Anomalous Magnetic Moment of the Muon

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Collaborators

- HVP & DWF simulations
  RBC/UKQCD (next page),
  P. Boyle, R. Hudspith, A. Juettner, K. Maltman, A. Porttelli, M. Spraggs
  ...
- HLbL
  T. Blum, N. Christ, M. Hayakawa, L. Jin, C. Jung, C. Lehner,
  ... (+ T. Ishiakwa, N. Yamada)
- NEDM : E. Shintani, T. Blum, C. Lehner, S. Syritsyn, A. Soni,
- Thank you for sending me emails for related works: R. Guputa, G. Pientka (Hotzel), G. Endrodi, G. Bali, J. Green, H. Wittig, A. Shindler, M. Golterman, H. Horch, C. Alexandrou, B. Chakraborty, C. Davies, E. Gregory

Part of calculation are done by resources from
USQCD (DOE), XSEDE, ANL BG/Q, Edinburgh BG/Q,
BNL BG/Q, RIKEN BG/Q and Cluster (RICC)

Support from US DOE, RIKEN, BNL, and JSPS

RBC / UKQCD 2015

The RBC & UKQCD collaborations

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Renwick Hudspith

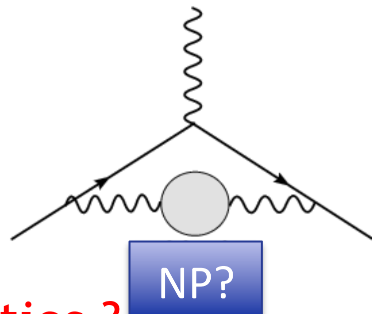
$(g-2)_\mu$ theory vs experiment

$$a_\mu^{\rm SM} = 116\ 591\ 803(1)_{\rm EW}(42)_{\rm HVP}(26)_{\rm HLbL} \times 10^{-11}$$

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 288(63)_{\rm exp}(49)_{\rm SM} \times 10^{-11} \quad [3.6\sigma]$$

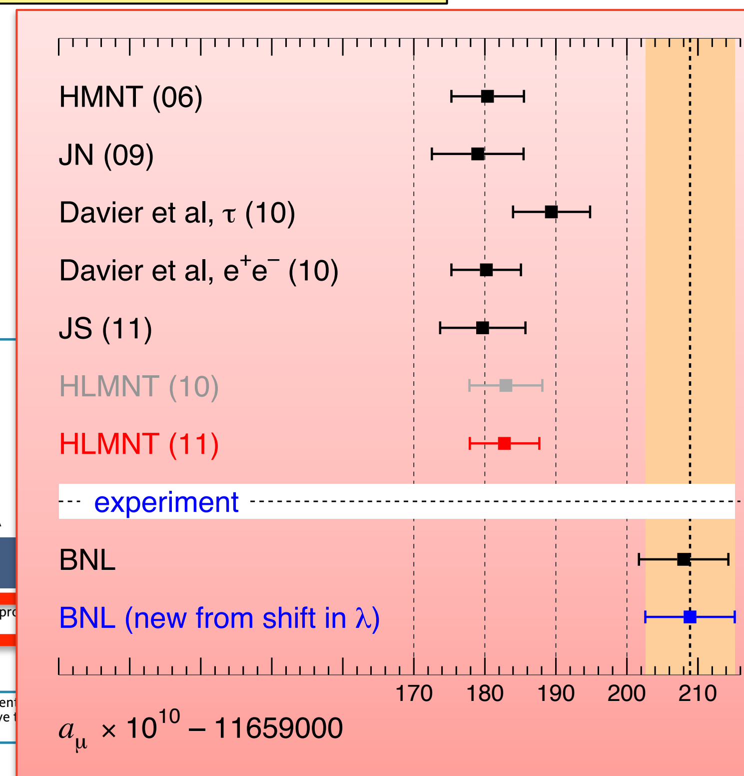
[PDG 2014, Hoecker & Marciano]

- ~ 3.6 σ discrepancy ?
- SM prediction
- New Physics
- → Hadronic uncertainties ?



P5 top recommendation

Project/Activity	Scenario A
Large Projects	
Muon program: Mu2e, Muon g-2	Y, Mu2e small repr... needed
HL-LHC	Y
LBNF + PIP-II	Y, LBNF component... delayed relative t... Scenario B.
ILC	R&D only

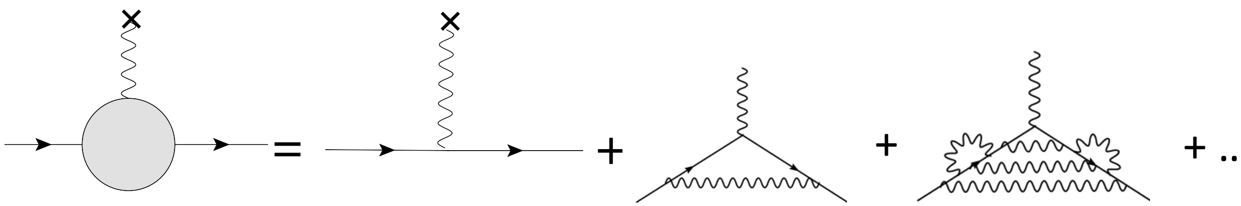


[FNAL, New (g-2) experiment (E989), is scheduled to taking data in 2016, x4 precision]

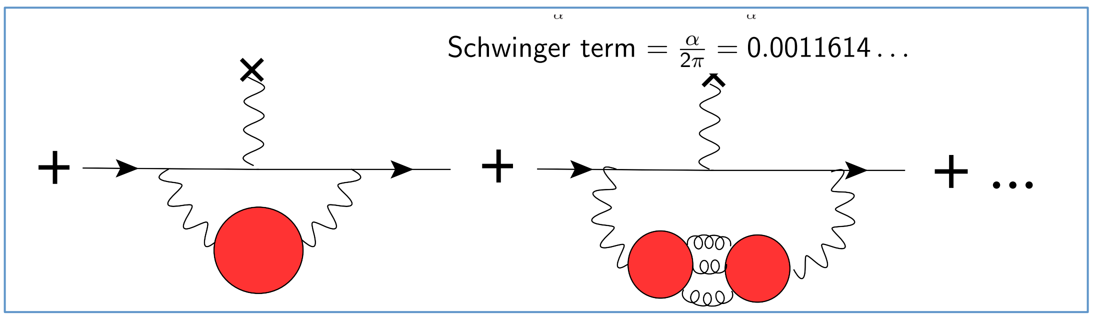
SM Theory

$$\gamma^\mu \;\rightarrow\; \Gamma^\mu(q) = \left(\gamma^\mu F_1(q^2) + \frac{i\,\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)\right)$$

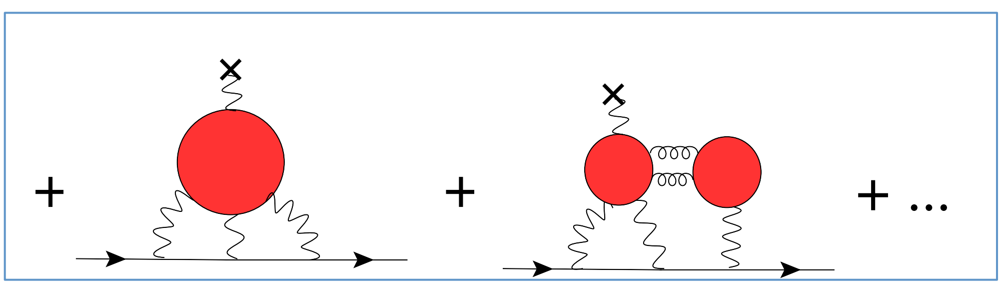
- QED, hadronic, EW contributions



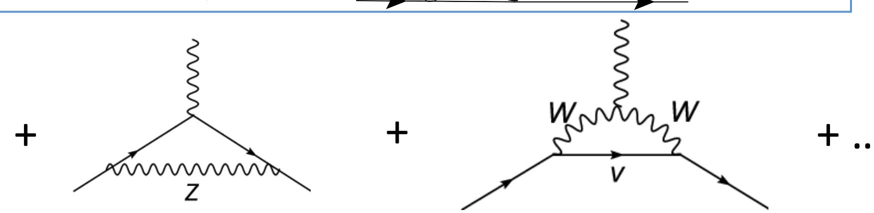
QED (5-loop)
Aoyama et al.
PRL109,111808 (2012)



Hadronic vacuum polarization (HVP)



Hadronic light-by-light (Hlbl)



Electroweak (EW)
Knecht et al 02
Czarnecki et al. 02

SM Theory prediction

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

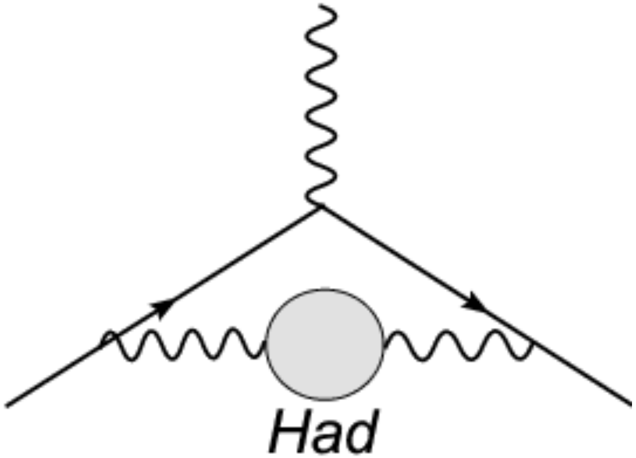
$$a_\mu^{\rm SM} = (11\ 659\ 182.8 \pm 4.9) \times 10^{-10}$$

$$a_\mu^{\rm QED} = (11\ 658\ 471.808 \pm 0.015) \times 10^{-10}$$
$$a_\mu^{\rm EW} = (15.4 \pm 0.2) \times 10^{-10}$$
$$\boxed{a_\mu^{\rm had,LOVP} = (694.91 \pm 4.27) \times 10^{-10}}$$
$$a_\mu^{\rm had,HOVP} = (-9.84 \pm 0.07) \times 10^{-10}$$
$$\boxed{a_\mu^{\rm had,lbl} = (10.5 \pm 2.6) \times 10^{-10}}$$

