

Generalising Calabi–Yau Geometries

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Introduction

Supersymmetric background with no flux

$$\nabla_m \epsilon = 0 \quad \implies \quad \text{special holonomy}$$

Classic case: Type II on Calabi–Yau

Geometry encoded in pair of **integrable** objects

$$d\omega = 0 \quad \text{symplectic structure}$$

$$d\Omega = 0 \quad \text{complex structure}$$

Supersymmetric background with flux (eg type II)?

$$\begin{aligned}(\nabla_m \mp \frac{1}{8} H_{mnp} \gamma^{np}) \varepsilon^\pm + \frac{1}{16} e^\phi \sum_i \not{F}_{(i)} \gamma_m \varepsilon^\mp &= 0 \\ \gamma^m (\nabla_m \mp \frac{1}{24} H_{mnp} \gamma^{np} - \partial_m \phi) \varepsilon^\pm &= 0\end{aligned}$$

What is the geometry?

- special holonomy? analogues of ω and Ω ?
- integrability?
- deformations? moduli spaces? ...

(n.b. no-go means non-compact/bdry for Minkowski)

G structures

Killing spinors ε_i^\pm invariant under

$$G = \text{Stab}(\{\varepsilon_i^\pm\}) \subset \text{SO}(6) \subset \text{GL}(6)$$

define G -structure and flux gives lack of integrability, eg

$$G = \text{SU}(3) \quad \begin{cases} d\omega \simeq \text{flux} & \text{Sp}(6, \mathbb{R}) \text{ structure} \\ d\Omega \simeq \text{flux} & \text{SL}(3, \mathbb{C}) \text{ structure} \end{cases}$$

- classification, new solutions, ...
- global questions: G can change, ..., moduli hard, ...

[Gauntlett, Martelli, Pakis & DW; Gauntlett, Gutowski, Hull, Pakis & Reall; Gauntlett & Pakis; ...,]

Is there some integrable geometry?

supersymmetry \Leftrightarrow *integrable G-structure in generalised geometry*

Generalised Calabi–Yau structures

Generalised tangent space $E \simeq TM \oplus T^*M$ with

$$\text{Stab}(\{\varepsilon_i^\pm\}) = \text{SU}(3) \times \text{SU}(3) \subset \text{SO}(6) \times \text{SO}(6) \subset \text{O}(6,6) \times \mathbb{R}^+$$

gives class of **pure NS-NS** backgrounds

$$G = \text{SU}(3) \times \text{SU}(3) \quad \begin{cases} d\Phi^+ = 0 & \text{SU}(3,3)_+ \text{ structure} \\ d\Phi^- = 0 & \text{SU}(3,3)_- \text{ structure} \end{cases}$$

for generalised **spinor** $\Phi^\pm \in S^\pm(E) \simeq \Lambda^\pm T^*M$

$$\Phi^+ = e^{-\phi} e^{-B-i\omega} \quad \Phi^- = e^{-\phi} e^{-B} (\Omega_1 + \Omega_3 + \Omega_5)$$

[Hitchin, Gualtieri; Graña, Minasian, Petrini and Tomasiello]

Generic $\mathcal{N} = 2$ backgrounds

Warped compactification

$$ds^2 = e^{2\Delta} ds^2(\mathbb{R}^{3,1}) + ds^2(M) \quad \begin{cases} M_6 & \text{type II} \\ M_7 & \text{M-theory} \end{cases}$$

- “exceptional generalised geometry” with $E_{7(7)} \times \mathbb{R}^+$
- spinors in $SU(8)$ vector rep $\epsilon = (\epsilon^+, \epsilon^-)$
- so for $\mathcal{N} = 2$ we have

$$\text{Stab}(\epsilon_1, \epsilon_2) = SU(6) \subset SU(8) \subset E_{7(7)} \times \mathbb{R}^+$$

The problem

Generalised ω and Ω

$$G = \mathrm{SU}(6) \quad \left\{ \begin{array}{l} ??? \\ ??? \end{array} \right. \quad \begin{array}{l} \text{“H structure”} \\ \text{“V structure”} \end{array}$$

- how do we **define structures**?
- what are **integrability conditions**?

