Chern-Simons Supergravity theories with torsion and non-relativistic limit

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Outline

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Introduction

- Teleparallel gravity is an alternative theory of gravity known to be considered equivalent to General Relativity.
- The teleparallel formulation of gravity is described by a non-vanishing torsion $\Longrightarrow$ Riemmanian-Cartan geometry.
- Three-dimensional gravity with torsion possesses black hole solutions (*Teleparallel black hole*) [M Blagojevic and M. Vasilic (2003)]
- The asymptotic symmetry is described by two Virasoro algebras. [M. Blagojevic and M. Vasilic (2003)]

Introduction

- It can be derived as a particular case of the MB gravity model.

$$I_{\text{MB}} = aI_1 + \Lambda I_2 + \beta_3 I_3 + \beta_4 I_4$$

where a, Λ, β_3 and β_4 are constants and

$$\begin{aligned} I_1 &= 2\int e_a R^a \\ I_2 &= -\frac{1}{3}\int \epsilon_{abc} e^a e^b e^c \\ I_3 &= \int \omega^a d\omega_a + \frac{1}{3}\epsilon^{abc}\omega_a\omega_b\omega_c \\ I_4 &= \int e_a T^a \end{aligned}$$

$$R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c \qquad T^a = de^a + \epsilon^{abc}\omega_b e_c$$

Three-dimensional teleparallel gravity can be obtained by fixing the parameters (a, Λ, β_4) appearing in the MB gravity as

$$a = \frac{1}{16\pi G} \qquad \Lambda = -\frac{1}{4\pi G\ell^2} \qquad \beta_4 = -\frac{1}{8\pi G\ell}$$

With this choice, the MB action takes the form

$$I_{\text{TG}} = \frac{1}{16\pi G}\int \tilde{\beta}_3 L(\omega) + \left(2e_a R^a + \frac{4}{3\ell^2}\epsilon_{abc}e^a e^b e^c - \frac{2}{\ell}e_a T^a\right)$$

$\tilde{\beta}_3 \equiv 16\pi G\beta_3$.

Outline

Chern-Simons gravity with torsion

The teleparallel gravity action can alternatively be obtained as a CS gravity action invariant under a deformation of the Poincaré algebra:

Teleparallel algebra

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{abc} J^c \\
[J_a, P_b] &= \epsilon_{abc} P^c \\
[P_a, P_b] &= -\frac{2}{\ell} \epsilon_{abc} P^c
\end{aligned}
$$

$$
L_a \equiv J_a + \frac{\ell}{2} P_a \,, \qquad S_a \equiv -\frac{\ell}{2} P_a
$$

$$
\begin{aligned}
[L_a, L_b] &= \epsilon_{abc} L^c \\
[S_a, S_b] &= \epsilon_{abc} S^c
\end{aligned}
$$

Chern-Simons gravity with torsion

The gauge field connection one-form A for the teleparallel algebra reads

$$A = \omega^a J_a + e^a P_a$$

The corresponding curvature two-form $F = dA + \frac{1}{2}[A, A]$ is given by

$$F = R^a J_a + \hat{T}^a P_a$$

with

$$\begin{aligned} R^a &= d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c \\ \hat{T}^a &= T^a - \frac{1}{\ell}\epsilon^{abc}e_b e_c \end{aligned}$$

where T^a is the usual torsion two-form. The teleparallel algebra admits the following invariant tensor

$$\langle J_a J_b \rangle = \alpha_0\,\eta_{ab} \qquad \langle J_a P_b \rangle = \alpha_1\,\eta_{ab} \qquad \langle P_a P_b \rangle = -\frac{2\alpha_1}{\ell}\,\eta_{ab}$$

Here α_0 and α_1 are arbitrary constants which are related to the $\mathfrak{so}(2,1)$ constants through $\alpha_0 = \mu + \tilde{\mu}$ and $\alpha_1 = -(2\tilde{\mu})/\ell$.

Chern-Simons gravity with torsion

Considering the non-vanishing components of the invariant tensor and the gauge potential one-form in the general definition of the CS action

$$I_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \langle AdA + \frac{2}{3} A^3 \rangle$$

we get

$$I_{\mathrm{TG}} = \frac{1}{16\pi G} \int_{\mathcal{M}} \alpha_0 L(\omega) + \alpha_1 \left(2R_a e^a + \frac{4}{3\ell^2} \epsilon^{abc} e_a e_b e_c - \frac{2}{\ell} T^a e_a \right)$$

The first term is the gravitational CS term with coupling constant α_0. The second term proportional to the constant α_1 contains the usual Einstein-Hilbert Lagrangian, a cosmological constant term and a torsional CS term.

Chern-Simons gravity with torsion

The corresponding equations of motion are given by

$$\delta e^a \quad : \qquad 0 = \alpha_1 \left(R_a - \frac{2}{\ell} \hat{T}_a \right)$$

$$\delta \omega^a \quad : \qquad 0 = \alpha_0 R_a + \alpha_1 \hat{T}_a$$

Since $\alpha_1 \neq 0$ and $\alpha_0 \neq -\frac{\ell}{2}\alpha_1$, the above equations reduce to the vanishing of the curvature two-forms

$$R^a = 0$$

$$T^a - \frac{1}{\ell}\epsilon^{abc} e_b e_c = 0$$

These field equations are geometrically dual to the AdS ones characterized by a Riemannian spacetime. Here, the CS gravity action describes a non-Riemannian geometry with a vanishing curvature and non-vanishing torsion $T^a \neq 0$.