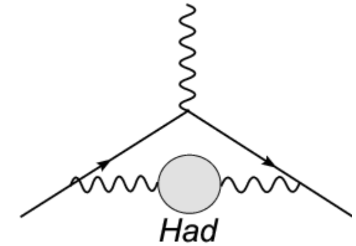
$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- x4 more accurate experiment
- Goal : sub 1% accuracy for HVP, and
  10% accuracy for HLbL

Hadronic Vacuum Polarization (HVP) contributions



Leading order of hadronic contribution (HVP)



- Hadronic vacuum polarization (HVP)

V_μ  V_ν $= (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_V(q^2)$

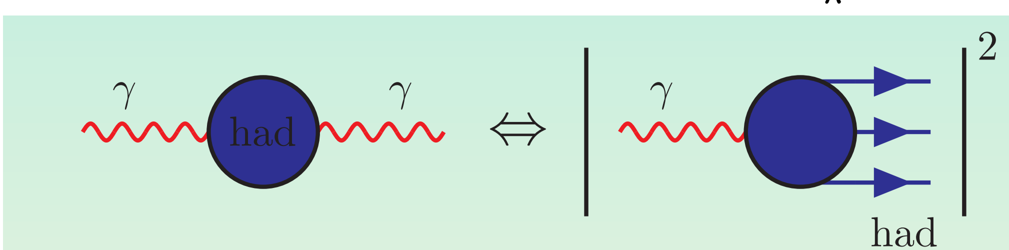
quark's EM current : $V_\mu = \sum_f Q_f \bar{f}\gamma_\mu f$

- Optical Theorem

$$\mathrm{Im}\Pi_V(s) = \frac{s}{4\pi\alpha}\sigma_{\mathrm{tot}}(e^+e^- \to X)$$

- Analyticity

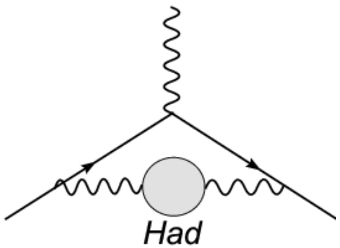
$$\Pi_V(s) - \Pi_V(0) = \frac{k^2}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}\Pi_V(s)}{s(s-k^2-i\epsilon)}$$



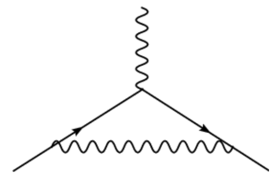
F. Jegerlehner's lecture

Leading order of hadronic contribution (HVP)

- Hadronic vacuum polarization (HVP)



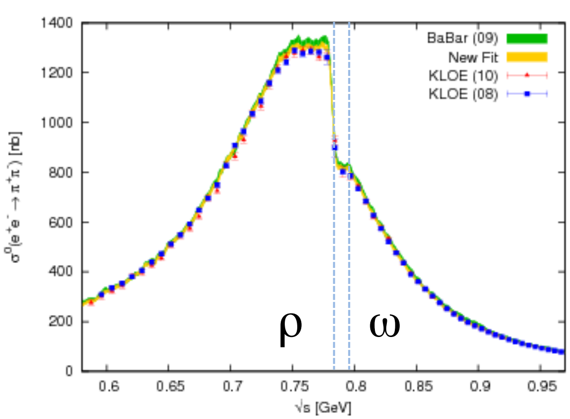
$$a_\mu^{\text{had}} = \int ds$$

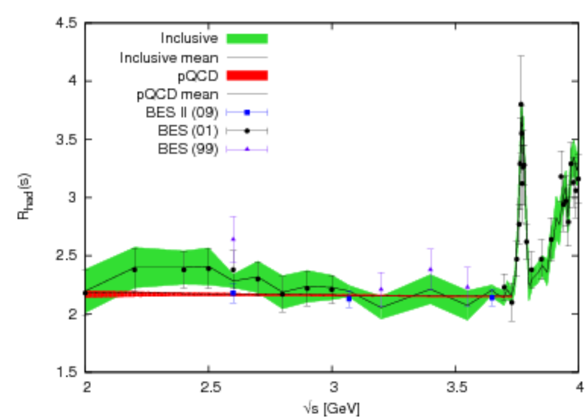


×

$$= \frac{\alpha}{\pi^2}\int_{m_\pi^2}^{\infty}\frac{ds}{s}\mathrm{Im}\Pi(s)K(s) \qquad K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2+(s/m_\mu^2)(1-x)}$$

$$= \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\left[\int_{m_\pi^2}^{s_{\text{cut}}} ds\frac{K(s)}{s}R_{\text{had}}^{\text{data}}(s) + \int_{s_{\text{cut}}}^{\infty} ds\frac{K(s)}{s}R_{\text{had}}^{\text{pQCD}}(s)\right]$$

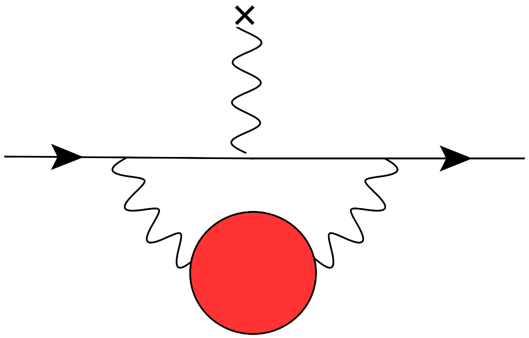




Hagiwara, et al. J.Phys. G38,085003 (2011)

HVP from experimental data

- From experimental e+ e- total cross section σ_{total}(e+e-) and dispersion relation



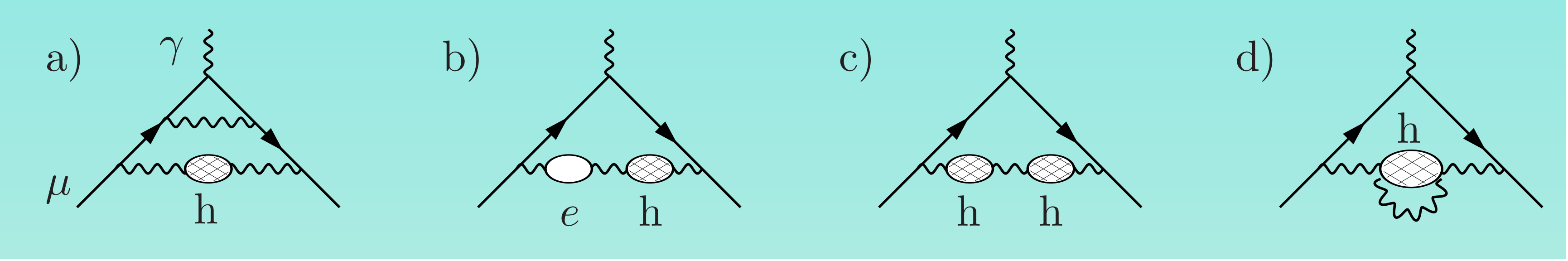
$$a_\mu^{\rm HVP} = \frac{1}{4\pi^2}\int_{4m_\pi^2}^{\infty} ds K(s)\sigma_{\rm total}(s)$$

time like $q^2 = s >= 4\, m_\pi^2$

$$a_\mu^{\rm HVP,LO} = (694.91 \pm 4.27) \times 10^{-10}$$

$$a_\mu^{\rm HVP,HO} = (-9.84 \pm 0.07) \times 10^{-10}$$

[~ 0.6 % err]



[T. Blum PRL91 (2003) 052001]

HVP from Lattice

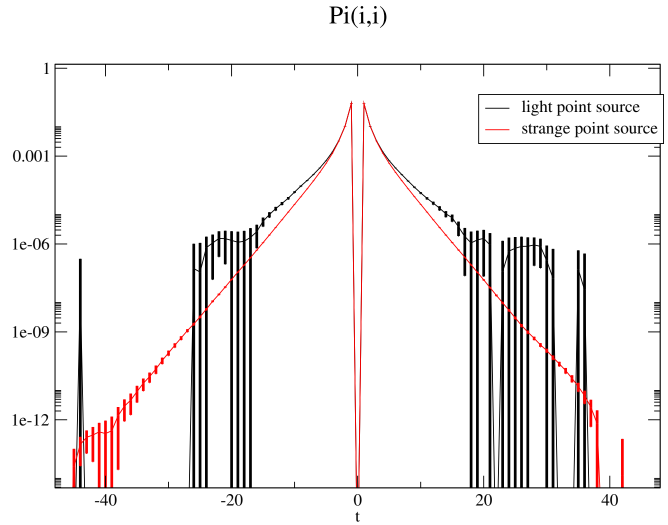
- Analytically continue to Euclidean/space-like momentum $K^2 = - q^2 > 0$
- Vector current 2pt function

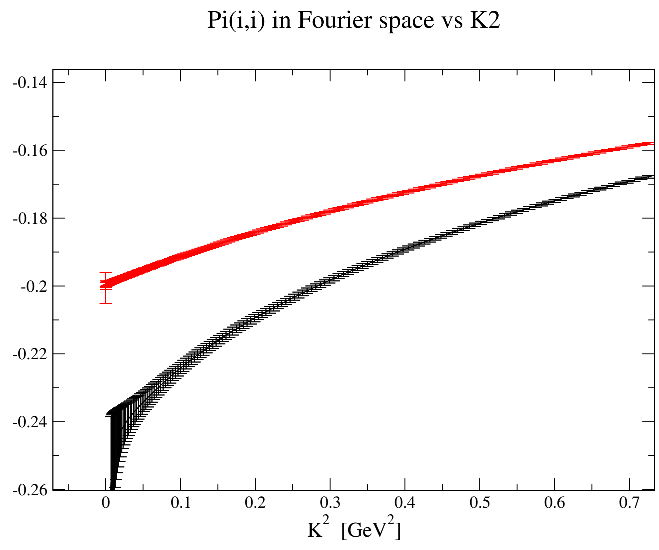
$$a_\mu = \frac{g-2}{2} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2)$$

$$\Pi^{\mu\nu}(q) = \int d^4x e^{iqx} \langle J^\mu(x) J^\nu(0) \rangle$$

- $J^\mu(x)$ conserved current
- $(\mu, \nu) = (i,i)$, q = 2 Pi / N x (real number) a good approximation if Fourier integrand is rapidly suppressed in large |x|