[cf. Graña, Louis, Sim & DW; Graña & Orsi; Graña & Triendl]

Generalised geometry

$E_{d(d)} \times \mathbb{R}^+$ generalised geometry ($d \leq 7$)

Unify all symmetries of fields restricted to M_{d-1} in type II

$$\begin{aligned}\delta g &= \mathcal{L}_v g & \delta C_{\pm} &= \mathcal{L}_v C_{\pm} + d\lambda^{\mp} + \dots \\ \delta B &= \mathcal{L}_v B + d\lambda & \delta \tilde{B} &= \mathcal{L}_v \tilde{B} + d\tilde{\lambda} + \dots\end{aligned}$$

gives generalised tangent space

$$\begin{aligned}E &\simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus \Lambda^{\pm} T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \\ V^M &= (v^m, \lambda_m, \tilde{\lambda}_{m_1 \dots m_5}, \lambda^{\pm}, \dots)\end{aligned}$$

Transforms under $E_{d(d)} \times \mathbb{R}^+$ rep with \mathbb{R}^+ weight $(\det T^*M)^{1/(9-d)}$

[Hull; Pacheco & DW]

In M-theory

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)$$
$$V^M = (v^m, \lambda_m, \tilde{\lambda}_{m_1 \dots m_5}, \dots)$$

Generalised Lie derivative

$$L_V = \text{diffeo} + \text{gauge transf} = V \cdot \partial - (\partial \times_{\text{ad}} V)$$

where type IIA, IIB and M-theory distinguished by

$$\partial_M f = (\partial_m f, 0, 0, \dots) \in E^*$$

[Coimbra, S-Constable & DW]

Generalised tensors: $E_{d(d)} \times \mathbb{R}^+$ representations

For example, adjoint includes potentials

$$\begin{aligned} \text{ad } \tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM \\ \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M \oplus \Lambda^\pm TM \oplus \Lambda^\pm T^*M, \end{aligned}$$

$$A^M_N = (\dots, B_{mn}, \dots, \tilde{B}_{m_1 \dots m_6}, \dots, C^\pm)$$

Gives “twisting” of generalised vector and adjoint

$$V = e^{B+\tilde{B}+C^\pm} \tilde{V} \quad R = e^{B+\tilde{B}+C^\pm} \tilde{R} e^{-B-\tilde{B}-C^\pm}$$

Generalised geometry and supergravity

Unified description of supergravity on M

- Generalised metric

G_{MN} invariant under max compact $H_d \subset E_{d(d)} \times \mathbb{R}^+$

equivalent to $\{g, \phi, B, \tilde{B}, C^\pm, \Delta\}$

- Generalised Levi-Civita connection $D_M V^N = \partial_M V^N + \Omega_M{}^N{}_P V^P$

exists gen. torsion-free connection D with $DG = 0$

but not unique

- Analogue of Ricci tensor is unique gives bosonic action

$$S_B = \int_M |\text{vol}_G| R \quad \text{eom} = \text{gen. Ricci flat}$$

where $|\text{vol}_G| = \sqrt{g} e^{2\Delta}$

- Leading-order fermions and supersymmetry

$$\delta\psi = D \Upsilon \epsilon \quad \delta\rho = \not{D}\epsilon \quad \text{etc}$$

unique operators, full theory has local H_d invariance

[CSW] (c.f [Berman & Perry; ...; Aldazabal, Graña, Marqués & Rosabal; ...] and [Siegel; Hohm, Kwak & Zweibach; Jeon, Lee & Park] for $O(d, d)$)

H and V structures

H and V structures

Generalised structures in $E_{7(7)} \times \mathbb{R}^+$

H structure	$G = \text{Spin}^*(12)$	“hypermultiplets”
V structure	$G = E_{6(2)}$	“vector-multiplets”

[*Graña, Louis, Sim & DW*]

Invariant tensor for V structure

Generalised vector in $\mathbf{56}_1$

$$K \in \Gamma(E) \quad \text{such that } q(K) > 0$$

where q is $E_{7(7)}$ quartic invariant, determines second vector \hat{K}

Invariant tensor for H structure

Weighted tensors in $\mathbf{133}_1$

$$J_\alpha(x) \in \Gamma(\text{ad } \tilde{F} \otimes (\det T^*M)^{1/2})$$

forming highest weight \mathfrak{su}_2 algebra

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma$$

$$\text{tr } J_\alpha J_\beta = -\kappa^2\delta_{\alpha\beta}$$

where $\kappa^2 \in \Gamma(\det T^*M)$

Compatible structures and SU(6)

The H and V structures are **compatible** if

$$J_\alpha \cdot K = 0 \qquad \sqrt{q(K)} = \frac{1}{2}\kappa^2$$

analogues of $\omega \wedge \Omega = 0$ and $\frac{1}{6}\omega^3 = \frac{1}{8}i\Omega \wedge \bar{\Omega}$

the compatible pair $\{J_\alpha, K\}$ define an **SU(6) structure**

J_α and K come from **spinor bilinears**.

