Numerical stochastic perturbation theory revisited

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Motivation

Uses of perturbation theory in lattice QCD

- Parameter matching at high energies

$$\alpha_{\overline{\rm MS}}(q) = \alpha(q) + k_1\alpha(q)^2 + k_2\alpha(q)^3 + \ldots$$
$$\qquad\qquad \sim 10\% \qquad \sim 1\%$$

- O(a) improvement

$$\tilde{M}_{\rm q} = \left\{[1+\bar{b}_m{\rm tr}(aM_{\rm q})]M_{\rm q} + b_m aM_{\rm q}^2\right\} - \tfrac{1}{3}{\rm tr}\{\ldots\}$$
$$+\tfrac{1}{3}r_m\left\{[1+\bar{d}_m{\rm tr}(aM_{\rm q})]{\rm tr}M_{\rm q} + d_m{\rm tr}(aM_{\rm q}^2)\right\}$$

$$b_m = -\tfrac{1}{2} + {\rm O}(g_0^2), \qquad \bar{b}_m = {\rm O}(g_0^4), \quad \text{etc.}$$

Lattice Feynman rules and many observables of interest are very complicated!

Numerical stochastic perturbation theory? Di Renzo et al. '94

- ⋆ Fully automated numerical approach
- ⋆ Effort tends to grow slowly with the loop order
- ⋆ But: systematic & statistical errors

Recent developments: NSPT with SF bc, ISPT, HSPT

Brambilla et al. '13, Dalla Brida & Hesse '13, M.L. '14, Dalla Brida, Kennedy & Garofalo '15

Not discussed here: very-high-order computations, renomalons, resurgence

Bali, Bauer & Pineda '14

Outline

Numerical stochastic perturbation theory

- *Standard NSPT*
- *Instantaneous stochastic perturbation theory*

Taking the continuum limit ...

- *How does ISPT scale in this limit?*
- *Power-divergent statistical errors*
- *Another Langevin miracle*

NSPT recap

For simplicity, consider

$$S = a^4 \sum_x \left\{ \frac{1}{2} \partial_\mu \varphi(x) \partial_\mu \varphi(x) + \frac{1}{2}(m^2 + \delta m^2) \varphi(x)^2 + \frac{g_0}{4!} \varphi(x)^4 \right\}$$

$$\delta m^2 = \sum_{k=1}^{\infty} (\delta m^2)^{(k)} g_0^k : \quad \text{additive mass counterterm}$$

Simulation based on Langevin equation

$$\partial_t \phi = -\frac{\delta S}{\delta \phi} + \eta$$

$$\langle \eta(t,x) \eta(s,y) \rangle_\eta = 2a^{-4} \delta_{xy} \delta(t-s)$$

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle = \langle \phi(t,x_1) \ldots \phi(t,x_n) \rangle_\eta$$

Stochastic perturbation theory   Parisi & Wu '81

$$\phi = \sum_{k=0}^{\infty} g_0^k \phi_k$$

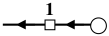
$$\partial_t \phi_0 = (\Delta - m^2)\phi_0 + \eta$$

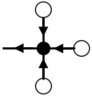
$$\partial_t \phi_1 = (\Delta - m^2)\phi_1 - (\delta m^2)^{(1)}\phi_0 - \tfrac{1}{3!}\phi_0^3$$

etc.

In frequency-momentum space

$\tilde{\phi}_0$ =  $= (\hat{p}^2 + m^2 - i\omega)^{-1}\tilde{\eta}(\omega, p)$

$\tilde{\phi}_1$ = 

$+ \frac{1}{6}$ 

NSPT   Di Renzo et al. '94

- Choose a finite lattice of size $T \times L^3$ with some boundary conditions
- Integrate equations for $\phi_0, \phi_1, \ldots$ numerically with random initial values
- Replace average over η by time average

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle_{\text{order } g_0^k}$$
$$= \frac{1}{N} \sum_{j=1}^{N} \{\phi(j\Delta t, x_1) \ldots \phi(j\Delta t, x_n)\}_{\text{order } g_0^k} + \mathrm{O}(N^{-1/2})$$

- Extrapolate results to vanishing integration step size ϵ

Main technical difficulties:

Extrapolation in ϵ, autocorrelations grow $\propto 1/a^2$, ...