X. Feng et al. 2013, C.Lehner 2014, L. del Debbio & A. Portelli 2015





Current conservation & subtractions

- conservation => transverse tensor
$$\Pi^{\mu\nu}(q) = (\hat{q}^2\delta^{\mu\nu} - \hat{q}^\mu\hat{q}^\nu)\Pi(\hat{q}^2)$$
- In infinite volume, q=0, $\Pi_{\mu\nu}(q) = 0$
- For finite volume, $\Pi_{\mu\nu}(0)$ is exponentially small (L.Jin, use also in HLbL)

$$\int_V dx^4 \langle V_\mu(x)\mathcal{O}(0)\rangle = \int_V dx^4\, \partial_{x_\nu}\left(x_\mu\langle V_\nu(x)\mathcal{O}(0)\rangle\right)$$
$$= \int_{\partial_\nu V} dx^3\, x_\mu\langle V_\nu(x)\mathcal{O}(0)\rangle \propto L^4\exp(-ML/2) \to 0$$

- e.g. DWF L=2, 3, 5 fm $\Pi_{\mu\nu}(0)$ = 8(3)e-4, 2(13)e-5, -1(5)e-8
- Subtract $\Pi_{\mu\nu}(0)$ alternates FVE, and reduce stat error
"-1" subtraction trick :

$$\Pi^{\mu\nu}(q) - \Pi^{\mu\nu}(0) = \int d^4x(e^{iqx} - 1)\langle J^\mu(x)J^\nu(0)\rangle$$

“-1” trick, DDS

- Current conservation is config-by-config.

$$\Pi^{\mu\nu}(q) - \Pi^{\mu\nu}(0) = \int d^4x\,(e^{iqx} - 1)\langle J^\mu(x) J^\nu(0)\rangle$$

- removes x=0 contact term from rhs, now local current could be used $J(x) = Z_V\ J^{(local)}(x)$ without finite correction

- Reduces statistical error for small q , dominant in integral of a_μ

- Further extended to $\Pi(0)$ subtraction (Direct Double Subtraction)

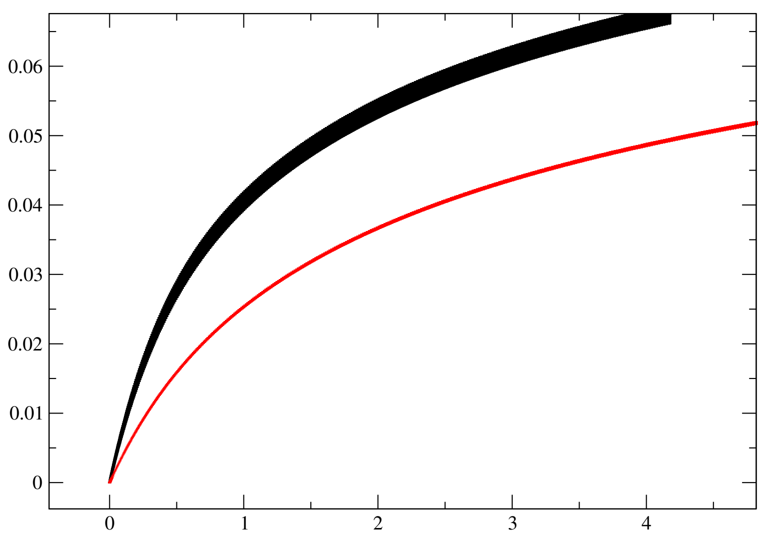
- Bernecker & Meyer 2011
- Lehner TI 2014
- BMWc (Malak et al) 2014
- HPQCD 2014
- Del Debbio & Portelli 2015
- L. Jin et al. 2015

$$\Pi(\hat{q}^2) - \Pi(0) = \langle \sum_t \Re\left(\frac{e^{iqt} - 1}{q^2} + \frac{t^2}{2}\right) \Re C_{ii}\rangle$$

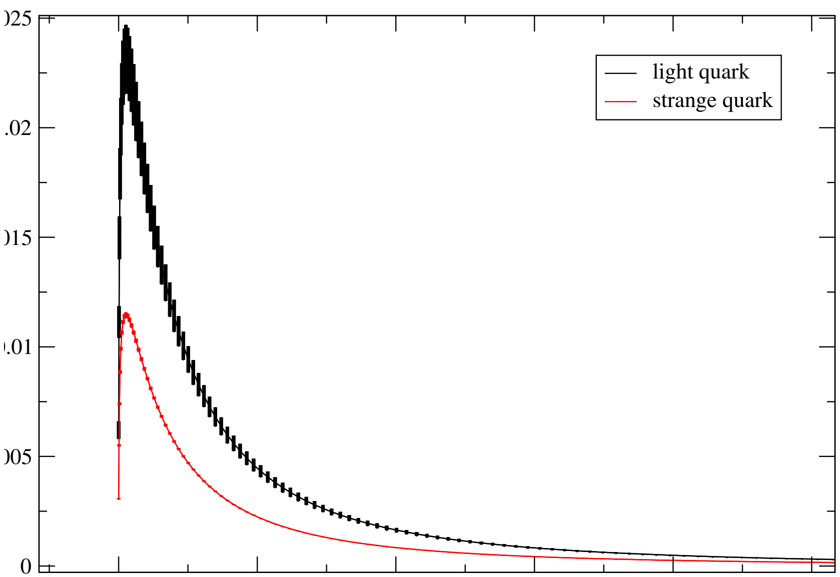
- The subtractions reduces noise significantly
- $$\hat{\Pi}(K^2) = \Pi(\hat{K}^2) - \Pi(0)$$
- a_μ Integrand peaks around $K^2 \sim (m_\mu/2)^2$

$$a_\mu = \frac{g-2}{2} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2)$$

Pihat vs K^2 [GeV^2]

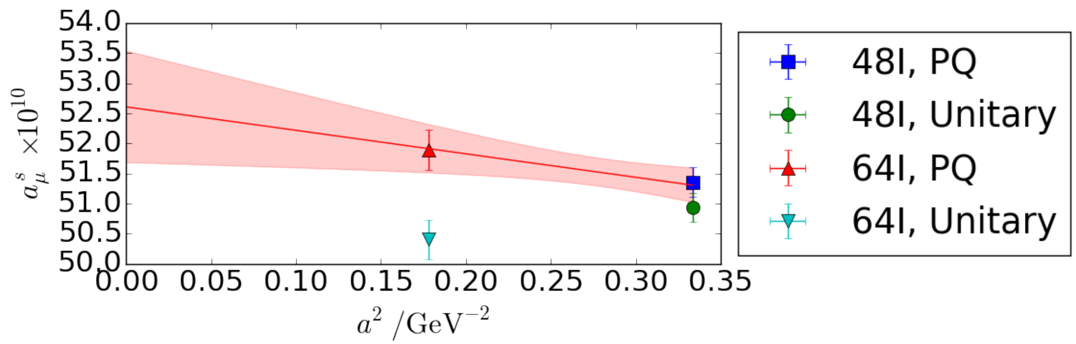


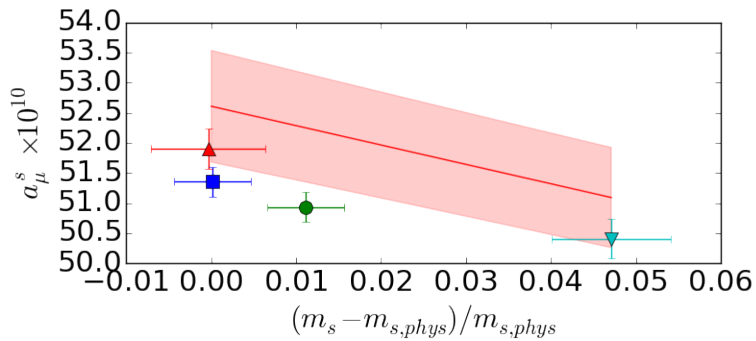
Integrand vs K^2 [GeV^2]



Strange quark contribution

- Matt Spragss (RBC/UKQCD) [Tue, 15:00]
- Mobius DWF, Nf=2+1,Physical mass, L=5.5fm, a=0.114, 0.09 fm
- Many fits, moment, and cuts are used to examine systematics
- parts of systematic errors are being estimated
- consistent with HPQCD's value (next page)





Use of Time-Moments
[HPQCD, PRD89(2014)114501]

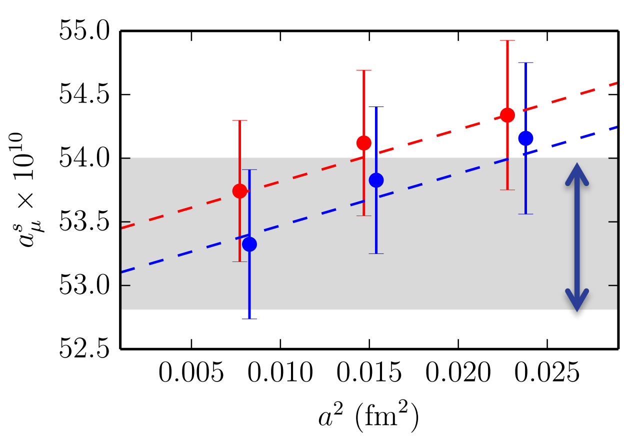
- Compute Time-moments of 2pt

$$G_{2n} \equiv a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_V^2 \langle j^i(\vec{x},t) j^i(0)\rangle$$
$$= (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2)\bigg|_{q^2=0}.$$

$$\hat{\Pi}(q^2) = \sum_{j=1}^{\infty} q^{2j}\Pi_j$$

$$\Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}.$$

- subtractions by taking derivatives, use local currents
- Pade approximation, <u>determined</u> from Π j , for high q2 integration



Strange

$$\boxed{a_\mu^s = 53.41(59) \times 10^{-10}.}$$

[1.1% ~ lattice spacing error]