Example: CY in IIA

$$J_+ = \frac{1}{2}\kappa \Omega - \frac{1}{2}\kappa \Omega^\sharp$$

$$J_3 = \frac{1}{2}\kappa I + \frac{1}{2}\kappa \text{vol}_6 - \frac{1}{2}\kappa \text{vol}_6^\sharp$$

$$\begin{aligned} \text{ad } \tilde{F} &\simeq (TM \otimes T^*M) \\ &\oplus \Lambda^2 T^*M \oplus \Lambda^2 TM \\ &\oplus \mathbb{R} \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M \\ &\oplus \Lambda^- TM \oplus \Lambda^- T^*M, \end{aligned}$$

where $\kappa^2 = \text{vol}_6 = \frac{1}{8}i\Omega \wedge \bar{\Omega}$ and I is complex structure

$$K + i\hat{K} = e^{-i\omega}$$

$$\begin{aligned} E &\simeq TM \oplus T^*M \oplus \Lambda^+ T^*M \\ &\oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \end{aligned}$$

Example: CY in IIB

$$J_+ = \frac{1}{2}\kappa e^{-i\omega} - \frac{1}{2}\kappa e^{-i\omega^\sharp}$$

$$J_3 = \frac{1}{2}\kappa\omega + \frac{1}{2}\kappa\omega^\sharp - \frac{1}{2}\kappa\text{vol}_6 - \frac{1}{2}\kappa\text{vol}_6^\sharp$$

where $\kappa^2 = \text{vol}_6 = \frac{1}{6}\omega^3$

$$K + i\hat{K} = \Omega$$

$$\text{ad } \tilde{F} \simeq (TM \otimes T^*M)$$

$$\oplus \Lambda^2 T^*M \oplus \Lambda^2 TM$$

$$\oplus \mathbb{R} \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M$$

$$\oplus \Lambda^+ TM \oplus \Lambda^+ T^*M,$$

$$E \simeq TM \oplus T^*M \oplus \Lambda^- T^*M$$

$$\oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M)$$

Example: D3-branes in IIB

Smearred branes on $M_{\text{SU}(2)} \times \mathbb{R}^2$

$$ds^2 = e^{2\Delta} ds^2(\mathbb{R}^{3,1}) + d\tilde{s}^2(M_{\text{SU}(2)}) + \zeta_1^2 + \zeta_2^2,$$

with integrability

$$d(e^\Delta \zeta_i) = 0, \quad d(e^{2\Delta} \omega_\alpha) = 0, \quad d\Delta = -\frac{1}{4} \star F,$$

where ω_α triplet of **two-forms** defining **SU(2) structure**. (Can also add anti-self-dual three-form flux.)

The H and V structures are

$$\begin{aligned}\tilde{J}_\alpha &= -\frac{1}{2}\kappa I_\alpha - \frac{1}{2}\kappa \omega_\alpha \wedge \zeta_1 \wedge \zeta_2 \\ &\quad + \frac{1}{2}\kappa \omega_\alpha^\# \wedge \zeta_1^\# \wedge \zeta_2^\#, \end{aligned}$$

$$\begin{aligned}\text{ad } \tilde{F} &\simeq (TM \otimes T^*M) \\ &\quad \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM \\ &\quad \oplus \mathbb{R} \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M \\ &\quad \oplus \Lambda^+ TM \oplus \Lambda^+ T^*M, \end{aligned}$$

where $\kappa^2 = e^{2\Delta} \text{vol}_6$ and I_α are complex structures

$$\begin{aligned}\tilde{K} + i\tilde{K} &= n^i e^\Delta (\zeta_1 - i\zeta_2) \\ &\quad + i n^i e^\Delta (\zeta_1 - i\zeta_2) \wedge \text{vol}_4 \end{aligned}$$

$$\begin{aligned}E &\simeq TM \oplus T^*M \oplus \Lambda^- T^*M \\ &\quad \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \end{aligned}$$

where $n^i = (i, 1)$ is S-duality doublet, then twist by C_4

$$K = e^{C_4} \tilde{K} \quad J_\alpha = e^{C_4} \tilde{J}_\alpha e^{C_4}$$

Generic form?