Minimal supersymmetric extension of gravity with torsion

Teleparallel superalgebra

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{abc} J^c \\
[J_a, P_b] &= \epsilon_{abc} P^c \\
[P_a, P_b] &= -\frac{2}{\ell}\epsilon_{abc} P^c \\
[J_a, Q_\alpha] &= -\frac{1}{2}(\gamma_a)_\alpha^{\ \beta} Q_\beta \\
\{Q_\alpha, Q_\beta\} &= -(\gamma^a C)_{\alpha\beta}\left(\frac{2}{\ell} J_a + P_a\right)
\end{aligned}
$$

$\ell \to \infty \implies$ Poincaré superalgebra

[R.Caroca, P.Concha, D.Peñafiel, E.Rodríguez (2021)]

Minimal supersymmetric extension of gravity with torsion

This superalgebra can be written as the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ superalgebra by considering the following identification of the generators

$$L_a \equiv J_a + \frac{\ell}{2} P_a \qquad S_a \equiv -\frac{\ell}{2} P_a \qquad \mathcal{G}_\alpha \equiv \sqrt{\frac{\ell}{2}} Q_\alpha$$

where $\{L_a, \mathcal{G}_\alpha\}$ satisfy the $\mathfrak{osp}(2|1)$ superalgebra, while S_a are $\mathfrak{sp}(2)$ generators,

$$\begin{aligned}
[L_a, L_b] &= \epsilon_{abc} L^c \\
[S_a, S_b] &= \epsilon_{abc} S^c \\
[L_a, \mathcal{G}_\alpha] &= -\frac{1}{2} (\gamma_a)_\alpha^{\ \beta} \mathcal{G}_\alpha \\
\{\mathcal{G}_\alpha, \mathcal{G}_\beta\} &= -(\gamma^a C)_{\alpha\beta} L_a
\end{aligned}$$

Although the teleparallel superalgebra is isomorphic to the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ one, they have noticeable differences at the dynamics and geometric level.

Minimal supersymmetric extension of gravity with torsion

The gauge field connection one-form for the teleparallel superalgebra reads

$$A = \omega^a J_a + e^a P_a + \bar{\psi}\, Q$$

The corresponding curvature two-form is

$$F = \mathcal{R}^a J_a + \mathcal{T}^a P_a + \nabla\bar{\psi}\, Q$$

where

$$\begin{aligned} \mathcal{R}^a &= R^a + \frac{1}{\ell}\bar{\psi}\gamma^a\psi \\ \mathcal{T}^a &= \hat{T}^a + \frac{1}{2}\bar{\psi}\gamma^a\psi \\ \nabla\psi &= d\psi + \frac{1}{2}\omega^a\gamma_a\psi \end{aligned}$$

In particular, $\nabla\psi$ defines the covariant derivative of the gravitino.

Minimal supersymmetric extension of gravity with torsion

The teleparallel superalgebra admits the following invariant tensor,

$$
\begin{aligned}
\langle J_a J_b \rangle &= \alpha_0 \, \eta_{ab} \\
\langle J_a P_b \rangle &= \alpha_1 \, \eta_{ab} \\
\langle P_a P_b \rangle &= -\frac{2\alpha_1}{\ell} \, \eta_{ab} \\
\langle Q_\alpha, Q_\beta \rangle &= 2 \left(\frac{2\alpha_0}{\ell} + \alpha_1 \right) C_{\alpha\beta}
\end{aligned}
$$

where α_0 and α_1 are arbitrary constants. Then, the action reads

$$
\begin{aligned}
I_{\mathrm{TSG}} = \frac{1}{16\pi G} \int_{\mathcal{M}} & \alpha_0 \left(\omega^a d\omega_a + \frac{1}{3} \epsilon^{abc} \omega_a \omega_b \omega_c - \frac{4}{\ell} \bar{\psi} \nabla \psi \right) \\
& + \alpha_1 \left(2 R_a e^a + \frac{4}{3\ell^2} \epsilon^{abc} e_a e_b e_c - \frac{2}{\ell} T^a e_a - 2 \bar{\psi} \nabla \psi \right)
\end{aligned}
$$

Minimal supersymmetric extension of gravity with torsion

The corresponding field equations read

$$\begin{aligned} \delta e^a &: \quad 0 = \alpha_1 \left(\mathcal{R}_a - \frac{2}{\ell} \mathcal{T}_a \right) \\ \delta \omega^a &: \quad 0 = \alpha_0 \mathcal{R}_a + \alpha_1 \mathcal{T}_a \\ \delta \bar{\psi} &: \quad 0 = \frac{2\alpha_0}{\ell} \nabla \psi + \alpha_1 \nabla \psi \end{aligned}$$

The non-degeneracy of the invariant tensor requires $\alpha_1 \neq 0$ and $\alpha_0 \neq -\frac{\ell}{2}\alpha_1$ which implies that the equations of motion are given by the vanishing of the curvature two-forms. In particular, this theory corresponds to a supersymmetric extension of the teleparallel gravity being characterized by a non-vanishing super-torsion

$$T^a + \frac{1}{2}\bar{\psi}\gamma^a\psi = \frac{1}{\ell}\epsilon^{abc}e_b e_c$$

