# *IIB* Supergravity and the $\mathrm{E}_{6(6)}$ covariant vector-tensor hierarchy

MITP PROGRAM STRING THEORY

24 September 2015

Bernard de Wit

Nikhef Amsterdam

Utrecht University

The question whether the duality invariances of the low-dimensional maximal supergravities are already reflected in the higher-dimensional theories, is an old one.

Thirty years ago it was shown in the case of *11D* dimensional supergravity and its *4D* descendant that one can rewrite the former in a *4D* perspective while retaining all the *11D* degrees of freedom. In that case the higher-dimensional theory indeed shows a pattern that is consistent with $\mathrm{E}_{7(7)}$.

*dW, Nicolai, 1984*

In supergravity and string theory it is relevant to compare theories living in space-times of different dimensions. Hence it is important to know whether solutions can be 'uplifted' and whether truncations can be consistent.

Here I intend to return to the original approach and apply it to *IIB* supergravity, while taking many of the more recent developments into account.

in collaboration with Franz Ciceri and Oscar Varela, *JHEP 1505*

The initial motivation for the present work was to demonstrate that the approach followed for *11D* supergravity can also be applied to other theories. As compared to *IIB* supergravity the *11D* theory is rather simple. Unlike the latter the *IIB* theory is reducible. Besides the gravitini and the graviton, there are four types of bosonic fields, and one matter fermion (*the dilatino*). But even worse, the *IIB* theory posseses two independant supersymmetries (*i.e. N=2*). These two features give rise to many subtleties in the analysis.

From the point of view of *D=5* maximal supergravity, the tensor fields are expected to play a more dominant role. This indicates that the vector-tensor hierarchy must enter at an earlier stage!

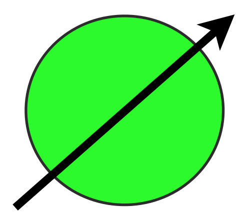
*dW, Samtleben, Trigiante, 2004*
*dW, Samtleben, 2005*
*dW, Nicolai, Samtleben, 2008*

# The embedding tensor formalism

| | rank ⇨ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 7 | SL(5) | $\overline{\mathbf{10}}$ | $\mathbf{5}$ | $\overline{\mathbf{5}}$ | $\mathbf{10}$ | $\mathbf{24}$ | $\overline{\mathbf{15}}+\mathbf{40}$ |
| 6 | SO(5,5) | $\mathbf{16}_c$ | $\mathbf{10}$ | $\overline{\mathbf{16}}_s$ | $\mathbf{45}$ | $\mathbf{144}_s$ | $\mathbf{10}+\mathbf{126}_s+\mathbf{320}$ |
| 5 | $E_{6(+6)}$ | $\overline{\mathbf{27}}$ | $\mathbf{27}$ | $\mathbf{78}$ | $\mathbf{351}$ | $\mathbf{27}+\mathbf{1728}$ | |
| 4 | $E_{7(+7)}$ | $\mathbf{56}$ | $\mathbf{133}$ | $\mathbf{912}$ | $\mathbf{133}+\mathbf{8165}$ | | |
| 3 | $E_{8(+8)}$ | $\mathbf{248}$ | $\mathbf{3875}$ | $\mathbf{3875}+\mathbf{147250}$ | | | |

Implicit connection between space-time electric/magnetic (Hodge) duality and the U-duality group

Probes new states in M-Theory!



Θ dial

**Meanwhile there has been quite a variety of new developments, such as generalized geometry, double field theory, exceptional field theory, vector-tensor hierarchies, and more:**

*Generalized geometry*
*Double field theory*
*Exceptional geometry*
*Exceptional field theory*

*etc.*

*Koepsell, Nicolai, Samtleben, 2000*
*West, 2001*
*Hillmann, 2009*
*Hohm, Hull, Zwiebach, 2010*
*Coimbra, Strickland-Constable, Waldram, 2011*
*Berman, Godazgar, Perry, West, 2011*
*Berman, Cederwall, Kleinschmidt, Waldram, 2012*
*Hohm, Samtleben, 2013*
*Cederwall, Edlund, Karlsson, 2013*
*Aldazaba, Graña, Marqués, Rosabal, 2013*

*etc.*

**As it turns out, all these schemes do have certain common features and relations, although their initial starting points are sometimes rather different.**

Exceptional Field Theory is in some sense the opposite of what I will be presenting. In that case one extends the *D=5* maximal supergravity by introducing 27 extra coordinates transforming according to the fundamental representation of $\mathrm{E}_{6(6)}$. For consistency the space must subsequently be constrained by a covariant section condition that enables one to obtain a conventional supergravity. One theory that one can obtain in this way is *IIB* supergravity.

*Hohm, Samtleben, 2013*
*Samtleben, Musaev, 2014*

We shall also take advantage of many recent advances and extensions of the *11D* supergravity program, when applying the same strategy in the context of *IIB* supergravity!