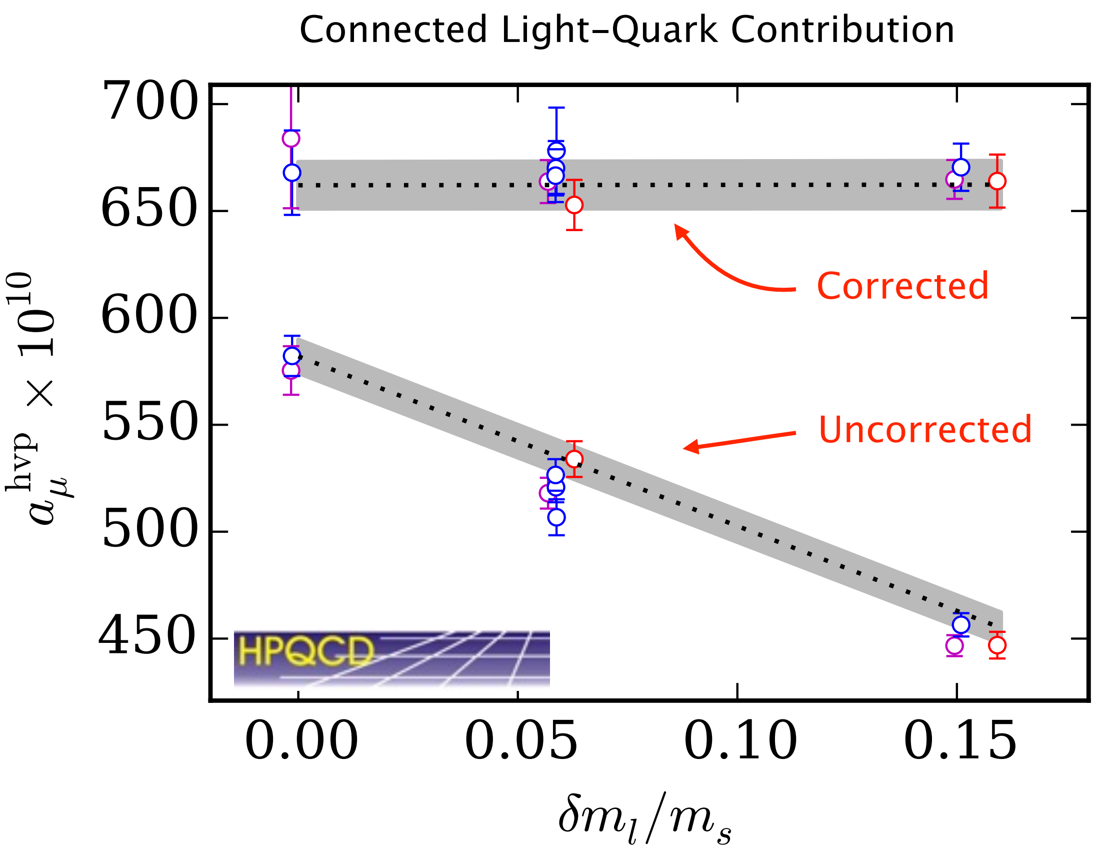
charm

$$\boxed{a_\mu^c = 14.42(39) \times 10^{-10}.}$$

[2.7% ~ Z_V error]

HPQCD light quark HVP

Bipasha Chakraborty at Lattice 2015



- a=0.09, 0.12, 0.16 fm
- switch to multi-exp at t*=1.5fm
- sub 2% total error !
- ETMC-type ρ correction + ChPT pipi sub/add

→ 10% correction at physical point

- Large finite volume effects, even for L~ 5.8fm, 5.1 fm at physical poit
- also from taste pion effects to pipi amplitude
- estimate disc. loop phenomenological, -3% of connected

$$\hat{\Pi}_j^{\mathrm{latt}} \to \left(\hat{\Pi}_j^{\mathrm{latt}} - \hat{\Pi}_j^{\mathrm{latt}}(\pi\pi)\right) \left[\frac{m_\rho^{2+2j}}{f_\rho^2}\right]_{\mathrm{latt}} \left[\frac{f_\rho^2}{m_\rho^{2+2j}}\right]_{\mathrm{expt}} + \hat{\Pi}_j^{\mathrm{cont}}(\pi\pi)$$

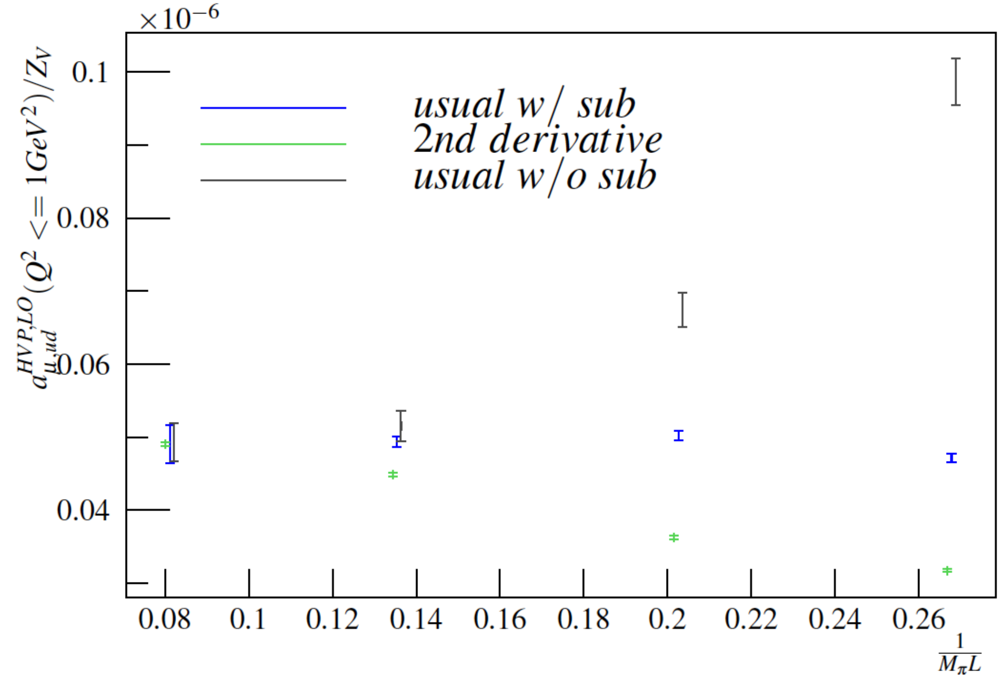
Finite Volume effects

- 2011 Bernecker Meyer
- Malak et al. (15, BMWc)

  w/o $\Pi_{\mu\nu}(0)$ subtraction,
  +40% FVE at Mpi L=5

- FVE for $\Pi_{\mu\nu}(0)$ subtracted ones get small

- t^2 moment undershoots
  -30% or so at Mpi L =5



- Maarten Golterman Lattice 2015

  Compares different H4 Irrepps, find 10+% difference.
  Also ChPT analysis for different FV treatment (Irreps, subtractions)

A_1:

[0,1] Padé: $a_\mu^{\mathrm{HVP}}(1\ \mathrm{GeV}^2) = 8.4(4) \times 10^{-8}$

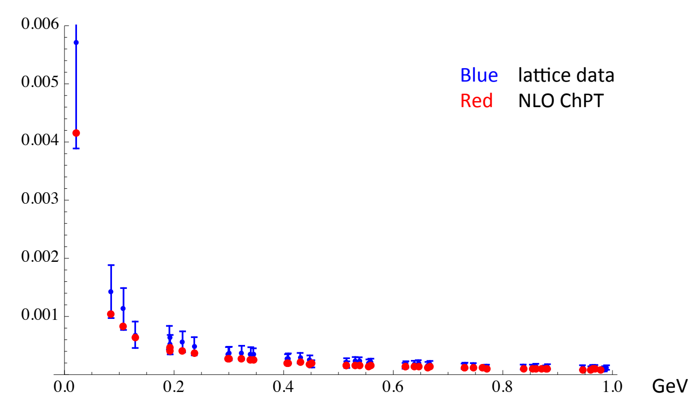
quadr. conf. pol.: $a_\mu^{\mathrm{HVP}}(1\ \mathrm{GeV}^2) = 8.4(5) \times 10^{-8}$

A_1^{44}:

[0,1] Padé: $a_\mu^{\mathrm{HVP}}(1\ \mathrm{GeV}^2) = 9.2(3) \times 10^{-8}$

quadr. conf. pol.: $a_\mu^{\mathrm{HVP}}(1\ \mathrm{GeV}^2) = 9.6(4) \times 10^{-8}$

Difference of 9 – 13% as a consequence of finite volume effects

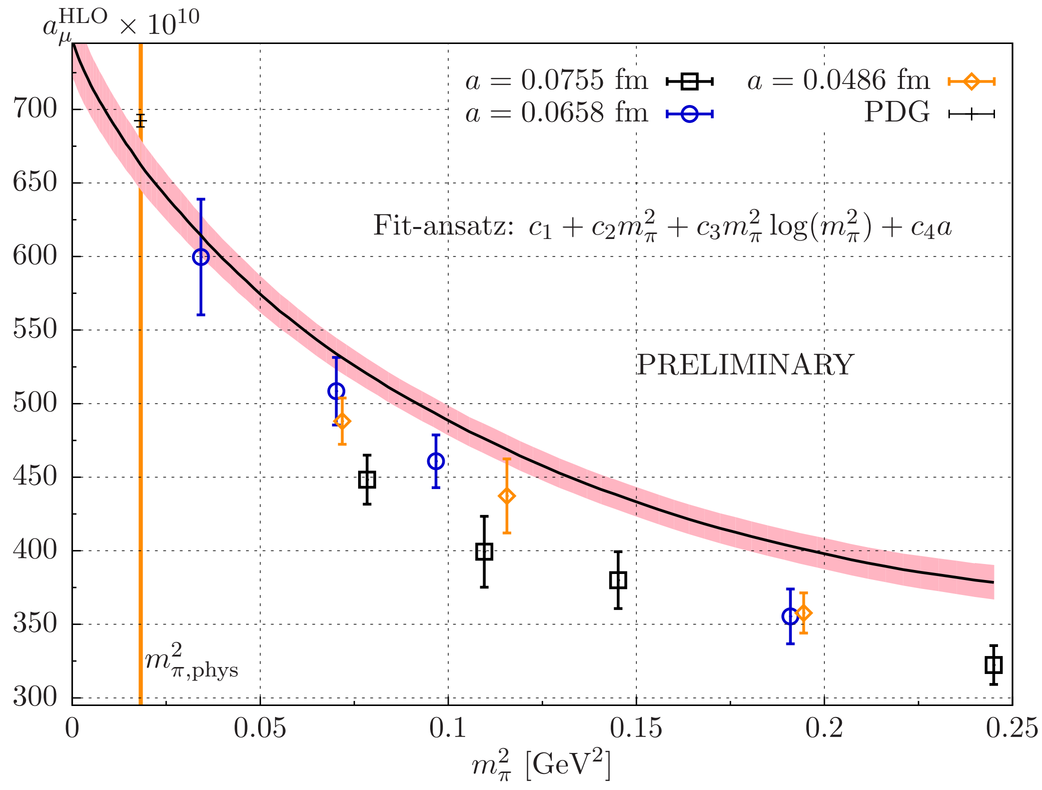


Difference of $\Pi_{A_1}(Q^2)$ (unsubtracted) and $\hat{\Pi}_{A_1}(Q^2)$ (subtracted)