Further developments

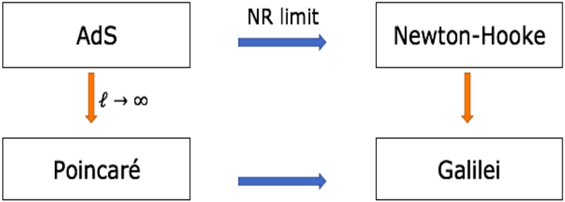
The results presented here could serve as a starting point for diverse further studies:

- To study appropriate boundary conditions to our teleparallel supergravity theory and analyze its boundary dynamics. One could expect to obtain a superconformal structure.
- To explore the non-relativistic counterpart of the teleparallel (super)gravity theory. To study the role of torsion in a non-relativistic environment.
- To explore a Maxwellian version of the teleparallel gravity.

Outline

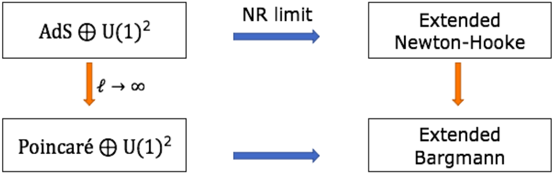
Non-relativistic gravity with torsion

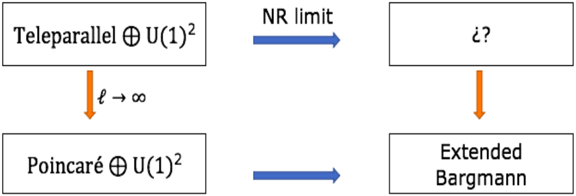
- There has been a renewed interest in non-relativistic (NR) theories due to their utilities to approach strongly coupled condensed matter systems as well as NR effective field theories.
- A NR theory can be obtained by a suitable **limiting process** from a relativistic theory.
- In particular, through this talk, the NR limit corresponds to the limit in which $c \to \infty$.



Non-relativistic gravity with torsion

- In the limit $c \to \infty$ there might appear infinities in the expansion of the original Lagrangian $\Longrightarrow$ enlargement of the field content





U(1)-enlargements

A torsional NR algebra can be obtained by applying the NR limit to a particular $U(1)$-enlargement of the teleparallel algebra. To this end, we introduced two extra $U(1)$ gauge fields to the field content

$$A = W^A J_A + E^A P_A + MY_1 + SY_2$$

The extra generators yield the following non-vanishing invariant tensors

$$\langle Y_2 Y_2 \rangle = \alpha_0 \qquad \langle Y_1 Y_2 \rangle = \alpha_1 \qquad \langle Y_1 Y_1 \rangle = -\frac{2\alpha_1}{\ell}$$

The relativistic CS action based on the [teleparallel]$\oplus\mathfrak{u}(1)^2$ algebra is

$$\begin{aligned} I_{\mathrm{R}} &= \frac{1}{16\pi G}\int_{\mathcal{M}} \alpha_0 \left(W^A dW_A + \frac{1}{3}\epsilon^{ABC} W_A W_B W_C + SdS \right) \\ &+ \alpha_1 \left(2E_A R^A(W) + \frac{4}{3\ell^2}\epsilon^{ABC} E_A E_B E_C - \frac{2}{\ell} T^A E_A + 2MdS - \frac{2}{\ell} MdM \right) \end{aligned}$$

Torsional NR gravity theory

The NR counterpart of the relativistic [teleparallel]$\oplus\mathfrak{u}(1)^2$ algebra can be derived through an Inönü-Wigner procedure . To this end, we shall first consider the decomposition of the A index as follows:

$$A \to (0, a) \qquad a = 1, 2$$

Then, we will introduce a dimensionless parameter ξ which will allows us to identify the relativistic generators in terms of the NR ones (denoted with a tilde) as:

$$\begin{aligned} J_0 &= \frac{\tilde{J}}{2} + \xi^2 \tilde{S} \qquad J_a = \xi \tilde{G}_a \qquad Y_2 = \frac{\tilde{J}}{2} - \xi^2 \tilde{S} \\ P_0 &= \frac{\tilde{H}}{2\xi} + \xi \tilde{M} \quad P_a = \tilde{P}_a \qquad Y_1 = \frac{\tilde{H}}{2\xi} - \xi \tilde{M} \end{aligned}$$