*dW, Nicolai, 2013*
*Godazgar, Godazgar, Nicolai, 2013, 2014*
*Godazgar, Godazgar, Hohm, Nicolai, Samtleben, 2014*

## *IIB* SUPERGRAVITY

The existence of this theory was inferred from the *IIB* superstring theory. The theory has a non-linearly realized $\mathrm{SL}(2) \cong \mathrm{SU}(1,1)$ symmetry. Its field configuration contains the vielbein, a complex chiral gravitino, a complex anti-chiral fermion (dilatino), a complex scalar, and a number of anti-symmetric tensor gauge fields:

$$E_M{}^A \quad \phi^\alpha \quad A_{MN}{}^\alpha \quad A_{MNPQ} \qquad\qquad \psi_M \quad \lambda$$

*Green, Schwarz, 1982*
*Schwarz, West, 1983*
*Schwarz, 1983*
*Howe, West, 1984*

Upon truncation:
Its compactification on a five-torus leads to ungauged *5D* maximal supergravity with a non-linear realized $\mathrm{E}_{6(6)}$ invariance.

*Cremmer, 1980*

Its compactification on the five-sphere is expected to lead to SO(6) gauged supergravity.

*Günaydin, Romans, Warner, 1986*

**Highly reducible field representation !**

*The Lagrangian description is subtle. It involves a Chern-Simons term and there is a supersymmetric constraint on the five-form field strength:*

$$F_{MNPQR} = 5\,\partial_{[M} A_{NPQR]} - \tfrac{15}{8}\mathrm{i}\varepsilon_{\alpha\beta}\, A^{\alpha}{}_{[MN}\,\partial_P A^{\beta}{}_{QR]}$$

$$\begin{aligned}\tfrac{1}{120}\mathrm{i}\,\varepsilon_{ABCDEFGHIJ}\,F^{FGHIJ} = F_{ABCDE} &- \tfrac{1}{8}\mathrm{i}\,\bar\psi_M \breve\Gamma^{[M}\breve\Gamma_{ABCDE}\breve\Gamma^{N]}\psi_N \\ &+ \tfrac{1}{16}\mathrm{i}\,\bar\lambda\,\breve\Gamma_{ABCDE}\,\lambda\end{aligned}$$

*Bosonic supersymmetry variations*

$$\begin{aligned}
\delta E_M{}^A &= \tfrac{1}{2}(\bar\epsilon\,\breve\Gamma^A\psi_M + \bar\epsilon^{\mathrm{c}}\,\breve\Gamma^A\psi^{\mathrm{c}}_M) \\
\delta\phi^\alpha &= \tfrac{1}{2}\varepsilon^{\alpha\beta}\phi_\beta\,\bar\epsilon^{\mathrm{c}}\lambda \\
\delta A^\alpha{}_{MN} &= -\tfrac{1}{2}\phi^\alpha\big(\bar\lambda\,\breve\Gamma_{MN}\epsilon - 4\,\bar\epsilon\,\breve\Gamma_{[M}\psi_{N]}{}^{\mathrm{c}}\big) + \tfrac{1}{2}\varepsilon^{\alpha\beta}\phi_\beta\,\big(\bar\epsilon\,\breve\Gamma_{MN}\lambda + 4\,\bar\psi_{[M}{}^{\mathrm{c}}\,\breve\Gamma_{N]}\epsilon\big) \\
\delta A_{MNPQ} &= \tfrac{1}{2}\mathrm{i}\bar\epsilon\,\breve\Gamma_{[MNP}\psi_{Q]} + \tfrac{1}{2}\mathrm{i}\bar\psi_{[M}\breve\Gamma_{NPQ]}\epsilon + \tfrac{3}{8}\mathrm{i}\,\varepsilon_{\alpha\beta}A^\alpha{}_{[MN}\,\delta A^\beta{}_{PQ]}
\end{aligned}$$

Note: $\psi_M, \psi_M{}^{\mathrm{c}}, \epsilon, \epsilon^{\mathrm{c}}$ positive chirality spinors

$\lambda, \lambda^{\mathrm{c}}$ negative chirality spinors

# THE *10 = 5 + 5* SPLIT :

Extended tangent space group:

by means of a gauge choice

$$\mathrm{Spin}(9,1)\times\mathrm{U}(1)\longrightarrow\mathrm{Spin}(4,1)\times\mathrm{USp}(4)\times\mathrm{U}(1)$$
$$\longrightarrow\mathrm{Spin}(4,1)\times\mathrm{USp}(8)$$

Fermion decomposition: $\psi_M\oplus\lambda\longrightarrow\psi_\mu\oplus\psi_a\oplus\lambda$

$(\mathbf{4}+\overline{\mathbf{4}})+(\mathbf{20}+\overline{\mathbf{20}}+\mathbf{4}+\overline{\mathbf{4}})$ ***5D spinors***

Identification with a $\mathrm{USp}(8)$ spinor and tri-spinor:

$$\mathbf{8}\overset{\mathrm{SU}(4)\times\mathrm{U}(1)}{\longrightarrow}(\mathbf{4},\tfrac{1}{2})\oplus(\overline{\mathbf{4}},-\tfrac{1}{2})$$
$$\mathbf{48}\overset{\mathrm{SU}(4)\times\mathrm{U}(1)}{\longrightarrow}(\overline{\mathbf{4}},\tfrac{3}{2})\oplus(\mathbf{4},-\tfrac{3}{2})\oplus(\mathbf{20},\tfrac{1}{2})\oplus(\overline{\mathbf{20}},-\tfrac{1}{2})$$

$\mathrm{USp}(8):\ \mathbf{8}+\mathbf{48}$

dilatini $\lambda$ gravitini $\psi_\mu\ \psi_a$

Make use of the standard Kaluza-Klein ansätze:

$$E_M{}^A(x,y) = \begin{pmatrix} \Delta^{-1/2} e_\mu{}^\alpha & B_\mu{}^m\, e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}$$

$$\Delta = \frac{\det[e_m{}^a(x,y)]}{\det[\mathring{e}_m{}^a(y)]}$$

and likewise for the other fields, including the fermion fields.

