- In connection to the section condition, $\partial^A\partial_A \equiv 0$, the former matches well with the choice, $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$, while the latter is natural when $\frac{\partial}{\partial x^\mu} \equiv 0$.

- Yet if we consider dimensional reductions from D to lower dimensions, there is no longer preferred parametrization $\Longrightarrow$ "Non-geometry"

  c.f. Other parametrizations: Lüst, Andriot, Betz, Blumenhagen, Fuchs, Sun *et al.*

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Parametrization: Reduction to Generalized Geometry

- **We re-emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also novel non-Riemannian ("metric-less") backgrounds.**

- For example, the Gomis-Ooguri non-relativistic string theory can be readily realized within DFT on such a non-Riemannian background.
  - The sigma model spectrum matches with the pertubations of DFT around the non-Riemannian background.

Ko-Melby-Thompson-Meyer-JHP 2015

Parametrization: Reduction to Generalized Geometry

- From now on, let us restrict ourselves to the former parametrization and impose $\frac{\partial}{\partial\tilde{x}_{\mu}} \equiv 0$.

- This reduces (S)DFT to 'Generalized Geometry'

  Hitchin; Grana, Minasian, Petrini, Waldram

- For example, the $\mathbf{O}(D,D)$ covariant Dirac operators become

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\begin{array}{l}
\sqrt{2}\gamma^{A}\mathcal{D}_{A}\rho \equiv \gamma^{m}\left(\partial_{m}\rho + \frac{1}{4}\omega_{mnp}\gamma^{np}\rho + \frac{1}{24}H_{mnp}\gamma^{np}\rho - \partial_{m}\phi\rho\right), \\
\sqrt{2}\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}} \equiv \gamma^{m}\left(\partial_{m}\psi_{\bar{p}} + \frac{1}{4}\omega_{mnp}\gamma^{np}\psi_{\bar{p}} + \bar{\omega}_{m\bar{p}\bar{q}}\psi^{\bar{q}} + \frac{1}{24}H_{mnp}\gamma^{np}\psi_{\bar{p}} + \frac{1}{2}H_{m\bar{p}\bar{q}}\psi^{\bar{q}} - \partial_{m}\phi\psi_{\bar{p}}\right), \\
\sqrt{2}\bar{V}^{A}{}_{\bar{p}}\mathcal{D}_{A}\rho \equiv \partial_{\bar{p}}\rho + \frac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\rho + \frac{1}{8}H_{\bar{p}qr}\gamma^{qr}\rho\,, \\
\sqrt{2}\mathcal{D}_{A}\psi^{A} \equiv \partial^{\bar{p}}\psi_{\bar{p}} + \frac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\psi^{\bar{p}} + \bar{\omega}^{\bar{p}}{}_{\bar{p}\bar{q}}\psi^{\bar{q}} + \frac{1}{8}H_{\bar{p}qr}\gamma^{qr}\psi^{\bar{p}} - 2\partial_{\bar{p}}\phi\psi^{\bar{p}}\,.
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- $\omega_{\mu} \pm \frac{1}{2}H_{\mu}$ and $\omega_{\mu} \pm \frac{1}{6}H_{\mu}$ naturally appear as spin connections. Liu, Minasian

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- Since the two zehnbeins correspond to the same spacetime metric, they are related by a Lorentz rotation,

$$(e^{-1}\bar{e})_{p}{}^{\bar{p}}(e^{-1}\bar{e})_{q}{}^{\bar{q}}\,\bar{\eta}_{\bar{p}\bar{q}}=-\eta_{pq}\,.$$

- There exists also a spinorial representation for this local Lorentz rotation, S_e,

$$S_{e}\bar{\gamma}^{\bar{p}}S_{e}^{-1}=\gamma^{(D+1)}\gamma^{p}(e^{-1}\bar{e})_{p}{}^{\bar{p}}\,,$$

such that, in particular,

$$S_{e}\bar{\gamma}^{(D+1)}S_{e}^{-1}=-\det(e^{-1}\bar{e})\gamma^{(D+1)}\,.$$

Unification of type IIA and IIB SUGRAs

- The $D=10$ maximal SDFT 'Riemannian' solutions are then classified into two groups,

$$\det(e^{-1}\bar{e}) = +1 \quad : \quad \text{type IIA}\,,$$

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- This identification with the ordinary IIA/IIB SUGRAs can be established, if we 'fix' the two zehnbeins equal to each other,

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- However, the theory contains two 'types' of Riemannian solutions, as classified above.
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Diagonal gauge fixing and Reduction to SUGRA

- Setting the diagonal gauge,

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with $\eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}}$, $\bar{\gamma}^{\bar{p}} = \gamma^{(D+1)}\gamma^p$, $\bar{\gamma}^{(D+1)} = -\gamma^{(D+1)}$, breaks the local Lorentz symmetry,

$$\mathbf{Spin}(1, D{-}1)_L \times \mathbf{Spin}(D{-}1, 1)_R \quad \Longrightarrow \quad \mathbf{Spin}(1, D{-}1)_D\,.$$

- And it reduces SDFT to SUGRA:

$\mathcal{N} = 2$ $D = 10$ **SDFT** $\Longrightarrow$ $10D$ **Type II democratic SUGRA**