Furthermore, in order to ensure a finite NR symmetry after the contraction process we shall consider the following scaling of the length parameter,

$$\ell \to \xi \ell$$

Then, considering the previous identification and the limit $\xi \to \infty$ in the relativistic commutation relations, we obtain a novel NR symmetry spanned by the set of generators $\{\tilde{J}, \tilde{G}_a, \tilde{H}, \tilde{P}_a, \tilde{S}, \tilde{M}\}$ which satisfy the following commutators: [P. Concha, L. Ravera, E. Rodríguez (2021)]

$$
\begin{aligned}
\left[\tilde{J}, \tilde{G}_a\right] &= \epsilon_{ab}\tilde{G}_b & \left[\tilde{G}_a, \tilde{G}_b\right] &= -\epsilon_{ab}\tilde{S} & \left[\tilde{H}, \tilde{G}_a\right] &= \epsilon_{ab}\tilde{P}_b \\
\left[\tilde{J}, \tilde{P}_a\right] &= \epsilon_{ab}\tilde{P}_b & \left[\tilde{G}_a, \tilde{P}_b\right] &= -\epsilon_{ab}\tilde{M} & \left[\tilde{H}, \tilde{P}_a\right] &= -\frac{2}{\ell}\epsilon_{ab}\tilde{P}_b \\
\left[\tilde{P}_a, \tilde{P}_b\right] &= \frac{2}{\ell}\epsilon_{ab}\tilde{M}
\end{aligned}
$$

$\ell \to \infty \implies$ Extended Bargmann algebra

It can be written as two copies of the Nappi-Witten algebra

$$
\begin{aligned}
\left[\tilde{J}^{\pm}, \tilde{G}_a^{\pm}\right] &= \epsilon_{ab}\tilde{G}_b^{\pm} \\
\left[\tilde{G}_a^{\pm}, \tilde{G}_b^{\pm}\right] &= -\epsilon_{ab}\tilde{S}^{\pm}
\end{aligned}
$$

Torsional NR gravity theory

The torsional NR algebra admits a non-degenerate invariant bilinear trace

$$
\begin{aligned}
\left\langle \tilde{J}\tilde{S} \right\rangle &= -\tilde{\alpha}_0 \\
\left\langle \tilde{G}_a\tilde{G}_b \right\rangle &= \tilde{\alpha}_0\delta_{ab} \\
\left\langle \tilde{G}_a\tilde{P}_b \right\rangle &= \tilde{\alpha}_1\delta_{ab} \\
\left\langle \tilde{H}\tilde{S} \right\rangle &= \left\langle \tilde{M}\tilde{J} \right\rangle = -\tilde{\alpha}_1 \\
\left\langle \tilde{P}_a\tilde{P}_b \right\rangle &= -\frac{2\tilde{\alpha}_1}{\ell}\delta_{ab} \\
\left\langle \tilde{H}\tilde{M} \right\rangle &= \frac{2\tilde{\alpha}_1}{\ell}
\end{aligned}
$$

where the relativistic parameters α's have been rescaled as

$$\alpha_0 = \tilde{\alpha}_0\xi^2 \qquad \alpha_1 = \tilde{\alpha}_1\xi$$

Torsional NR gravity theory

The gauge connection one-form for the torsional NR algebra reads

$$\tilde{A} = \omega \tilde{J} + \omega^a \tilde{G}_a + \tau \tilde{H} + e^a \tilde{P}_a + m\tilde{M} + s\tilde{S}$$

The curvature associated to this gauge connection is given by

$$\tilde{F} \;=\; R(\omega)\tilde{J} + R^a(\omega^b)\tilde{G}_a + R(\tau)\tilde{H} + R^a(e^b)\tilde{P}_a + R(m)\tilde{M} + R(s)\tilde{S}$$

where

$$\begin{aligned}
R(\omega) &= d\omega \\
R^a(\omega^b) &= d\omega^a + \epsilon^{ac}\omega\omega_c \\
R(\tau) &= d\tau \\
R^a(e^b) &= de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c - \frac{2}{\ell}\epsilon^{ac}\tau e_c \\
R(m) &= dm + \epsilon^{ac}e_a\omega_c - \frac{1}{\ell}\epsilon^{ac}e_a e_c \\
R(s) &= ds + \frac{1}{2}\epsilon^{ac}\omega_a\omega_c
\end{aligned}$$

Torsional NR gravity theory

The CS action based on the torsional NR algebra is:

$$
\begin{aligned}
I_{NR} \;=\; & \int \tilde{\alpha}_0 \left[\omega_a R^a(\omega^b) - 2sR\left(\omega\right)\right] + \tilde{\alpha}_1 \left[2e_a R^a(\omega^b) - 2mR(\omega) - 2\tau R(s)\right. \\
& \left. - \frac{2}{\ell} e_a R^a\left(e^b\right) + \frac{2}{\ell} mR\left(\tau\right) + \frac{2}{\ell}\tau R\left(m\right) + \frac{2}{\ell^2}\tau\epsilon^{ac} e_a e_c\right]
\end{aligned}
$$

In this torsional NR gravity theory, the cosmological constant can be seen as a source for the spatial torsion $T^a\left(e^b\right) = de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c$ and for the curvature $T\left(m\right) = dm + \epsilon^{ac} e_a \omega_c$. Indeed, on-shell we find

$$
\begin{aligned}
T^a\left(e^b\right) &= \frac{2}{\ell}\epsilon^{ac}\tau e_c \\
T\left(m\right) &= \frac{1}{\ell}\epsilon^{ac} e_a e_c
\end{aligned}
$$

$\ell \to \infty \implies$ Extended Bargmann gravity

Torsional NR gravity theory

The NR gravity action can be alternatively recovered from the relativistic $U(1)$-enlarged teleparallel CS action. Indeed, one can express the relativistic gauge fields in terms of the NR ones as follows

$$
\begin{aligned}
W^0 &= \omega + \frac{s}{2\xi^2} \qquad W^a = \frac{\omega^a}{\xi} \qquad S = \omega - \frac{s}{2\xi^2}\,, \\
E^0 &= \xi\tau + \frac{m}{2\xi} \qquad E^a = e^a \qquad M = \xi\tau - \frac{m}{2\xi}
\end{aligned}
$$

which satisfy $A = \tilde{A}$. The NR CS action appears considering these last expressions along the rescaling of the relativistic parameters α_0 and α_1 on the relativistic CS action and then applying the limit $\xi \to \infty$.