*Cremmer, Julia, 1979*

In this way the fields transform consistently with respect to the diffeomorphisms of the lower-dimensional space-time. The diffeomorphisms in the internal space are not so systematic. They will be related to a form of exceptional geometry.

*Hohm, Samtleben, 2013*

To realize a local $\mathrm{USp}(8)$ covariance one needs compensating phases!

$$\Phi \in \mathrm{USp}(8)/[\mathrm{USp}(4)\times \mathrm{U}(1)]$$

# Counting vector and tensor fields

$$B_\mu{}^m \oplus A^\alpha{}_{\mu m} \oplus A_{\mu mnp} \qquad \mathbf{5+10+10}$$

$$A^\alpha{}_{\mu\nu} \oplus A_{\mu\nu mn} \qquad \mathbf{2+10}$$

*We expect* **27+27** *vectors and tensors! Some of them are provided by the dual six-form fields:*

(following e.g. *Godazgar, Godazgar, Nicolai, 2013*)

$$A_{\alpha\, MNPQRS} \longrightarrow A_{\alpha\, \mu mnpqr} \oplus A_{\alpha\, \mu\nu mnpq} \oplus \cdots$$

*Hence we obtain* **27** *vector fields and* **22** *tensor fields. The remaining* **5** *tensor fields can be provided by a descendant of the 10D dual graviton.*

*Hull, 2000*
*Curtright, 1985*
*Bekaert, Boulanger, Henneaux, 2001*

$A_{\mu\nu\, m;npqrs}$ *representation consistent with the vector-tensor hierarchy!*

# The dual six-form field

*The field equation for $A^\alpha{}_{MN}$ takes the following form*

$$\partial_{[M} F_{NPQRSTU]\,\alpha} = 0$$

*with*

$$\begin{aligned} F_{\alpha\,MNPQRST} = & -\tfrac{1}{7} E\,\varepsilon_{MNPQRSTUVW}\left(\varepsilon_{\alpha\gamma}\,\phi^\gamma\phi_\beta + \varepsilon_{\beta\gamma}\,\phi^\gamma\phi_\alpha\right)\partial^U A^{VW\,\beta} \\ & - 120\,\varepsilon_{\alpha\beta}\,A_{[MN}{}^\beta\left[\partial_P A_{QRST]} - \tfrac{1}{8}\mathrm{i}\varepsilon_{\gamma\delta}\,A_{PQ}{}^\gamma\,\partial_R A_{ST}{}^\delta\right] \\ & - \tfrac{1}{7}\mathrm{i}\,\varepsilon_{\alpha\beta}\phi^\beta\left[\bar\psi_U\,\breve\Gamma^{[U}\breve\Gamma_{MNPQRST}\,\breve\Gamma^{V]}\psi_V{}^{\mathrm{c}} + \bar\lambda\,\breve\Gamma^U\,\breve\Gamma_{MNPQRST}\psi_U\right] \\ & - \tfrac{1}{7}\mathrm{i}\,\phi_\alpha\left[\bar\psi_U{}^{\mathrm{c}}\,\breve\Gamma^{[U}\breve\Gamma_{MNPQRST}\,\breve\Gamma^{V]}\psi_V - \bar\psi_U\,\breve\Gamma_{MNPQRST}\breve\Gamma^U\,\lambda\right] \end{aligned}$$

*Now apply a supersymmetry transformation,*

$$\delta F_{\alpha\,MNPQRST} = 6\,\partial_{[M}\delta A_{\alpha\,NPQRST} + \cdots$$

*up to equations of motion.*

*In this way we find*

$$\delta A_{\alpha\, MNPQRS} = -\tfrac{1}{6}\mathrm{i}\varepsilon_{\alpha\beta}\phi^{\beta}\big(\bar\lambda\Gamma_{MNPQRS}\epsilon + 2\bar\epsilon\Gamma_{[MNPQR}\psi^{\mathrm{c}}_{S]}\big) + \tfrac{1}{6}\mathrm{i}\phi_{\alpha}\big(\epsilon\Gamma_{MNPQRS}\lambda - 2\psi^{\mathrm{c}}_{[M}\Gamma_{NPQRS]}\epsilon\big) - 20\,\varepsilon_{\alpha\beta}A^{\beta}_{[MN}\,\big(\delta A_{PQRS]} - \tfrac{1}{8}\mathrm{i}\varepsilon_{\gamma\delta}A^{\gamma}_{PQ}\,\delta A^{\delta}_{RS]}\big)$$

*which can be treated in the same manner as the previous vector and tensor fields.*

The fact that the vector fields are complete is an interesting feature of the *IIB* supergravity. Furthermore the tensor fields will play a more major role in this case (as is to be expected)!