Bergshoeff, *et al.*; Coimbra, Strickland-Constable, Waldram

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- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \textstyle\sum_p' \frac{1}{p!}\, \mathcal{C}_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p}$$

and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \textstyle\sum_p' \frac{1}{(p+1)!}\, \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}}$$

where $\sum_p'$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1 a_2 \cdots a_p} = p\left(D_{[a_1}\mathcal{C}_{a_2 \cdots a_p]} - \partial_{[a_1}\phi\, \mathcal{C}_{a_2 \cdots a_p]}\right) + \frac{p!}{3!(p-3)!}\, H_{[a_1 a_2 a_3}\mathcal{C}_{a_4 \cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to a ‘twisted K-theory’ exterior derivative and its dual, after the diagonal gauge fixing,

$$\mathcal{D}^0_+ \quad \Longrightarrow \quad \mathrm{d} + (H - \mathrm{d}\phi)\wedge$$

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- **In this way,** **ordinary SUGRA $\equiv$ gauge-fixed SDFT,**

$$\textbf{Spin}(1,D{-}1)_{L}\times\textbf{Spin}(D{-}1,1)_{R}\quad\Longrightarrow\quad\textbf{Spin}(1,D{-}1)_{D}\,.$$

Modifying $\mathbf{O}(D,D)$ transformation rule

- The diagonal gauge, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, is incompatible with the vectorial $\mathbf{O}(D,D)$ transformation rule of the DFT-vielbein.

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Modifying $\mathbf{O}(D,D)$ transformation rule

- The $\mathbf{O}(D,D)$ rotation must accompany a compensating $\mathbf{Pin}(D{-}1,1)_R$ local Lorentz rotation, $\bar{L}_{\bar{q}}{}^{\bar{p}}$, $S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}}$ which we can construct explicitly,

$$\bar{L}=\bar{e}^{-1}\left[\mathbf{a}^{t}-(g+B)\mathbf{b}^{t}\right]\left[\mathbf{a}^{t}+(g-B)\mathbf{b}^{t}\right]^{-1}\bar{e}\,,\qquad \bar{\gamma}^{\bar{q}}\bar{L}_{\bar{q}}{}^{\bar{p}}=S_{\bar{L}}^{-1}\bar{\gamma}^{\bar{p}}S_{\bar{L}}\,,$$

where $\mathbf{a}$ and $\mathbf{b}$ are parameters of a given $\mathbf{O}(D,D)$ group element,

$$M_{A}{}^{B}=\begin{pmatrix}\mathbf{a}^{\mu}{}_{\nu} & \mathbf{b}^{\mu\sigma}\\ \mathbf{c}_{\rho\nu} & \mathbf{d}_{\rho}{}^{\sigma}\end{pmatrix}.$$

Modified O(D,D) Transformation Rule After The Diagonal Gauge Fixing

$$
\begin{array}{ccc}
d & \longrightarrow & d \\
V_{A}{}^{p} & \longrightarrow & M_{A}{}^{B}\,V_{B}{}^{p} \\
\bar{V}_{A}{}^{\bar{p}} & \longrightarrow & M_{A}{}^{B}\,\bar{V}_{B}{}^{\bar{q}}\,\bar{L}_{\bar{q}}{}^{\bar{p}} \\
\mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\alpha}{}_{\bar{\alpha}} & \longrightarrow & \mathcal{C}^{\alpha}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\alpha}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}} \\
\rho^{\alpha} & \longrightarrow & \rho^{\alpha} \\
\rho'^{\bar{\alpha}} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\rho'^{\bar{\beta}} \\
\psi^{\alpha}_{\bar{p}} & \longrightarrow & (\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}}\,\psi^{\alpha}_{\bar{q}} \\
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\end{array}
$$

- **All the barred indices are now to be rotated.** Consistent with Hassan
- **The R-R sector can be also mapped to O(D,D) spinors.**

Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach

Flipping the chirality: IIA $\Leftrightarrow$ IIB

- **If and only if $\det(\bar{L})=-1$, the modified $\mathbf{O}(D,D)$ rotation flips the chirality of the theory, since**

$$\bar{\gamma}^{(D+1)}S_{\bar{L}}=\det(\bar{L})\,S_{\bar{L}}\bar{\gamma}^{(D+1)}\,.$$

- Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $\mathbf{O}(D,D)$ T-duality.

- However, since $\bar{L}$ explicitly depends on the parametrization of V_{Ap} and $\bar{V}_{A\bar{p}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $\mathbf{O}(D,D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.

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Twofold spin and Standard Model

1506.05277

- In principle, fermions live on a locally inertial frame.
- Local Lorentz symmetry means the arbitrariness of the locally inertial frame at each spacetime point.
- SDFT manifests twofold local Lorentz symmetries: $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$, and as a consequence it unifies type IIA and IIB supergravities.
- Left and right string modes perceive/live on two different locally inertial frames. Duff
- **SDFT predicts the fermions in Standard Model are twofold: $\mathbf{Spin}(1,3)_L \times \mathbf{Spin}(3,1)_R$.**
  (Even after Scherk-Schwarz compactification, the spin group remains still twofold.)
- Employing the completely covariant DFT-geometry described above, we can couple Standard Model to stringy backgrounds in a covariant way: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degrees.**
- Doing so, one has to decide the spin group for each fermion (Yukawa coupling). No experimental evidence of proton decay seems to indicate that **the quarks and the leptons may belong to different spin groups.**
- If so, this constrains the possible higher order corrections to SM.
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- Employing the completely covariant DFT-geometry described above, we can couple Standard Model to stringy backgrounds in a covariant way: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degrees.**
- Doing so, one has to decide the spin group for each fermion (Yukawa coupling).
  No experimental evidence of proton decay seems to indicate that
  the quarks and the leptons may belong to different spin groups.
- If so, this constrains the possible higher order corrections to SM.
- Further, DFT forbids the theta term and hence solves the strong CP problem.

Outlook : things to do

Revist and Double Field Theorize 20th centrury physics covariantly,

including string theory itself.