Non-relativistic CS Supergravity with torsion

The formulation of NR supergravity theories has only been approached recently and remains as a challenging task mainly motivated by the diverse applications of these models in the context of holography and relativistic field theory. One way to circumvent the difficulty to establish a well-defined NR limit in presence of supersymmetry is through the Lie algebra expansion method which allows to recover the respective NR version of a Lie (super)algebra.

- A torsional NR superalgebra can be obtained by applying the S-expansion method to the $\mathcal{N}=2$ teleparallel superalgebra.

- The S-expansion will also provide us with a non-degenerate invariant tensor and therefore be able to produce a well-defined torsional NR CS supergravity action which can be seen as the corresponding NR counterpart of the $\mathcal{N}=2$ teleparallel supergravity action.

Torsional non-relativistic superalgebra

- The $\mathcal{N}=2$ teleparallel superalgebra is spanned by the generators $(J_A, P_A, Z, Q_\alpha^i)$.
- In order to have a well-defined flat limit $\ell \to \infty$, we performed the redefinition $\mathcal{T} = Z - \frac{\ell}{2}\mathcal{S}$.

$$
\begin{aligned}
\left[J_A, Q_\alpha^i\right] &= -\frac{1}{2}\left(\gamma_A\right)_\alpha^{\ \beta} Q_\beta^i \\
\left[\mathcal{T}, Q_\alpha^i\right] &= -\epsilon^{ij} Q_\alpha^j \\
\left\{Q_\alpha^i, Q_\beta^j\right\} &= -\delta^{ij}\left(\gamma^A C\right)_{\alpha\beta}\left(P_A + \frac{2}{\ell}J_A\right) + C_{\alpha\beta}\epsilon^{ij}\left(\frac{2}{\ell}\mathcal{T} + \mathcal{S}\right)
\end{aligned}
$$

- The $\mathcal{N}=2$ teleparallel superalgebra is endowed with the following non-vanishing components of the (non-degenerate) invariant tensor:

$$
\begin{aligned}
\langle \mathcal{T}\mathcal{T} \rangle &= -2\alpha_0 \qquad \langle \mathcal{T}\mathcal{S} \rangle = -2\alpha_1 \\
\langle \mathcal{S}\mathcal{S} \rangle &= \frac{4\alpha_1}{\ell} \qquad \langle Q_\alpha^i Q_\beta^j \rangle = 2\left(\frac{2\alpha_0}{\ell} + \alpha_1\right) C_{\alpha\beta}\delta^{ij}
\end{aligned}
$$

Torsional non-relativistic superalgebra

We first proceed as before considering the decomposition of the A index as $A \rightarrow (0, a)$. Besides, we perform a redefinition of the supercharges by defining

$$Q_{\alpha}^{\pm} = \frac{1}{\sqrt{2}} \left(Q_{\alpha}^{1} \pm \left(\gamma^{0} \right)_{\alpha\beta} Q_{\beta}^{2} \right)$$

Then, we consider $S_{E}^{(2)} = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3 \}$ as the relevant semigroup

λ_3	λ_3	λ_3	λ_3	λ_3
λ_2	λ_2	λ_3	λ_3	λ_3
λ_1	λ_1	λ_2	λ_3	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
	λ_0	λ_1	λ_2	λ_3

where $\lambda_3 = 0_s$ is the zero element of the semigroup such that $0_s \lambda_k = 0_s$, $k = 0, 1, 2, 3$.

Torsional NR superalgebra

Before applying the S-expansion procedure, we consider the following subspace decomposition: $V_0 = \{J_0, P_0, \mathcal{T}, \mathcal{S}, Q_\alpha^+\}$ and $V_1 = \{J_a, P_a, Q_\alpha^-\}$, which satisfies

$$[V_0, V_0] \subset V_0 \qquad [V_0, V_1] \subset V_1 \qquad [V_1, V_1] \subset V_0$$

Let us consider now $S_E^{(2)} = S_0 \cup S_1$ as decomposition of the relevant semigroup $S_E^{(2)}$, where

$$\begin{aligned} S_0 &= \{\lambda_0, \lambda_2, \lambda_3\} \\ S_1 &= \{\lambda_1, \lambda_3\} \end{aligned}$$

This decomposition is said to be resonant since it satisfies the same structure as the subspaces, that is

$$S_0 \cdot S_0 \subset S_0 \qquad S_0 \cdot S_1 \subset S_1 \qquad S_1 \cdot S_1 \subset S_0$$