## *Determination of the ‘proper’ vector fields:*

Kaluza-Klein decompositions (example):

$$A^{\alpha}{}_{mn}{}^{\mathrm{KK}} = A^{\alpha}{}_{mn}$$

$$A^{\alpha}{}_{\mu m}{}^{\mathrm{KK}} = A^{\alpha}{}_{\mu m} - B_{\mu}{}^{p}\, A^{\alpha}{}_{pm}$$

$$A^{\alpha}{}_{\mu\nu}{}^{\mathrm{KK}} = A^{\alpha}{}_{\mu\nu} + 2\, B_{[\mu}{}^{p}\, A^{\alpha}{}_{\nu]p} + B_{\mu}{}^{p}\, B_{\nu}{}^{q} A^{\alpha}{}_{pq}$$

*Cremmer, Julia, 1979*

Further redefinitions required by the vector-tensor hierarchy:

$$C_{\mu}{}^{m} = B_{\mu}{}^{m}$$

$$C_{\mu}{}^{\alpha}{}_{m} = A^{\alpha}{}_{\mu m}{}^{\mathrm{KK}}$$

$$C_{\mu\, mnp} = A_{\mu mnp}{}^{\mathrm{KK}} - \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta} A^{\alpha}{}_{\mu[m}{}^{\mathrm{KK}}\, A^{\beta}{}_{np]}$$

ESSENTIAL!

*dW, Samtleben, Trigiante, 2004*

## *Supersymmetry variations of some of the vectors*

$$\delta C_\mu{}^m = \tfrac{1}{2}\Delta^{-1/3} e_a{}^m \Big[\mathrm{i}\big(\bar\epsilon\,\Gamma^a\psi_\mu + \bar\epsilon^{\mathrm{c}}\,\Gamma^a\psi_\mu{}^{\mathrm{c}}\big) + \bar\epsilon\,\gamma_\mu(\delta^a{}_b + \tfrac{1}{3}\Gamma^a\Gamma_b)\psi^b + \bar\epsilon^{\mathrm{c}}\,\gamma_\mu(\delta^a{}_b + \tfrac{1}{3}\Gamma^a\Gamma_b)\psi^{b\mathrm{c}}\Big]$$

$$\begin{aligned}\delta C_\mu{}^\alpha{}_m = &-\tfrac{1}{2}\Delta^{-1/3}\phi^\alpha\Big[2\mathrm{i}\,\bar\epsilon\,\Gamma_m\psi_\mu{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_\mu(\delta_m{}^n - \tfrac{1}{3}\Gamma_m\Gamma^n)\psi_n{}^{\mathrm{c}} + \bar\epsilon^{\mathrm{c}}\,\Gamma_m\gamma_\mu\lambda^{\mathrm{c}}\Big]\\ &-\tfrac{1}{2}\Delta^{-1/3}\varepsilon^{\alpha\beta}\phi_\beta\Big[2\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\Gamma_m\psi_\mu - 2\,\bar\epsilon^{\mathrm{c}}\gamma_\mu(\delta_m{}^n - \tfrac{1}{3}\Gamma_m\Gamma^n)\psi_n + \bar\epsilon\,\Gamma_m\gamma_\mu\lambda\Big]\\ &+\tfrac{1}{2}\mathrm{i}\,\Delta^{-1/3}A^\alpha{}_{mp}\Big[\bar\epsilon\,\Gamma^p\psi_\mu + \bar\epsilon^{\mathrm{c}}\,\Gamma^p\psi_\mu{}^{\mathrm{c}}\Big]\\ &+\tfrac{1}{2}\Delta^{-1/3}A^\alpha{}_{mp}\Big[\bar\epsilon\,\gamma_\mu(e_a{}^p + \tfrac{1}{3}\Gamma^p\Gamma_a)\psi^a + \bar\epsilon^{\mathrm{c}}\,\gamma_\mu(e_a{}^p + \tfrac{1}{3}\Gamma^p\Gamma_a)\psi^{a\mathrm{c}}\Big]\end{aligned}$$

where $\Delta = \dfrac{\det[e_m{}^a(x,y)]}{\det[\mathring{e}_m{}^a(y)]}$

Note: spinors will eventually be written as eight-component symplectic Majorana spinors.

## *Determination of the proper vector and tensor fields:*

$$C_{\mu\nu}{}^{\alpha} = A^{\alpha}{}_{\mu\nu}{}^{\mathrm{KK}} - C_{[\mu}{}^{p}\, C_{\nu]}{}^{\alpha}{}_{p}$$

$$C_{\mu\nu\, mn} = A_{\mu\nu mn}{}^{\mathrm{KK}} - \tfrac{1}{16}\mathrm{i}\varepsilon_{\alpha\beta} A^{\alpha}{}_{\mu\nu}{}^{\mathrm{KK}}\, A^{\beta}{}_{mn} - C_{[\mu}{}^{p}\, C_{\nu]pmn}$$