Mainz HVP

Hanno Horch Lattice 2015

- Nf=2 O(a)-imp Wilson, CLS,
  Mpi = 185-495 MeV, a=0.05,0.07, 0.08 fm
- TBC
- ETMC rho rescaling
- extended frequentist's method

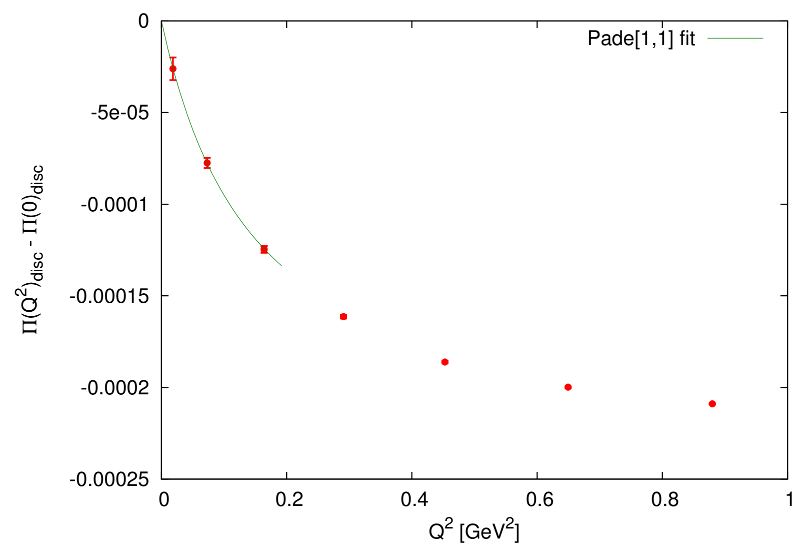


BMWc staggered
A. Sastre, B. Toth Lattice 2015

- Nf=2+1+1 4-Stout staggered, Symanzik tree Gauge, 10 ensemble @ physical quark mass, a=0.063-0.133 fm, L ~ 6fm
- ~1,000 lattice, a few K measurements / lattice = a few Million measurements !
- connected : Hybrid (low+high q2) method
- disconnected : Hopping Parameter Expansion (with tuned coeffs), TSM, SU(3) cancellation

	Fit	Derivative
$\Pi_1^{ud}(\mathrm{GeV}^{-2})$	0.162(2)	0.167(3)
$\Pi_2^{ud}(\mathrm{GeV}^{-4})$	-0.29(2)	-0.34(4)
$a_\mu^{\mathrm{hi},ud}$	$52.05(8)\times 10^{-10}$	

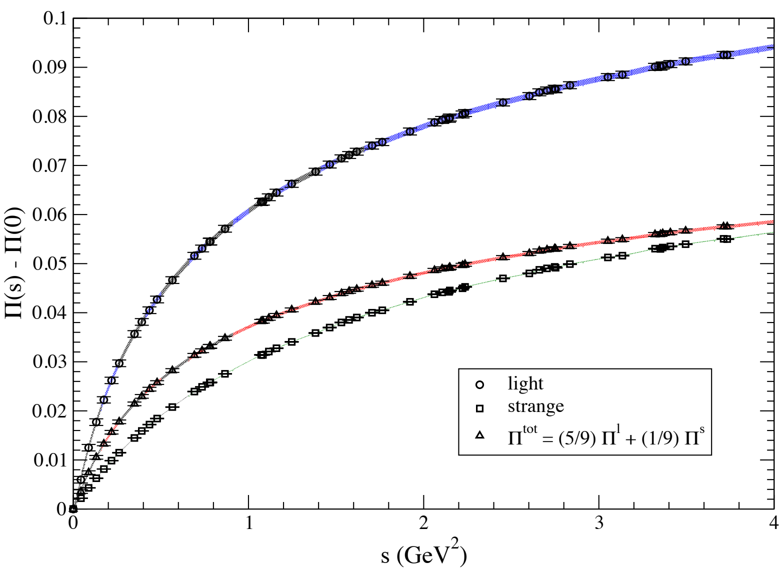
$a = 0.095$ fm, physical quark masses, $64^3 \times 96$

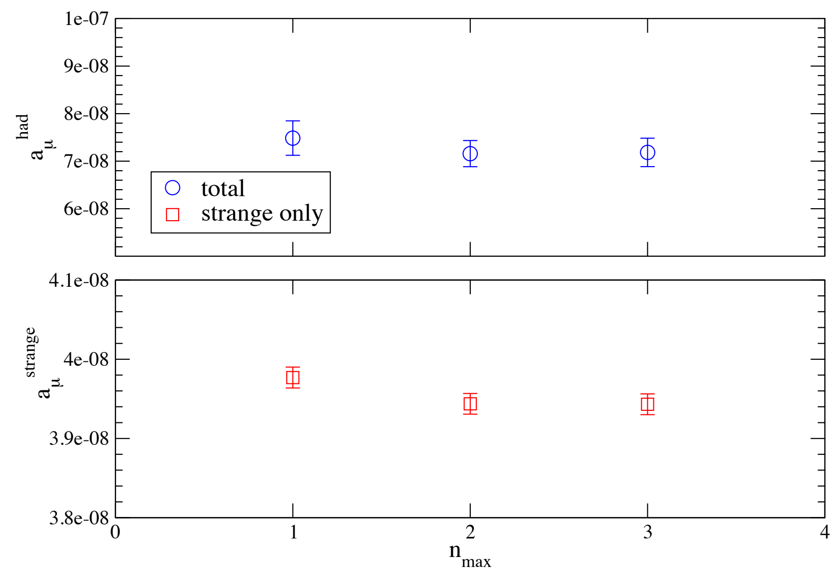


HVP on BMWc ensemble

Eric Gregory Lattice 2015

- Extract smooth function π(s) from Taylor expansion, with derivatives measured from vector correlator.
- 1065 config @ physical Mpi, 1/a=2.131 GeV, ~6fm, 2HEX-smeared Wilson-type
- strange contribution ~15% smaller than HPQCD, RBC/UKQCD



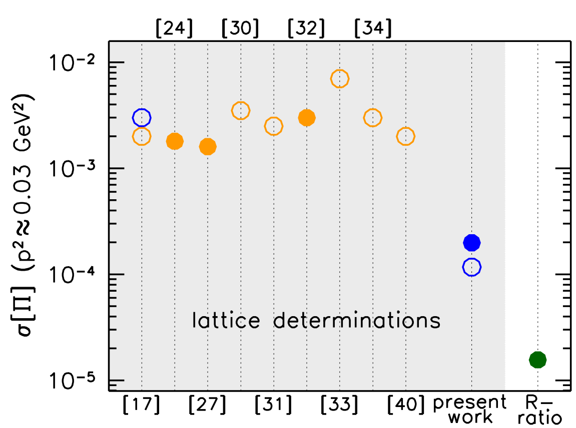


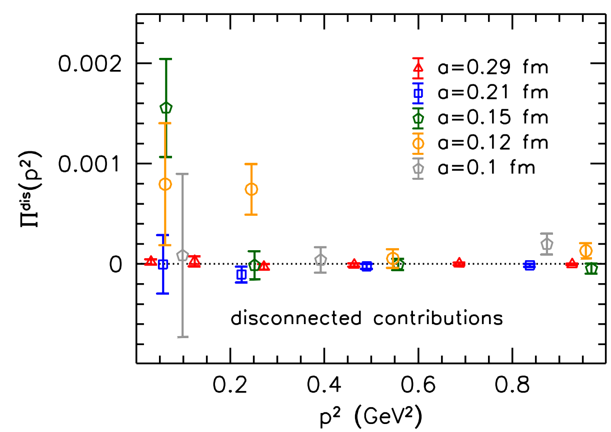
HVP & magnetic susceptibilities

- Gunnar Bali, Gergely Endrodi,arXiv:1506.08638
  Relates magnetic susceptibilities with oscillatory magnetic background and constant one, extract HVP. Also include disconnected loop

$$2\chi_p = \Pi(p^2)\,, \qquad \chi_0 = \Pi(0)\,. \qquad \chi_p = -\frac{\partial^2 f[\mathbf{B}^p]}{\partial(eB)^2}\,.$$

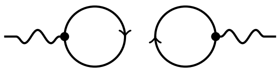
$$\mathbf{B}^p(x) = B\sin(px)\,\mathbf{e}_3\,, \qquad \mathbf{B}^0 = B\,\mathbf{e}_3\,,$$



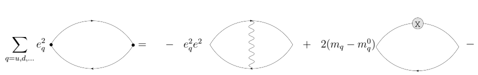


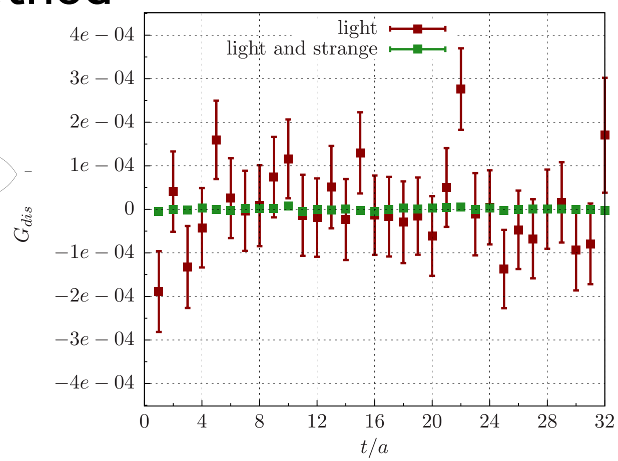
Compared to older results

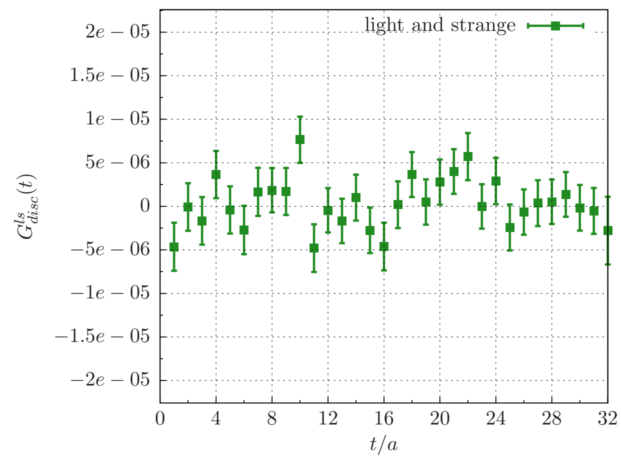
disconnected loop, isospin breaking effects