Torsional NR superalgebra

After extracting a resonant subalgebra of the $S_E^{(2)}$-expansion of the $\mathcal{N}=2$ teleparallel superalgebra and applying a 0_s-reduction, one ends up with a new NR expanded superalgebra spanned by the set of generators

$$\{\tilde{J}, \tilde{G}_a, \tilde{S}, \tilde{H}, \tilde{P}_a, \tilde{M}, \tilde{T}_1, \tilde{T}_2, \tilde{U}_1, \tilde{U}_2, \tilde{Q}^+_\alpha, \tilde{R}_\alpha, \tilde{Q}^-_\alpha\}$$

which are related to the relativistic ones through the semigroup elements as

λ_3		
λ_2	$\tilde{S},\ \tilde{M},\ \tilde{T}_2,\ \tilde{U}_2,\ \tilde{R}_\alpha$	
λ_1		$\tilde{G}_a,\ \tilde{P}_a,\ \tilde{Q}^-_\alpha$
λ_0	$\tilde{J},\ \tilde{H},\ \tilde{T}_1,\ \tilde{U}_1,\ \tilde{Q}^+_\alpha$	
	$J_0,\ P_0,\ \mathcal{T},\ \mathcal{S},\ Q^+_\alpha$	$J_a,\ P_a,\ Q^-_\alpha$

Torsional NR superalgebra

The NR generators satisfy precisely the purely bosonic subalgebra along with the following (anti-)commutation relations:

$$\left[\tilde{J}, \tilde{Q}^{\pm}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_0\right)_{\alpha}^{\ \beta}\tilde{Q}^{\pm}_{\beta} \qquad \left[\tilde{J}, \tilde{R}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_0\right)_{\alpha}^{\ \beta}\tilde{R}_{\beta} \qquad \left[\tilde{S}, \tilde{Q}^{+}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_0\right)_{\alpha}^{\ \beta}\tilde{R}_{\beta}$$

$$\left[\tilde{G}_a, \tilde{Q}^{+}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_a\right)_{\alpha}^{\ \beta}\tilde{Q}^{-}_{\beta} \qquad \left[\tilde{G}_a, \tilde{Q}^{-}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_a\right)_{\alpha}^{\ \beta}\tilde{R}_{\beta}$$

$$\left[\tilde{T}_1, \tilde{Q}^{\pm}_{\alpha}\right] = \pm\left(\gamma^0\right)_{\alpha\beta}\tilde{Q}^{\pm}_{\beta}\,, \qquad \left[\tilde{T}_2, \tilde{Q}^{+}_{\alpha}\right] = \left(\gamma^0\right)_{\alpha\beta}\tilde{R}_{\beta} \qquad \left[\tilde{T}_1, \tilde{R}_{\alpha}\right] = \left(\gamma^0\right)_{\alpha\beta}\tilde{R}_{\beta}$$

$$\left\{\tilde{Q}^{+}_{\alpha}, \tilde{Q}^{+}_{\beta}\right\} = -\left(\gamma^0 C\right)_{\alpha\beta}\left(\tilde{H} + \frac{2}{\ell}\tilde{J}\right) - \left(\gamma^0 C\right)_{\alpha\beta}\left(\frac{2}{\ell}\tilde{T}_1 + \tilde{U}_1\right)\,,$$

$$\left\{\tilde{Q}^{-}_{\alpha}, \tilde{Q}^{-}_{\beta}\right\} = -\left(\gamma^0 C\right)_{\alpha\beta}\left(\tilde{M} + \frac{2}{\ell}\tilde{S}\right) + \left(\gamma^0 C\right)_{\alpha\beta}\left(\frac{2}{\ell}\tilde{T}_2 + \tilde{U}_2\right)$$

$$\left\{\tilde{Q}^{+}_{\alpha}, \tilde{R}_{\beta}\right\} = -\left(\gamma^0 C\right)_{\alpha\beta}\left(\tilde{M} + \frac{2}{\ell}\tilde{S}\right) - \left(\gamma^0 C\right)_{\alpha\beta}\left(\frac{2}{\ell}\tilde{T}_2 + \tilde{U}_2\right)$$

$$\left\{\tilde{Q}^{+}_{\alpha}, \tilde{Q}^{-}_{\beta}\right\} = -\left(\gamma^a C\right)_{\alpha\beta}\left(\tilde{P}_a + \frac{2}{\ell}\tilde{G}_a\right)$$

$\ell \to \infty \implies$ Extended Bargmann superalgebra

Torsional NR supergravity action

The gauge connection one-form for the torsional NR superalgebra reads

$$\begin{aligned}\tilde{A} &= \omega\tilde{J} + \omega^{a}\tilde{G}_{a} + \tau\tilde{H} + e^{a}\tilde{P}_{a} + m\tilde{M} + s\tilde{S} + t_{1}\tilde{T}_{1} + t_{2}\tilde{T}_{2} + u_{1}\tilde{U}_{1} \\ &\quad + u_{2}\tilde{U}_{2} + \bar{\psi}^{+}\tilde{Q}^{+} + \bar{\psi}^{-}\tilde{Q}^{-} + \bar{\rho}\tilde{R}\end{aligned}$$