*such that*

$$\delta C_{\mu\nu}{}^{\alpha} + C_{[\mu}{}^{p}\,\delta C^{\alpha}{}_{\nu]p} + C^{\alpha}{}_{[\mu\, p}\,\delta C_{\nu]}{}^{p}$$
$$= -\tfrac{1}{2}\Delta^{-2/3}\phi^{\alpha}\big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \tfrac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}} + \mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big]$$
$$-\tfrac{1}{2}\Delta^{-2/3}\varepsilon^{\alpha\beta}\phi_{\beta}\big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \tfrac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m} + \mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\big]$$

$$\delta C_{\mu\nu\, mn} + C_{[\mu}{}^{p}\,\delta C_{\nu]pmn} + C_{[\mu\, pmn}\,\delta C_{\nu]}{}^{p} + \tfrac{1}{4}\mathrm{i}\varepsilon_{\alpha\beta}\, C_{[\mu}{}^{\alpha}{}_{[m}\,\delta C_{\nu]}{}^{\beta}{}_{n]}$$
$$= \tfrac{1}{4}\Delta^{-2/3}\big[\mathrm{i}\bar\epsilon\,\Gamma_{mn}\gamma_{[\mu}\psi_{\nu]} - \bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[m}(\delta^{p}_{n]} - \tfrac{1}{3}\Gamma_{n]}\Gamma^{p})\psi_{p}\big]$$
$$+ \tfrac{1}{4}\Delta^{-2/3}\big[-\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mn}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma_{[m}(\delta_{n]}{}^{p} - \tfrac{1}{3}\Gamma_{n]}\Gamma^{p})\psi_{p}{}^{\mathrm{c}}\big]$$
$$- \tfrac{1}{16}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\, A^{\alpha}{}_{mn}\,\phi^{\beta}\big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \tfrac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}} + \mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big]$$
$$+ \tfrac{1}{16}\mathrm{i}\,\Delta^{-2/3}\, A^{\alpha}{}_{mn}\,\phi_{\alpha}\big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \tfrac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m} + \mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\big]$$

**Note: agreement with the vector-tensor hierarchy is essential for these results!**

Likewise for the dual vector and tensor field!

## Dual $\mathrm{E}_{6(6)}$ representations for vectors and tensors

$$C_\mu{}^m = C_\mu{}^m$$
$$C_{\mu\, mnp} = \frac{1}{128}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\,C_\mu{}^{qr}$$
$$C_\mu{}^\alpha{}_m = \mathrm{i}\,\varepsilon^{\alpha\beta}\,C_{\mu\,\beta m}$$
$$C_{\mu\,\alpha mnpqr} = -\tfrac{1}{6}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\,C_{\mu\,\alpha}$$

$$C_\mu{}^M$$

$$\overline{\mathbf{27}} \xrightarrow{\mathrm{SL}(2)\times\mathrm{SL}(6)} (\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{6}) \xrightarrow{\mathrm{SL}(2)\times\mathrm{SO}(5)} (\mathbf{1},\mathbf{5}) + (\mathbf{1},\mathbf{10}) + (\mathbf{2},\mathbf{5}) + (\mathbf{2},\mathbf{1})$$

$$C_{\mu\nu\, m;npqrs} \propto \mathring{e}\,\varepsilon_{npqrs}\,C_{\mu\nu\, m}$$
$$C_{\mu\nu\, mn} = C_{\mu\nu\, mn}$$
$$C_{\mu\nu\,\alpha mnpq} = \tfrac{1}{6}\sqrt{5}\mathrm{i}\,\mathring{e}\,\varepsilon_{mnpqr}\,\varepsilon_{\alpha\beta}\,C_{\mu\nu}{}^{\beta r}$$
$$C_{\mu\nu}{}^\alpha = C_{\mu\nu}{}^\alpha$$

← dual graviton

$$C_{\mu\nu\, M}$$

$$\mathbf{27} \xrightarrow{\mathrm{SL}(2)\times\mathrm{SL}(6)} (\mathbf{1},\mathbf{15}) + (\mathbf{2},\overline{\mathbf{6}}) \xrightarrow{\mathrm{SL}(2)\times\mathrm{SO}(5)} (\mathbf{1},\mathbf{5}) + (\mathbf{1},\mathbf{10}) + (\mathbf{2},\mathbf{5}) + (\mathbf{2},\mathbf{1})$$

**DECOMPOSE**

$$\delta C_{\mu\nu\,M} - 2\, d_{MNP}\, C_{[\mu}{}^{N}\, \delta C_{\nu]}{}^{P}$$

$$\delta C_{\mu\nu}{}^{\alpha m} - \tfrac{1}{8}\mathrm{i}\,\varepsilon^{\alpha\beta}\left[C_{[\mu\,\beta n}\,\delta C_{\nu]}{}^{mn} + C_{[\mu}{}^{mn}\,\delta C_{\nu]\,\beta n}\right] - \mathrm{i}\,\varepsilon^{\alpha\beta}\left[C_{[\mu}{}^{m}\,\delta C_{\nu]\beta} + C_{[\mu\beta}\,\delta C_{\nu]}{}^{m}\right]$$

$$\delta C_{\mu\nu}{}^{\alpha} + \mathrm{i}\,\varepsilon^{\alpha\beta}\left[C_{[\mu}{}^{m}\,\delta C_{\nu]\,\beta m} + C_{[\mu\,\beta m}\,\delta C_{\nu]}{}^{m}\right]$$