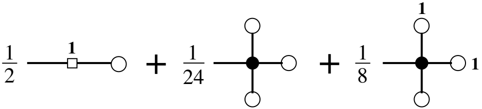
ISPT ML '14

Again

$$\phi = \sum_{k=0}^{\infty} g_0^k \phi_k$$

but with

$\tilde{\phi}_0 =$  $= (\hat{p}^2 + m^2)^{-1/2} \tilde{\eta}_0(p)$

$\tilde{\phi}_1 =$ 

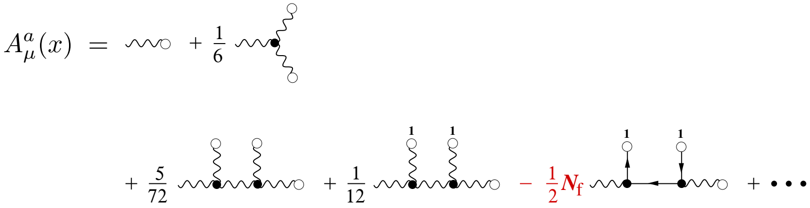
given instantaneously by Gaussian random fields $\eta_0(x), \eta_1(x), \ldots$

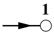
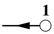
⇒ *No autocorrelations and no integration errors!*

ISPT in lattice QCD

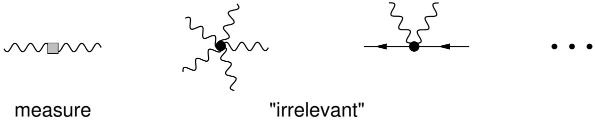
Gauge potential, gauge fixing, etc. as usual

Expansion of the gauge potential



 $= D^{-1}\chi_1(x)$,  $= \chi_1^*(x)$, $\chi_1, \chi_2, \ldots =$ pseudo-fermion noises

There are many more vertices than in continuum QCD



⇒ Large number of tree diagrams

Order	No. of diagrams
1	1
2	10
3	19
4	141
5	489
6	3524
7	16851
8	127143

However

- *Nothing is done by hand*
- *2nd order is already interesting*
- *May organize computations efficiently*

Does ISPT work out in practice?

Consider pure SU(3) gauge theory

L^4 lattice with Schrödinger-functional boundary conditions

Compute running coupling   Fritzsch & Ramos '13

$$\bar{g}^2(q) = \text{const} \times t^2 \,\langle E(t,x)\rangle_{x_0=L/2,\sqrt{8t}=0.3\times L} \quad \text{at} \quad q = 1/\sqrt{8t}$$

$E(t,x)$: YM action density at gradient-flow time t

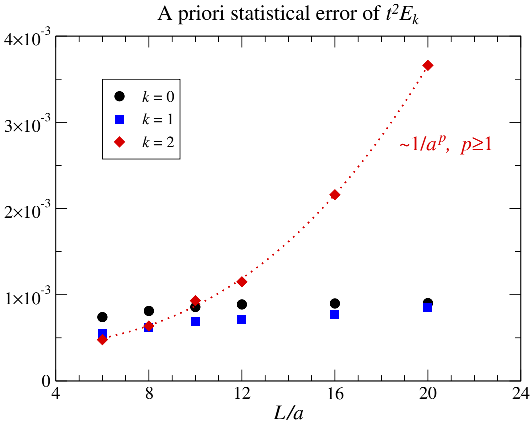
In perturbation theory

$$\langle E\rangle = E_0 g_0^2 + E_1 g_0^4 + E_2 g_0^6 + \ldots$$

$$\Rightarrow \; \alpha_{\overline{\rm MS}}(q) = \alpha(q) + k_1 \alpha(q)^2 + k_2 \alpha(q)^3 + \ldots$$