- -10% of connected Pi-Pi in ChPT [Della Morte, Juettner, 2010]
- Mainz group $$\frac{1}{9}\frac{G^{\ell s}_{\rm disc}(t)}{G^{\rho\rho}(t)} = \underbrace{\frac{G^{\gamma\gamma}(t)-G^{\rho\rho}(t)}{G^{\rho\rho}(t)}} - \frac{1}{9}\underbrace{\left(1+2\frac{G^{s}_{\rm con}(t)}{G^{\ell}_{\rm con}(t)}\right)} \longrightarrow -\frac{1}{9},$$
- HPQCD 2015, -3% of connected based on phenomenological estimate of ρ ω difference
- ETMC 2011, 2015
- Francis et al (Mainz) use same random source for strange and light for error reduction to exploit the SU(3) limit, noise ~ (ml – ms)
- Marina Marikovic (latice2015) , isospin breaking effect using ROME123 method







Recent results

3+1 new results
for Lattice15

a_μ(strange) x 10^{10}

53.41(59) HPQCD 14

53.6(1.9) ETMC (Grit Hotzel et al)

also RBC/UKQCD & BMW

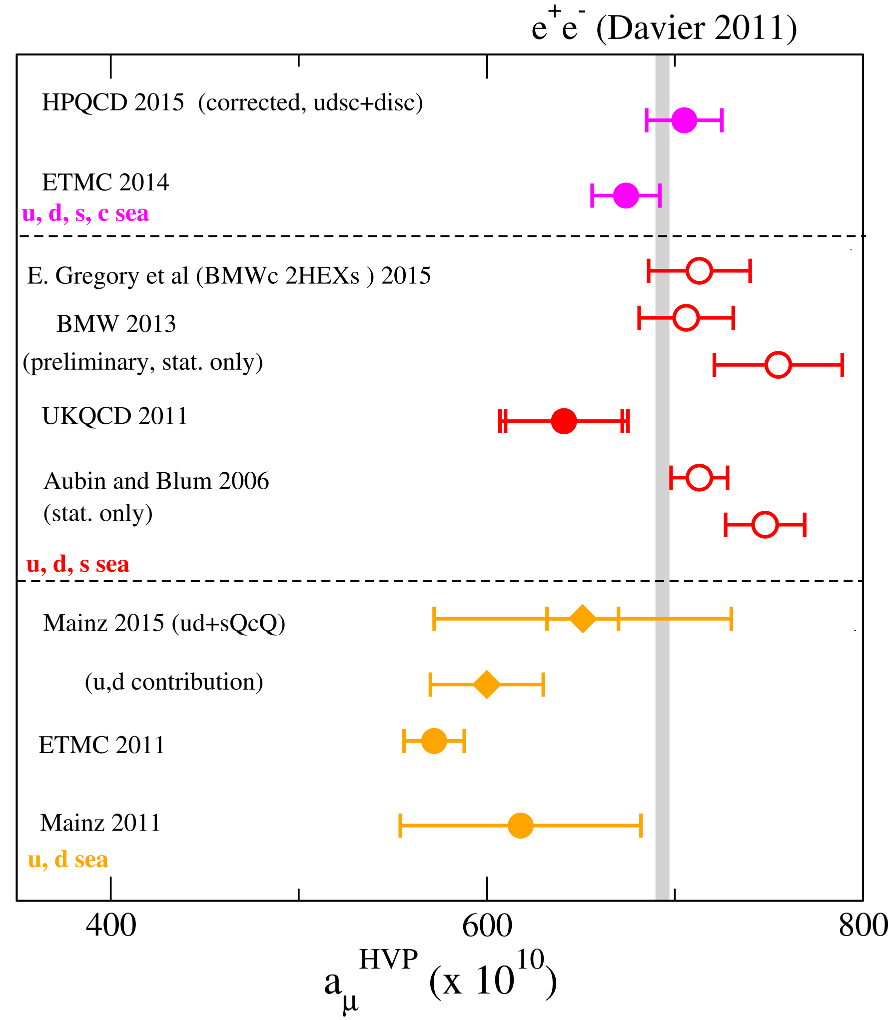
~ 44 E. Gregory et al.

a_μ (charm) x 10^{10}

_ 14.42(39) HPQCD 14

14.18(61) ETMC

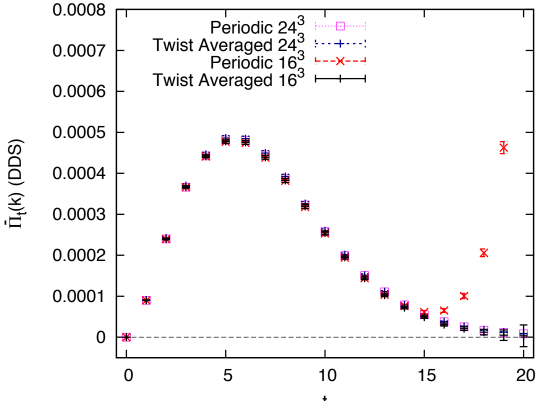
also BMWc



[added to Ruth Van de Water' 15-04 compilation, which was on Blum TI's compilation 14]

HVP Summary and future prospects

- Direct subtraction methods (Mainz formula, Doulbe Direct Subtraction, Sin-cardinal, moment method) avoids fit procedures.
One needs to check the Tmax truncation error, could fail.



- Error reduction technique
(AMA, TSM, A2A, HPE, noise method
with dilution) continue to help

- Consistent results, with increasing accuracies !

- Sizable finite volume effects 10+% at physical point, MpiL = 4 ? → need larger volume lattice, or Twist Averaging (C. Lehner TI 2014) ?

- Disconnected quark loop
- EM Isospin, ud mass difference

- Magnetic susceptibility
- Also electron, tau anomalous moment, ETMC arXiv:1501.05110
coupling runnings Gregorio Herdoiza,
Vera Guelpers lattice 2015

Twist angle Averaging (TA)

C. Lehner TI, Lattice 2014

QCD setup

CL lattice 2014

$U_\mu(x)$ $\Psi(x+\hat{L}_1+\hat{L}_2)$	$U_\mu(x)$ $\Psi(x+2\hat{L}_1+\hat{L}_2)$	$U_\mu(x)$ $\Psi(x+3\hat{L}_1+\hat{L}_2)$
$U_\mu(x)$ $\Psi(x+\hat{L}_1)$	$U_\mu(x)$ $\Psi(x+2\hat{L}_1)$	$U_\mu(x)$ $\Psi(x+3\hat{L}_1)$

Valence fermions Ψ living on a repeated gluon background U_μ with periodicity L_1, L_2 and vectors $\hat{L}_1=(L_1,0)$, $\hat{L}_2=(0,L_2)$

Let ψ^θ be the quark fields of your finite-volume action with twisted-boundary conditions

$$\psi^\theta_{x+L} = e^{i\theta} \psi^\theta_x \,.$$

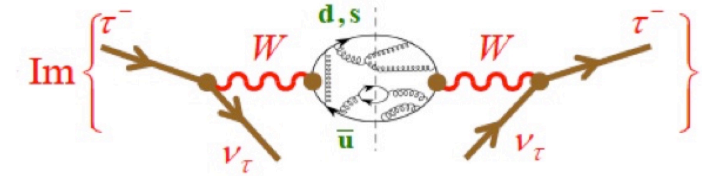
Then one can show that

$$\boxed{\left\langle \Psi_{x+nL} \bar{\Psi}_{y+mL} \right\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(n-m)} \left\langle \psi^\theta_x \bar{\psi}^\theta_y \right\rangle \,, \qquad (2)}$$

where the $\langle \cdot \rangle$ denotes the fermionic contraction in a fixed background gauge field $U_\mu(x)$. (4d proof available.)

This specific prescription produces exactly the setup of the previous page, it allows for the definition of a conserved current, and allows for a prescription for flavor-diagonal states.

Tau decay & Lattice HVPs



- tau -> had + nu decay dispersive analysis

$$\int_C ds\, \hat{\Pi}(s) w(s) = I$$

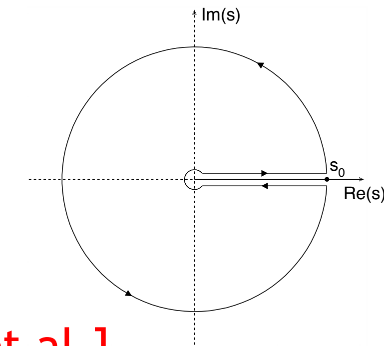
- current methods : analytic w(s) => I=0

  cut integral from experiment

  no data above $m_\tau{}^2$, larger error for higher s

  circle integral from pQCD+OPE

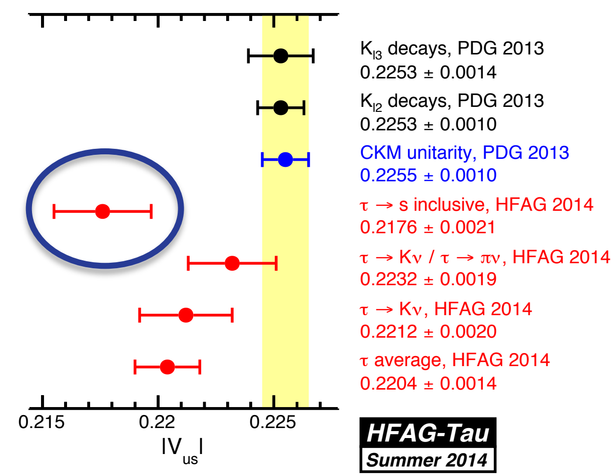
  slow α_s convergence, D>4 OPE contributions [K. Maltman et al.]