$$\delta C_{\mu\nu\,mn} + \tfrac{1}{128}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\left[C_{[\mu}{}^{p}\,\delta C_{\nu]}{}^{qr} + C_{[\mu}{}^{qr}\,\delta C_{\nu]}{}^{p}\right] - \tfrac{1}{4}\mathrm{i}\,\varepsilon^{\alpha\beta}\,C_{[\mu\alpha[m}\,\delta C_{\nu]\beta n]}$$

$$\delta C_{\mu\nu\,m} - \mathrm{i}\,\varepsilon^{\alpha\beta}\left[C_{[\mu\,\alpha m}\,\delta C_{\nu]\,\beta} - C_{[\mu\,\alpha}\,\delta C_{\nu]\,\beta m}\right] + \tfrac{1}{256}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\,C_{[\mu}{}^{np}\,\delta C_{\nu]}{}^{qr}$$

$$d_{MNP} \propto \begin{cases} d(_{mn}|^{\alpha p}|^{\beta q}) = \delta_{mn}{}^{pq}\,\varepsilon^{\alpha\beta} \\ d(_{mn}|_{pq}|_{r}) = \mathring{e}\,\varepsilon_{mnpqr} \\ d(_{m}|^{\alpha n}|^{\beta}) = \delta_{m}{}^{n}\,\varepsilon^{\alpha\beta} \end{cases} \neq 0$$

## Comparison to the *5D* transformation rules

$$\delta e_\mu{}^a = \tfrac{1}{2}\bar\epsilon_i\gamma^a\psi_\mu{}^i$$

$$\delta\mathcal{V}_M{}^{ij} = \mathrm{i}\,\mathcal{V}_M{}^{kl}\Big[4\,\Omega_{p[k}\bar\chi_{lmn]}\epsilon^p + 3\,\Omega_{[kl}\bar\chi_{mn]p}\epsilon^p\Big]\,\Omega^{mi}\,\Omega^{nj}$$

$$\delta A_\mu{}^M = 2\,\Big[\mathrm{i}\,\Omega^{ik}\,\bar\epsilon_k\psi_\mu{}^j + \bar\epsilon_k\gamma_\mu\chi^{ijk}\Big]\,\mathcal{V}_{ij}{}^M$$

$$\delta B_{\mu\nu\,M} = \frac{4}{\sqrt{5}}\,\mathcal{V}_M{}^{ij}\Big[2\,\bar\psi_{[\mu\,i}\gamma_{\nu]}\epsilon^k\,\Omega_{jk} - \mathrm{i}\,\bar\chi_{ijk}\gamma_{\mu\nu}\epsilon^k\Big] + 2\,d_{MNP}\,A_{[\mu}{}^N\,\delta A_{\nu]}{}^P$$

*dW, Samtleben, Trigiante, 2004*

*Enables you to read off the generalized vielbeine from the variations proportional to the gravitini.*
*Combining all the information you can also determine the expression for the spinor field $\chi^{ijk}$ in terms of the 10D fields $\psi_a$ and $\lambda$, generalized vielbein postulate, etc.*

Note: In the generalized vielbeine one has to include the local compensating phase factor

$$\Phi \in \mathrm{USp}(8)/[\mathrm{USp}(4)\times\mathrm{U}(1)]$$

## The generalized vielbeine $\mathcal{V}_{ij}{}^{M}$:

$$\mathcal{V}_{ij}{}^{m} = -\tfrac{1}{4}\mathrm{i}\,\Delta^{-1/3}\left(\Phi^{\mathrm{T}}\Omega\,\mathbf{\Gamma}^{m6}\mathbf{\Gamma}_7\,\Phi\right)_{ij}$$

$$\begin{aligned}\mathcal{V}_{ij}{}^{mn} = & -\tfrac{2}{5}\sqrt{5}\mathrm{i}\,\Delta^{2/3}\left(\Phi^{\mathrm{T}}\Omega\,\mathbf{\Gamma}^{mn}\mathbf{\Gamma}_7\,\Phi\right)_{ij} \\ & + \tfrac{2}{5}\sqrt{5}\mathrm{i}\,\mathring{e}^{-1}\varepsilon^{mnpqr}A^{\alpha}{}_{pq}\,\mathcal{V}_{ij\,\alpha r} \\ & + \tfrac{16}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\left[A_{pqrs} - \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{pq}A^{\beta}{}_{rs}\right]\mathcal{V}_{ij}{}^{s}\end{aligned}$$

$$\begin{aligned}\mathcal{V}_{ij\,\alpha m} = & -\tfrac{1}{4}\Delta^{-1/3}\left[(\phi_\alpha - \varepsilon_{\alpha\beta}\phi^{\beta})\left(\Phi^{\mathrm{T}}\Omega\,\mathbf{\Gamma}_m\,\Phi\right)_{ij} + (\phi_\alpha + \varepsilon_{\alpha\beta}\phi^{\beta})\left(\Phi^{\mathrm{T}}\Omega\,\mathbf{\Gamma}_m\mathbf{\Gamma}_7\,\Phi\right)_{ij}\right] \\ & - \varepsilon_{\alpha\beta}A^{\beta}{}_{mn}\,\mathcal{V}_{ij}{}^{n}\end{aligned}$$