The non-vanishing components of the invariant tensor for the NR teleparallel superalgebra are

$$\begin{aligned}\langle\tilde{T}_{1}\tilde{T}_{2}\rangle &= -2\tilde{\alpha}_{0} \\ \langle\tilde{T}_{1}\tilde{U}_{2}\rangle &= \langle\tilde{T}_{2}\tilde{U}_{1}\rangle = -2\tilde{\alpha}_{1} \\ \langle\tilde{U}_{1}\tilde{U}_{2}\rangle &= \frac{4\tilde{\alpha}_{1}}{\ell} \\ \langle Q_{\alpha}^{+}R_{\beta}\rangle &= 2\left(\frac{2\tilde{\alpha}_{0}}{\ell} + \tilde{\alpha}_{1}\right)C_{\alpha\beta} \\ \langle Q_{\alpha}^{-}Q_{\beta}^{-}\rangle &= 2\left(\frac{2\tilde{\alpha}_{0}}{\ell} + \tilde{\alpha}_{1}\right)C_{\alpha\beta}\end{aligned}$$

Torsional NR supergravity action

The NR supergravity action is

$$
\begin{aligned}
I_{NR}^{\text{super}} \quad = \quad & \frac{k}{4\pi}\int \tilde{\alpha}_0 \left[\omega_a R^a(\omega^b) - 2sR(\omega) - 4t_1 dt_2 - \frac{4}{\ell}\bar{\psi}^+\nabla\rho - \frac{4}{\ell}\bar{\rho}\nabla\psi^+ - \frac{4}{\ell}\bar{\psi}^-\nabla\psi^-\right] \\
& + \tilde{\alpha}_1 \left[2e_a R^a(\omega^b) - 2mR(\omega) - 2\tau R(s) - \frac{2}{\ell}e_a R^a\left(e^b\right) + \frac{2}{\ell}mR(\tau) + \frac{2}{\ell}\tau R(m)\right. \\
& \left. + \frac{2}{\ell^2}\tau\epsilon^{ac}e_a e_c - 4t_1 du_2 - 4t_2 du_1 + \frac{8}{\ell}u_1 du_2 - 2\bar{\psi}^+\nabla\rho - 2\bar{\rho}\nabla\psi^+ - 2\bar{\psi}^-\nabla\psi^-\right]
\end{aligned}
$$

where the fermionic field-strenghts are

$$
\begin{aligned}
\nabla\psi^+ &= d\psi^+ + \frac{1}{2}\omega\gamma_0\psi^+ + t_1\gamma_0\psi^+ \\
\nabla\psi^- &= d\psi^- + \frac{1}{2}\omega\gamma_0\psi^- + \frac{1}{2}\omega^a\gamma_a\psi^+ - t_1\gamma_0\psi^- \\
\nabla\rho &= d\rho + \frac{1}{2}\omega\gamma_0\rho + \frac{1}{2}s\gamma_0\psi^+ + \frac{1}{2}\omega^a\gamma_a\psi^- + t_2\gamma_0\psi^+ + t_1\gamma_0\rho
\end{aligned}
$$

$\ell \rightarrow \infty \implies$ most general extended Bargmann supergravity

Torsional NR supergravity action

Analogously to the bosonic case the cosmological constant can be seen as a source for the spatial super-torsion $\hat{T}^a\left(e^b\right) = de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c + \bar{\psi}^+\gamma^a\psi^-$ and for the curvature $\hat{T}\left(m\right) = dm + \epsilon^{ac}e_a\omega_c + \frac{1}{2}\bar{\psi}^-\gamma^0\psi^- + \bar{\psi}^+\gamma^0\rho$. In particular, on-shell we have

$$\boxed{\begin{aligned} \hat{T}^a\left(e^b\right) &= \frac{2}{\ell}\epsilon^{ac}\tau e_c \\ \hat{T}\left(m\right) &= \frac{1}{\ell}\epsilon^{ac}e_a e_c \end{aligned}}$$

Outline

Maxwell CS gravity

A particular extension and deformation of the Poincaré algebra is given by the **Maxwell algebra**, which is characterized by the non-vanishing commutator of the translational generator P_a:

$$[P_a, P_b] = \epsilon_{abc} Z^c$$

Three-dimensional gravity based on this symmetry describes (as Poincaré) a Riemannian and locally flat geometry. However, the presence of an additional gauge field in the Maxwell case leads to new effects compared to GR. In particular, the gravitational Maxwell gauge field modifies not only the vacuum energy and angular momentum of the stationary configuration but also the asymptotic structure.

Maxwell CS gravity with torsion

Using the CS formalism, we presented the three-dimensional gravity theory based on a particular deformation of the Maxwell algebra. The deformed Maxwell algebra is spanned by the generators $\{J_a, P_a, Z_a\}$, which satisfy the following non-vanishing commutation relations [P. Concha, H.Safari (2019); H. Adami, P. Concha, E. Rodríguez, H. Safari (2020)]

Deformed Maxwell algebra

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{abc} J^c \\
[J_a, P_b] &= \epsilon_{abc} P^c \\
[J_a, Z_b] &= \epsilon_{abc} Z^c \\
[P_a, P_b] &= \epsilon_{abc}\left(Z^c - \frac{2}{\ell} P^c\right)
\end{aligned}
$$

$\ell \to \infty \implies$ Maxwell algebra

Maxwell CS gravity with torsion

Considering the following redefinition of the generators

$$L_a \equiv J_a + \frac{\ell}{2}P_a - \frac{\ell^2}{4}Z_a$$
$$S_a \equiv -\frac{\ell}{2}P_a + \frac{\ell^2}{4}Z_a$$
$$T_a \equiv \frac{\ell}{2}Z_a$$

the deformed Maxwell algebra can be rewritten as

$$[L_a, L_b] = \epsilon_{abc}L^c$$
$$[L_a, T_b] = \epsilon_{abc}T^c$$
$$[S_a, S_b] = \epsilon_{abc}S^c$$

The motivation to use the basis $\{J_a, P_a, Z_a\}$ is twofold. First, it allows us to recover the Maxwell CS gravity theory in a particular limit. Second, it reproduces the Maxwell field equations with a non-vanishing torsion.