Complicated but

- **interpolates** between **symplectic**, **complex**, **product** and **hyper-Kähler** structures
- can construct from bilinears and twisting

Integrability

GDiff and moment maps

Symmetries of supergravity give **generalised diffeomorphisms**

$$\text{GDiff} = \text{Diff} \ltimes \text{gauge transf.}$$

acts on the spaces of H and V structures

$$\text{integrability} \Leftrightarrow \text{vanishing moment map}$$

Ubiquitous in supersymmetry equations

- flat connections on Riemann surface (Atiyah–Bott)
- Hermitian Yang–Mills (Donaldson–Uhlenbeck–Yau)
- Hitchin equations, Kähler–Einstein, ...

Space of H structures, \mathcal{A}_H

Consider infinite-dimensional space of structures, $J_\alpha(x)$ give coordinates

\mathcal{A}_H has hyper-Kähler metric

inherited fibrewise since at each $x \in M$

$$J_\alpha(x) \in W = \frac{E_{7(7)} \times \mathbb{R}^+}{\text{Spin}^*(12)}$$

and W is HK cone over homogenous quaternionic-Kähler (Wolf) space (n.b. \mathcal{A}_H itself is HK cone by *global* $\mathbb{H}^+ = \text{SU}(2) \times \mathbb{R}^+$)

Triplet of moment maps

Infinitesimally parametrised by $V \in \Gamma(E) \simeq \mathfrak{g}\text{diff}$ and acts by

$$\delta J_\alpha = L_V J_\alpha \quad \in \Gamma(T\mathcal{A}_H)$$

preserves HK structure giving maps $\mu_\alpha : \mathcal{A}_H \rightarrow \mathfrak{g}\text{diff}^* \otimes \mathbb{R}^3$

$$\mu_\alpha(V) = -\frac{1}{2} \epsilon_{\alpha\beta\gamma} \int_M \text{tr} J_\beta(L_V J_\gamma)$$

functions of coordinates $J_\alpha(x) \in \mathcal{A}_H$

Integrability

$$\text{integrable H structure} \Leftrightarrow \mu_\alpha(V) = 0, \quad \forall V$$

for CY gives $d\omega = 0$ or $d\Omega = 0$

Moduli space

Since structures related by GDifff are equivalent

$$\mathcal{M}_H = \mathcal{A}_H // \text{GDifff} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / \text{GDifff}.$$

moduli space is HK quotient, actually HK cone over QK space of hypermultiplets (as for CY $4h_{1,1} + 4$ or $4h_{2,1} + 4$)

Space of V structures, \mathcal{A}_V

Consider infinite-dimensional space of structures, $K(x)$ give coordinates

\mathcal{A}_H has (affine) special-Kähler metric

(explains \hat{K}) inherited fibrewise since at each $x \in M$

$$K(x) \in P = \frac{E_{7(7)} \times \mathbb{R}^+}{E_{6(2)}}$$

and P is homogenous special-Kähler space

Moment map

Infinitesimally $\mathfrak{g}\text{diff}$ and acts as

$$\delta K = L_V K \quad \in \Gamma(T\mathcal{A}_V)$$

preserves SK structure giving maps $\mu : \mathcal{A}_V \rightarrow \mathfrak{g}\text{diff}^*$

$$\mu(V) = -\frac{1}{2} \int_M \text{tr } s(K, L_V K)$$

where $s(\cdot, \cdot)$ is $E_{7(7)}$ symplectic invariant

Integrability

$$\text{integrable V structure} \Leftrightarrow \mu(V) = 0, \quad \forall V$$

for CY gives $\omega \wedge d\omega = 0$ or $(d\Omega)_{3,1} = 0$ Weak!

Moduli space

Since structures related by GDiff are equivalent

$$\mathcal{M}_H = \mathcal{A}_H // \text{GDiff} = \mu^{-1}(0) / \text{GDiff}.$$

moduli space is **symplectic quotient**, giving SK space

SU(6) structure integrability

Integrable H and V structures **not sufficient**: need **extra condition**

$$\mu_\alpha(V) = \mu(V) = 0 \quad \text{and} \quad L_K J_\alpha = L_{\hat{K}} J_\alpha = 0$$

for CY give $d\omega = d\Omega = 0$

Integrability and generalised intrinsic torsion

- in all cases integrability implies exists **torsion-free, compatible** D
- integrable **SU(6)** is **equivalent to KS equations** [CSW]

Why moment maps?