$$\begin{aligned}\mathcal{V}_{ij\,\alpha} = & -\tfrac{1}{10}\sqrt{5}\,\Delta^{2/3}\Big[(\phi_\alpha - \varepsilon_{\alpha\beta}\phi^{\beta})\left(\Phi^{\mathrm{T}}\Omega\,\mathbf{\Gamma}_6\,\Phi\right)_{ij} + (\phi_\alpha + \varepsilon_{\alpha\beta}\phi^{\beta})\left(\Phi^{\mathrm{T}}\Omega\,\mathbf{\Gamma}_6\mathbf{\Gamma}_7\,\Phi\right)_{i} \\ & - \tfrac{1}{8}\varepsilon_{\alpha\beta}A^{\beta}{}_{mn}\,\mathcal{V}_{ij}{}^{mn} \\ & - \tfrac{1}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\left[A_{mnpq}\,\mathcal{V}_{ij\alpha r} - 2\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{mn}\,A_{pqrs}\,\mathcal{V}_{ij}{}^{s}\right] \\ & - \tfrac{1}{40}\sqrt{5}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\left[A^{\beta}{}_{mn}\,A^{\gamma}{}_{pq}\mathcal{V}_{ij\,\gamma r} + \tfrac{1}{3}\varepsilon_{\gamma\delta}\,A^{\gamma}{}_{sm}\,A^{\delta}{}_{np}\,A^{\beta}{}_{qr}\,\mathcal{V}_{ij}{}^{s}\right]\end{aligned}$$

Note the presence of the phase $\Phi$.

We found several (equivalent) expressions for the tri-spinor $\chi^{ijk}$. The most elegant and efficient one is

$$\begin{aligned}\chi^{ABC} = &-\tfrac{3}{8}\mathrm{i}\Big[\big(\mathbf{\Gamma}_6\,\bar{\Omega}\big)^{[AB}\,\big(\mathbf{\Gamma}_7\lambda\big)^{C]} + \big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar{\Omega}\big)^{[AB}\,\lambda^{C]}\Big] \\ &-\tfrac{3}{4}\mathrm{i}\,\big(\mathbf{\Gamma}^a\mathbf{\Gamma}_6\mathbf{\Gamma}_7\,\bar{\Omega}\big)^{[AB}\,\psi_a{}^{C]} - \tfrac{1}{4}\mathrm{i}\,\bar{\Omega}^{[AB}\,\big(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\mathbf{\Gamma}^a\psi_a\big)^{C]}\end{aligned}$$

Here we combined the spinors on the right-hand side to eight-component symplectic Majorana spinors. For these extended spinors it was convenient to extend the SO(5) gamma matrices to SO(6) gamma matrices. We still have to include the phase factor $\Phi$, which will convert the indices $A, B, \ldots$ into $i, j, \ldots$.

Symplectic Majorana condition:

$$C^{-1}\bar{\chi}_{ijk}{}^{\mathrm{T}} = \Omega_{il}\,\Omega_{jm}\,\Omega_{kn}\,\chi^{lmn}$$

Likewise one determines the vielbeine $\mathcal{V}_M{}^{ij}$ from the supersymmetry transformations of the tensor fields.

As it turns out the vielbeine $\mathcal{V}_{ij}{}^M$ and $\mathcal{V}_M{}^{ij}$ are both $27 \times 27$ matrices, which are each others inverse (up to a phase) just as in five dimensions!

Under supersymmetry the vielbeine transform in the same way as in the five-dimensional theory, up to a field-dependent infinitesimal USp(8) transformation:

$$\begin{aligned}\Lambda^A{}_B = &-\tfrac{1}{16}\bar\epsilon\,\mathbf{\Gamma}_7[\mathbf{\Gamma}_{ab}\lambda + 4\,\mathbf{\Gamma}_{[a}\psi_{b]}]\left(\mathbf{\Gamma}^{ab6}\right)^A{}_B \\ &+\tfrac{1}{48}\bar\epsilon\,\mathbf{\Gamma}_7[\mathbf{\Gamma}_{abc6}\lambda + 2\,\mathbf{\Gamma}_{abcd6}\psi^d]\left(\mathbf{\Gamma}^{abc}\right)^A{}_B \\ &+\tfrac{1}{4}\bar\epsilon\,\mathbf{\Gamma}_7\mathbf{\Gamma}_{ac}\psi^c\left(\mathbf{\Gamma}^{a6}\right)^A{}_B + \tfrac{1}{4}\bar\epsilon\,\mathbf{\Gamma}_7\mathbf{\Gamma}_{6[a}\psi_{b]}\left(\mathbf{\Gamma}^{ab}\right)^A{}_B\end{aligned}$$

All bosons now transform as in *5D* supergravity.

# Consistent truncation

To establish that the maximal five-dimensional SO(6) gauged supergravity can be viewed as a consistent truncation of *IIB* supergravity compactified on the five-sphere, one can follow the same procedure as before. In this case the Killing spinors must be solutions of

$$\Big( \overset{\circ}{D}_m + \overset{\circ}{e}_m{}^a\, \mathbf{\Gamma}_a \mathbf{\Gamma}_6 \Big) \eta = 0$$

These Killing spinors will capture the $y^m$ dependence of the various fields in such a way that the supersymmetry transformations are consistent. The $x^\mu$ dependence of the generalized vielbeine is captured in terms of the corresponding expressions of the five-dimensional theory.