- By using w(s) with space-like poles, and take circle radius to infinity

$$w(s) = \frac{1}{(s+Q_1^2)(s+Q_2^2)\cdots(s+Q_m^2)}$$

and evaluate the residues using Lattice,
one may suppress the problems



- Provide checks for lattice vs exp data

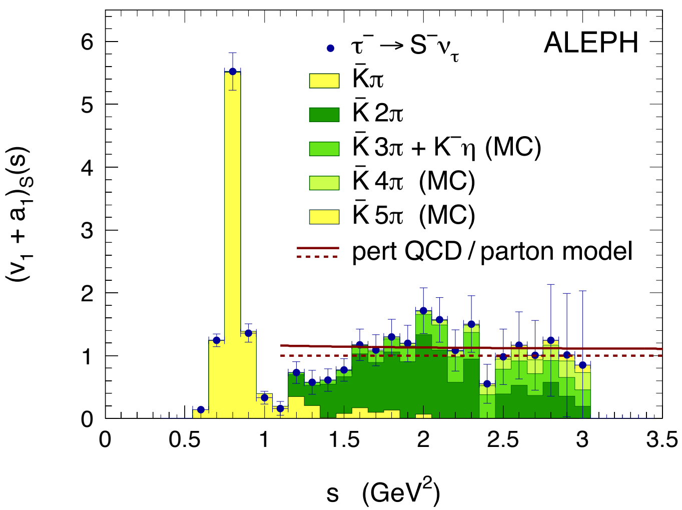
  (no disconnected, could study isospin breaking effects)

- alternative determinations for Vus

OPAL, ALEPH

20

18



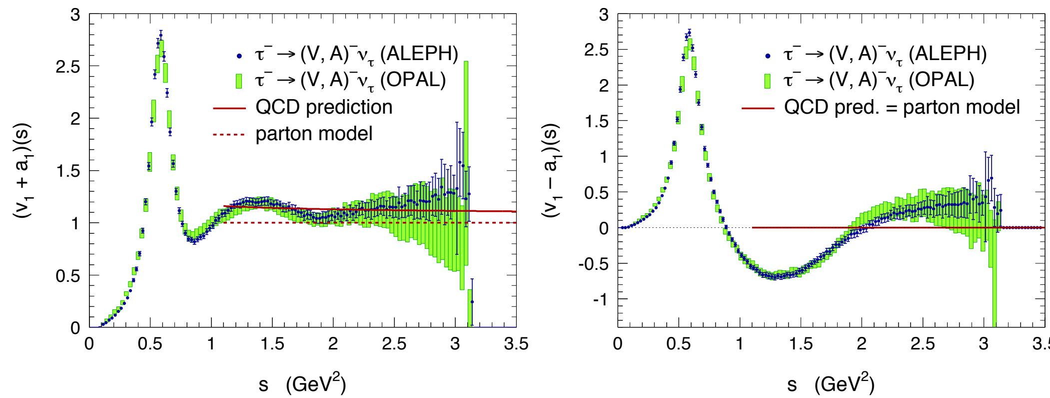
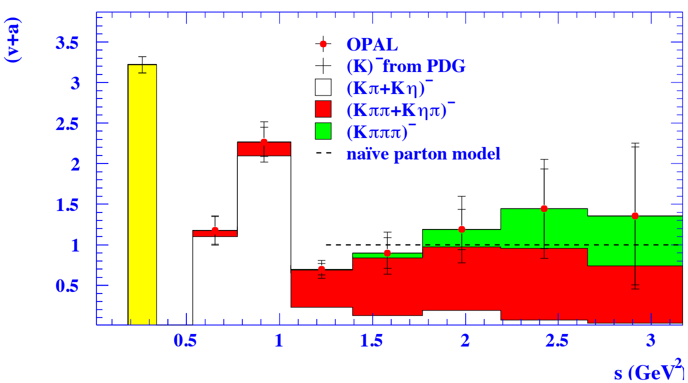
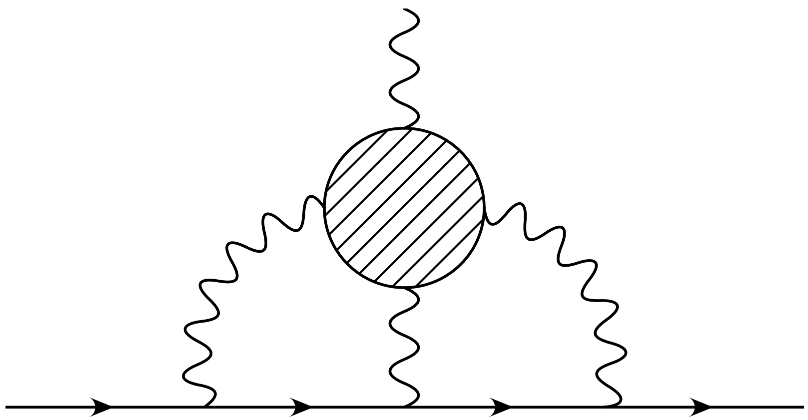


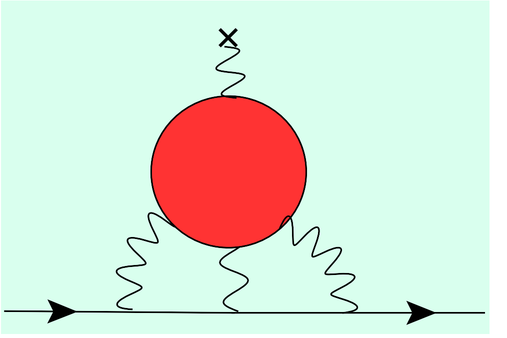
FIG. 5 Inclusive vector plus axial-vector (left) and vector minus axial-vector spectral function (right) as measured in (ALEPH Coll., 2005) (dots with errors bars) and (OPAL Coll., 1999b) (shaded one standard deviation errors). The lines show the predictions from the parton model (dotted) and from massless perturbative QCD using $\alpha_S(M_Z^2) = 0.120$ (solid). They cancel to all orders in the difference.



Hadronic Light-by-Light (HLbL) contributions



Hadronic Light-by-Light



- 4pt function of EM currents
- No experimental data directly help
- many Lorentz structure

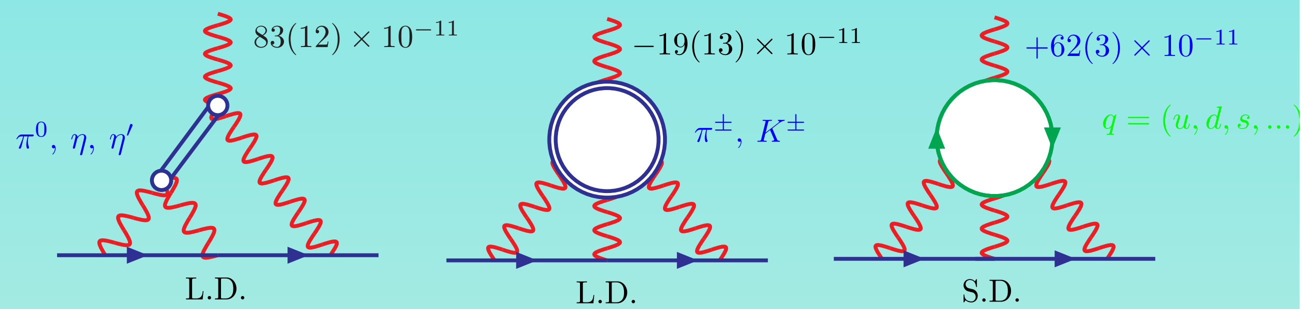
$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2,p_1) = ie^6\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2)}{k_1^2\,k_2^2\,k_3^2} \times\gamma_\nu S^{(\mu)}(\not{p}_2+\not{k}_2)\gamma_\rho S^{(\mu)}(\not{p}_1+\not{k}_1)\gamma_\sigma$$

$$\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2) = \int d^4x_1\,d^4x_2\,d^4x_3\;\exp[-i(k_1\cdot x_1+k_2\cdot x_2+k_3\cdot x_3)] \times\langle 0|T[j_\mu(0)j_\nu(x_1)j_\rho(x_2)j_\sigma(x_3)]|0\rangle$$

$$\text{Form factor}:\;\Gamma_\mu(q)=\gamma_\mu\,F_1(q^2)+\frac{i\sigma^{\mu\nu}q_\nu}{2\,m_l}\,F_2(q^2)$$

HLbL from Models

- Model estimate with non-perturbative constraints at the chiral / low energy limits using anomaly : (9—12) x 10^{-10} with 25-40% uncertainty

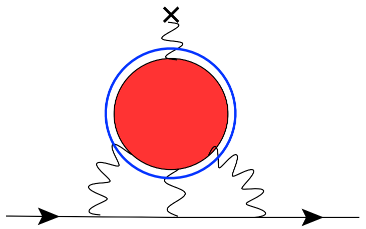


F. Jegerlehner

Contribution	BPP	HKS	KN	MV	PdRV	N/JN
π^0,η,η'	85±13	82.7±6.4	83±12	114±10	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	0±10	−19±19	−19±13
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	15±10	22± 5
scalars	−6.8±2.0	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	105±26	116±39

HVP like approach on lattice ?

- Calculate 4pt of EM currents



$$\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2) = \int d^4x_1\, d^4x_2\, d^4x_3\, \exp[-i(k_1\cdot x_1 + k_2\cdot x_2 + k_3\cdot x_3)] \times \underline{\langle 0|T[j_\mu(0)j_\nu(x_1)j_\rho(x_2)j_\sigma(x_3)]|0\rangle}$$

- One needs to calc. or fit all (q, k1,k2,k3) combination

$$\Gamma^{(\mathrm{Hlbl})}_\mu(p_2,p_1) = ie^6 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2)}{k_1^2\,k_2^2\,k_3^2} \times \gamma_\nu S^{(\mu)}(p_2+k_2)\gamma_\rho S^{(\mu)}(p_1+k_1)\gamma_\sigma$$

- Need to repeat (Volume)^3 times !