Reformulate IIA or M-theory as $D = 4$, $\mathcal{N} = 2$ but keep all KK modes

- ∞ -number of hyper- and vector multiplets: \mathcal{A}_H and \mathcal{A}_V
- gauged 4d supergravity with $G = \text{GDiff}$
- integrability is just $\mathcal{N} = 2$ vacuum conditions of [Hristov, Looyestijn, & Vandoren; Louis, Smyth & Triandl]

[Graña, Louis & DW; GLSW]

Other cases

- ▶ In **M-theory**, same H and V structures, generalises notion of $CY \times S^1$
- ▶ For type II or M-theory in $D = 5, 6$ with **eight supercharges**

$$\begin{array}{l} E_{6(6)} \\ E_{5(5)} \simeq \text{Spin}(5, 5) \end{array} \begin{cases} J_\alpha : SU^*(6) \\ K : F_{4(4)} \end{cases} \begin{array}{l} \text{hyper-Kähler} \\ \text{very special real} \end{array}$$

$$\begin{array}{l} E_{5(5)} \simeq \text{Spin}(5, 5) \end{array} \begin{cases} J_\alpha : SU(2) \times \text{Spin}(1, 5) \\ Q : \text{Spin}(4, 5) \end{cases} \begin{array}{l} \text{hyper-Kähler} \\ \text{flat (tensor)} \end{array}$$

- ▶ Extend to **AdS backgrounds**, generalises Sasaki–Einstein

Application : marginal deformations

with Ashmore, Gabella, Graña, Petrini

$\mathcal{N} = 1$ marginal deformations for $\mathcal{N} = 4$ [Leigh & Strassler]

Superpotential deformation

$$W = \epsilon_{ijk} \text{tr} Z^i Z^j Z^k + f_{ijk} \text{tr} Z^i Z^j Z^k$$

- f_{ijk} symmetric giving 10 complex marginal deformations
- but beta-function constrains moment map for SU(3) symmetry

$$f_{ikl} \bar{f}^{jkl} - \frac{1}{3} \delta_i^j f_{klm} \bar{f}^{klm} = 0$$

- only 2 exactly marginal deformation as symplectic quotient

$$\widetilde{\mathcal{M}} = \{f_{ijk}\} // \text{SU}(3)$$

long calculation in supergravity [Aharony, Kol & Yankielowicz]

General analysis [*Green, Komargodski, Seiberg, Tachikawa & Wecht*]

1. “Kähler deformation” dual to bulk vector multiplets
2. “superpotential deformation” dual to bulk hypermultiplets

Field theory analysis

- no Kähler deformations
- every marginal superpotential deform. is exactly marginal unless . . .
- if global symmetry G (other than $U(1)_R$) then

$$\text{exactly marginal} = \text{marginal} // G$$

H and V structures in $E_{6(6)} \times \mathbb{R}^+$ generalised geometry

Bulk is $D = 5$, $\mathcal{N} = 1$ supergravity

$$\left. \begin{array}{l} \text{H structure, } J_\alpha \quad G = \text{SU}^*(6) \\ \text{V structure, } K \quad G = \text{F}_{4(4)} \end{array} \right\} G = \text{USp}(6)$$

with integrability to AdS

$$\begin{aligned} \mu_\alpha(V) &= \lambda_\alpha \int_M c(K, K, V) \\ L_K J_\alpha &= \epsilon_{\alpha\beta\gamma} \lambda_\beta J_\gamma & L_K K &= 0 \end{aligned}$$

where $c(\cdot, \cdot, \cdot)$ is $E_{6(6)}$ cubic invariant

Kähler deformations: $\delta K \neq 0$, $\delta J_\alpha = 0$

No solution to moment map equation . . .

Superpotential deformations: $\delta K = 0$, $\delta J_\alpha \neq 0$

For moment maps **no obstruction** to linearised solution unless **fixed point**

- **fixed point of GDiff** is a gen. Killing vector ie. **global symmetry G**
- **obstruction** is **moment map of G** on linearised problem

Summary

- supersymmetry is special holonomy in generalised geometry
- natural extension of CY geometry

Questions/Extensions

- $\mathcal{N} = 1$ backgrounds ?✓
- deformation theory: underlying DGLA, cohomology, ...?✓
- topological string?: J_α is generalisation of Kähler and Kodaira/Spencer gravity even to M-theory ...
- algebraic geometry?: CFT gives (non-commutative) algebraic description ...