The $y$-dependence is described by the coset representative of $S^5$. Apart from the Killing spinors, from which one constructs Killing vectors $K_{\hat{a}\hat{b}}{}^m(y)$, one has the vector fields $Y^{\hat{a}}(y)$, subject to $Y^{\hat{a}}(y)\, Y_{\hat{a}}(y) = 1$ .

Then one exploits a number of quadratic contractions between the generalized vielbeine, some of which explicitly contain some of the *IIB* supergravity fields:

$$\bar{\mathcal{V}}^{ik\,m}\,\mathcal{V}_{kj}{}^{n} + \bar{\mathcal{V}}^{ik\,n}\,\mathcal{V}_{kj}{}^{m} = -\tfrac{1}{4}\delta^{i}{}_{j}\,\bar{\mathcal{V}}^{kl\,m}\,\mathcal{V}_{kl}{}^{n}$$

$$\bar{\mathcal{V}}^{kl\,m}\,\mathcal{V}_{kl}{}^{n} \propto \Delta^{-2/3} g^{mn}$$

$$\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,\mathcal{V}_{ij}{}^{m}\,\mathcal{V}_{kl\,\alpha n} = \mathrm{i}\varepsilon_{\alpha\beta}A^{\beta}{}_{np}\,\bar{\mathcal{V}}^{ij\,m}\,\mathcal{V}_{ij}{}^{p}$$

$$\bar{\mathcal{V}}^{ij\,m}\,\mathcal{V}_{ij}{}^{np} = \tfrac{32}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{npqrs}\big[A_{qrst} + \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{qr}A^{\beta}{}_{st}\big]\,\bar{\mathcal{V}}^{ij\,m}\,\mathcal{V}_{ij}{}^{t}$$

$$\varepsilon_{\alpha\gamma}\,\Omega_{ik}\,\Omega_{jl}\,\mathcal{V}^{\gamma\,ij}\,\mathcal{V}^{\beta\,kl} = \tfrac{5}{4}\Delta^{-4/3}\big(\delta_{\alpha}{}^{\beta} - 2\,\phi_{\alpha}\phi^{\beta}\big)$$

Then expand the generalized vielbeine in terms of the $y$-dependent quantities indicated earlier.

The first two identities enable the determination of the internal metric:

$$\Delta^{-2/3}\, g^{mn}(x,y) = 2\,\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat{a}\hat{b}}(x)\,U_{kl}{}^{\hat{c}\hat{d}}(x)\;K^{m}{}_{\hat{a}\hat{b}}(y)\,K^{n}{}_{\hat{c}\hat{d}}(y)$$

5D 27-bein

The next two identities enable the determination of the remaining scalars:

$$\Delta^{-2/3}\left[A_{mnpq} + \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{[mn}A^{\beta}{}_{p]q}\right] = \tfrac{1}{64}\sqrt{5}\,\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat{a}\hat{b}}(x)\,U_{kl}{}^{\hat{c}\hat{d}}(x)\,g_{qr}(x,y) \times \mathring{e}\,\varepsilon_{mnptu}K^{r}{}_{\hat{a}\hat{b}}(y)\,K^{tu}{}_{\hat{c}\hat{d}}(y)$$

$$\Delta^{-2/3}\,A^{\alpha}{}_{mn} = 2\mathrm{i}\,\varepsilon^{\alpha\beta}\,\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat{a}\hat{b}}(x)\,U_{kl\,\beta\hat{c}}(x)\,K^{p}{}_{\hat{a}\hat{b}}(y)\,g_{p[m}(x,y)\,\partial_{n]}Y^{\hat{c}}(y)$$

The last identities determines the dilaton:

$$\Delta^{-4/3}\left(\delta_{\alpha}{}^{\beta} - 2\,\phi_{\alpha}\phi^{\beta}\right) = \tfrac{4}{5}\varepsilon_{\alpha\gamma}\,\Omega_{ik}\,\Omega_{jl}\,U^{\gamma\hat{a}\,ij}(x)\,U^{\beta\hat{b}\,kl}(x)\,Y_{\hat{a}}(y)\,Y_{\hat{b}}(y)$$

Note: for convenience we suppressed the background volume form

These results only reproduce the results that have already been determined by similar methods or on the basis of generalized geometry arguments. They have been partially confirmed by explictit comparison of five- and ten-dimensional supergravity solutions.

*Lee, Strickland-Constable, Waldram, 2014*

*Pilch, Warner, 2000*

A more complete analysis can be given along these lines.

# Conclusion

*The results of this analysis are qualitatively in line with what has been achieved for 11-dimensional supergravity. Apart from many complications of a technical nature, there are interesting new features, such as the role played by the vector-tensor hierarchy.*

*The higher-rank tensor fields do not constitute full representations of* $\mathrm{E}_{6(6)}$*. This is a generic phenomenon that will be come more dominant for increasing rank.*

*See, e.g. West, 2001*

*The results are still incomplete and there are still many open questions. Besides establishing a more complete set of truncation ansätze and verifying their mutual consistency, the relation with Exceptional Field Theory is especially worth pursuing. This especially because the geometry of the internal dimensions has traditionally been ignored.*

*See, however, Gadazgar, Gadazgar, Nicolai, 2014*