Maxwell CS gravity with torsion

The gauge connection one-form A for the deformed Maxwell algebra reads

$$A = e^a P_a + \omega^a J_a + f^a Z_a$$

The associated field strength can be written as

$$F = \hat{T}^a P_a + R^a J_a + W^a Z_a$$

where

$$R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c$$

$$\hat{T}^a = T^a - \frac{1}{\ell}\epsilon^{abc}e_b e_c$$

$$W^a = Df^a + \frac{1}{2}\epsilon^{abc}e_b e_c$$

Maxwell CS gravity with torsion

The non-degenerate bilinear form of the deformed Maxwell algebra reads

$$\begin{aligned}
\langle J_a J_b \rangle &= \alpha_0 \eta_{ab}, \qquad & \langle P_a P_b \rangle &= (-\frac{2\alpha_1}{\ell} + \alpha_2)\eta_{ab}, \\
\langle J_a P_b \rangle &= \alpha_1 \eta_{ab}, \qquad & \langle P_a Z_b \rangle &= 0, \\
\langle J_a Z_b \rangle &= \alpha_2 \eta_{ab}, \qquad & \langle Z_a Z_b \rangle &= 0,
\end{aligned}$$

where α_0, α_1 and α_2 are arbitrary constants.
The CS gravity action invariant under the deformed Maxwell algebra reads

$$\begin{aligned}
I_{\text{DefMax}} = \frac{k}{4\pi} \int_{\mathcal{M}} \alpha_0 L(\omega) + \alpha_1 \left(2R_a e^a + \frac{4}{3\ell^2} \epsilon^{abc} e_a e_b e_c - \frac{2}{\ell} T^a e_a \right) \\
+ \alpha_2 \left(T^a e_a + 2R^a f_a - \frac{2}{3\ell} \epsilon^{abc} e_a e_b e_c \right)
\end{aligned}$$

$\ell \to \infty \implies$ Maxwell CS gravity theory

Maxwell CS gravity with torsion

When $\alpha_2 \neq 0$ the e.o.m are given by the vanishing of the curvature two-forms

$$R^a = 0 \qquad \hat{T}^a = 0 \qquad W^a = 0$$

The first two equations are those corresponding to the three-dimensional teleparallel gravity theory, in which the cosmological constant can be seen as a source for the torsion

$$T^a - \frac{1}{\ell}\epsilon^{abc} e_b e_c = 0$$

In the flat limit $\ell \to \infty$ the field equation for f_a remains untouched and is analogue to the constancy of the electromagnetic field in flat spacetime

$$\begin{aligned} d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c &= 0 \\ T^a &= 0 \\ Df^a + \frac{1}{2}\epsilon^{abc} e_b e_c &= 0 \end{aligned}$$

Maxwell CS gravity with torsion

By considering suitable boundary conditions, we found the following **asymptotic symmetry** algebra for the **Maxwell CS gravity with torsion**:

[H. Adami, P. Concha, E. Rodríguez, H. Safari (2020)]

$$
\begin{aligned}
i\{J_m, J_n\} &= (m-n)\,J_{m+n} + \frac{c_1}{12} m^3 \delta_{m+n,0} \\
i\{J_m, P_n\} &= (m-n)\,P_{m+n} + \frac{c_2}{12} m^3 \delta_{m+n,0} \\
i\{J_m, M_n\} &= (m-n)\,M_{m+n} + \frac{c_3}{12} m^3 \delta_{m+n,0} \\
i\{P_m, P_n\} &= (m-n)\,M_{m+n} - \frac{2}{\ell}(m-n)\,P_{m+n} + \frac{1}{12}\left(-\frac{2c_2}{\ell} + c_3\right) m^3 \delta_{m+n,0} \\
i\{P_m, M_n\} &= 0 \\
i\{M_m, M_n\} &= 0
\end{aligned}
$$

This algebra corresponds to an infinite-dimensional lift of the deformed Maxwell algebra

$\ell \to \infty \implies$ deformed $\mathfrak{bms}_3$ algebra

Outline

Comments and further developments

Now that we have seen how to include a non-vanishing torsion in a CS (super)gravity theory, it would be interesting to go further.

- It would be interesting to study appropriate boundary conditions to the teleparallel (super)gravity theory and analyze its boundary dynamics. One could expect to obtain a (super)conformal structure.
- Another aspect that deserves further investigation is the Maxwellian version of the teleparallel supergravity.
- The study of the black hole solution and thermodynamics of the Maxwellian teleparallel gravity could bring valuable information about the physical implications of a non-vanishing torsion in Maxwell gravity theory.

Thank you!