Computation of tree diagrams is cheaper than the measurement of E_0, E_1, E_2

However



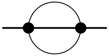
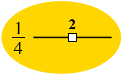
Same behaviour is observed in the ϕ^4 theory   Dalla Brida, Kennedy & Garofalo '15

Statistical variances are not guaranteed to be free of power divergences!

$$\sum_{j=0}^{k} \langle \phi_{k-j}(x)\phi_j(y)\rangle = \langle \varphi(x)\varphi(y)\rangle_{\text{order } g_0^k}$$

logarithmically divergent

but

$\langle \phi_1(x)\phi_1(y)\rangle = \frac{1}{24}$  $+$ 

$+ \cdots$

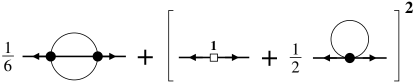
quadratically divergent

missing

$\Rightarrow$ The continuum limit is difficult to reach beyond the lowest orders in g_0

How about NSPT?

The power counting is different in this case

$\langle \phi_1(t,x)\phi_1(s,y)\rangle =$ 

$=$ log divergent

 $= (\hat{p}^2 + m^2 - i\omega)^{-1}$

 $= [(\hat{p}^2 + m^2)^2 + \omega^2]^{-1}$

Actually, all correlation functions of $\phi_0, \phi_1, \ldots$ are only logarithmically divergent!

Proof of the absence of power divergences

The Langevin equation has the form

$$\mathcal{D}\phi_0 = \eta, \qquad \mathcal{D} = \partial_t - \Delta + m^2$$

$$\mathcal{D}\phi_k = \mathcal{R}_k \ \text{ for all } \ k \geq 1$$

where

$$\mathcal{R}_k = -\sum_{j=0}^{k-1} (\delta m^2)^{(k-j)}\phi_j - \tfrac{1}{3!}\sum_{j_1,j_2,j_3=0}^{k-1} \delta_{k,j_1+j_2+j_3+1}\phi_{j_1}\phi_{j_2}\phi_{j_3}.$$

Would like to show that the correlation functions

$$\langle \phi_{k_1}(t_1,x_1)\dots\phi_{k_n}(t_n,x_n)\rangle$$

are at most logarithmically divergent

Functional integral in 5d   Zinn-Justin '86

Add Lagrange-multiplier fields $L_0(t,x), L_1(t,x), \ldots, L_n(t,x)$

$$\langle \phi_{k_1} \ldots \phi_{k_m} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[\phi_0] \ldots \mathrm{D}[\phi_n] \mathrm{D}[L_0] \ldots \mathrm{D}[L_n] \, \mathrm{e}^{-S} \phi_{k_1} \ldots \phi_{k_m}$$

$$S = \int \mathrm{d}t \, a^4 \sum_x \Big\{ L_0(\mathcal{D}\phi_0 - L_0) + \sum_{k=1}^{n} L_k(\mathcal{D}\phi_k - \mathcal{R}_k) \Big\}$$

Power counting shows that

- *Theory is renormalizable*
- *Power divergences can be canceled by the counterterms*

$$\int \mathrm{d}t \, a^4 \sum_x \sum_{k>j=0}^{n} c_{kj} L_k \phi_j, \qquad c_{kj} \propto 1/a^2$$

However, the **fixed-order two-point functions**

$$\sum_{j=0}^{k} \langle \phi_{k-j}(t,x) \phi_j(s,y) \rangle$$

are known to be only logarithmically divergent

⇒ *The contributions of the counterterms must cancel in these functions*

⇒ *The coefficients c_{kj} can recursively be shown to vanish*

⇒ *There are in fact no power divergences!*

Conclusions

NSPT

Pros:

Suitable for high-order computations

Statistical errors scale well

Cons:

Autocorrelations grow like $1/a^2$

Integration step size errors

Future:

Replace Langevin by HMC or SMD

ISPT

Pros:

Exact simulation

No autocorrelations

Cons:

Power-divergent statistical errors

Purely diagrammatic approach, little theoretical control

Future:

Try to get rid of power divergences