Direct 4pt calculation for selected kinematical range

- Jeremy Green (Mainz) Latice 2015, arXiv: 1507.01577
- Compute connected contribution of 4 pt function in momentum space
- forward amplitudes related to $\gamma^*\gamma^*$-> hadron cross section via dispersion relation

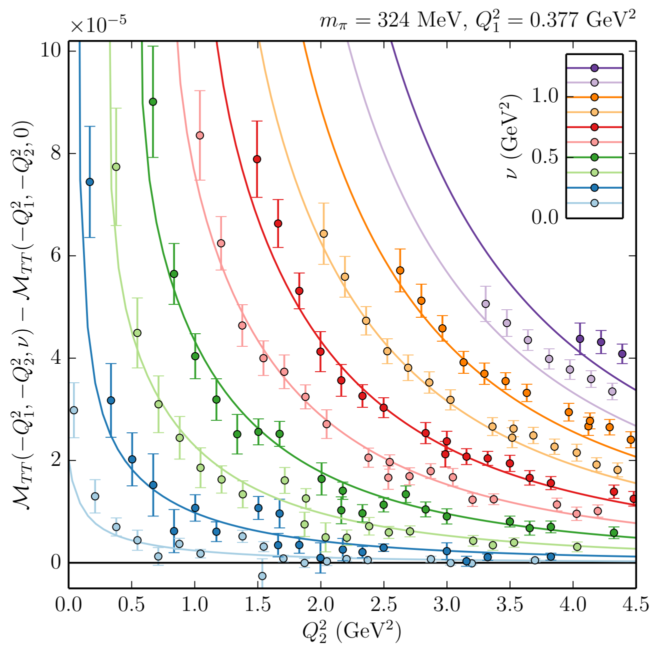


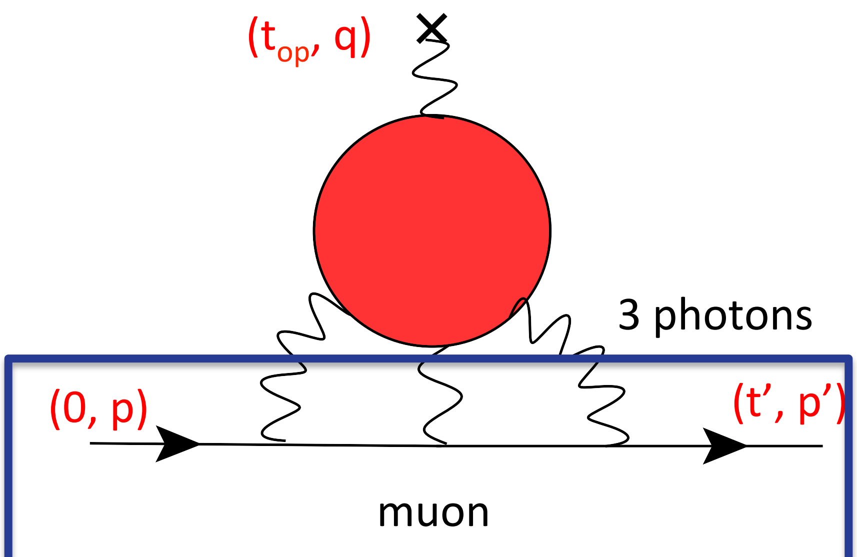
FIG. 3. The forward scattering amplitude $\mathcal{M}_{\rm TT}$ at a fixed virtuality $Q_1^2 = 0.377\mathrm{GeV}^2$, as a function of the other photon virtuality Q_2^2, for different values of ν. The curves represent the predictions based on Eq. (10), see the text for details.

$$\nu \equiv q_1 \cdot q_2.$$

Our strategy

- 4pt function has too much information to parameterize
- Do Monte Carlo integration for QED two-loop with 4 pt function $\pi^{(4)}$ which is sampled in lattice QCD
- Photon & lepton part of diagram is derived either in lattice QED+QCD [Blum et al 2014] (stat noise from QED), or exactly derive for given loop momenta [L. Jin et al 2015] (no noise from QED+lepton).

$$\Gamma_{\mu}^{(\mathrm{Hlbl})}(p_2, p_1) = ie^6 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \Pi^{(4)}_{\mu\nu\rho\sigma}(q, k_1, k_2, k_3)$$
$$\times \left[S(p_2)\gamma_\nu S(p_2 + k_2)\gamma_\rho S(p_1 + k_1)\gamma_\sigma S(p_1) + (\mathrm{perm.}) \right]$$



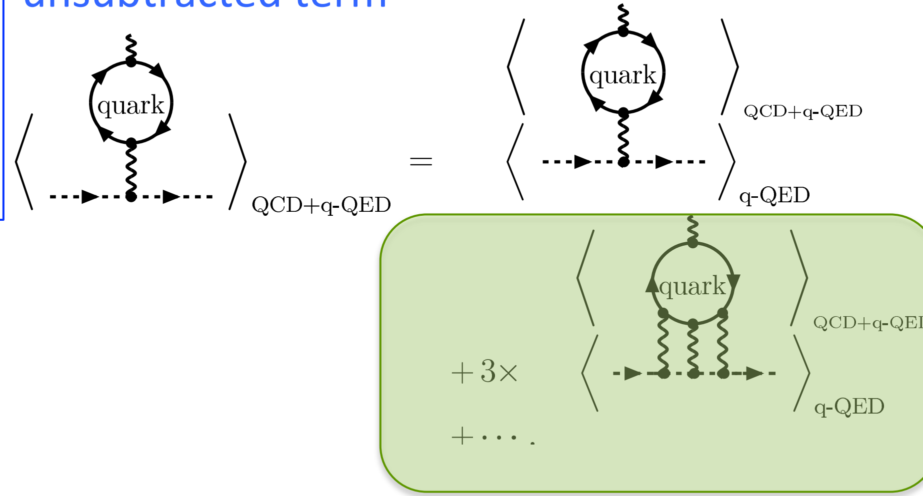
- set spacial momentum for
  - external EM vertex q
  - in- and out- muon p, p'
    q = p-p'
- set time slice of muon source(t=0), sink(t') and operator (t_{op})
- take large time separation for ground state matrix element

QCD+QED method [Blum et al 2014]

- One photon is treated analytically
- other two sampled stochastically a la LQED
- needs subtraction
- use AMA for error reduction
- use Furry's theorem to reduce α^2 noise

unsubtracted term

Subtraction term

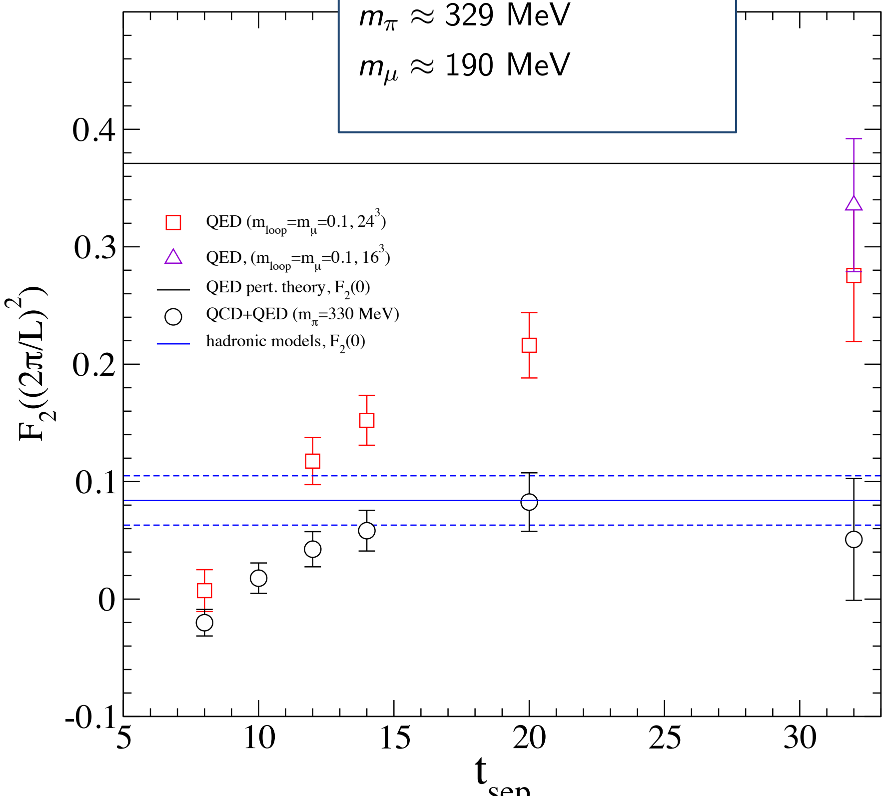


24^3 lattice size

$Q^2 = 0.11$ and $0.18\ \mathrm{GeV}^2$

$m_\pi \approx 329$ MeV

$m_\mu \approx 190$ MeV



- Connected part only

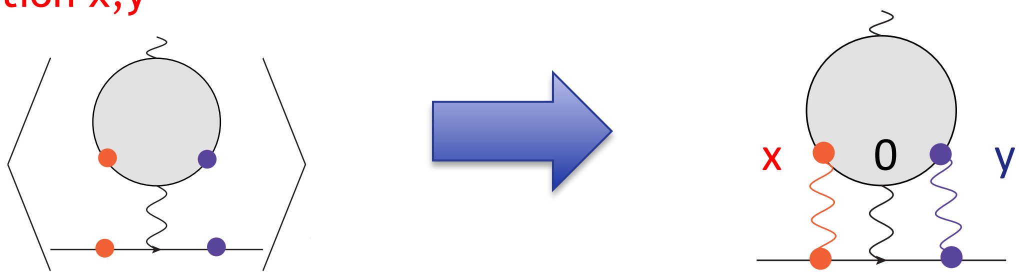
- QED only calculation consistent with QED loop calculation for larger volume

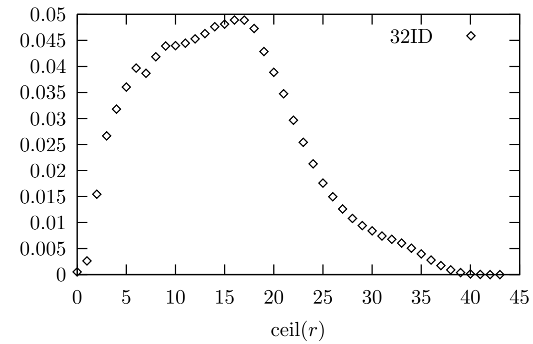
- QED+QCD
  - ball park of model values
  - significant exited state effects ?

Coordinate space Point photon method

[Jin, Blum, Christ, Hayakawa, TI, Lehner, et al. 2015]

- Treat all 3 photon propagators exactly (3 analytical photons) , which makes the quark loop and the lepton line connected :
  disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, (two of) quark-photon vertex location x,y





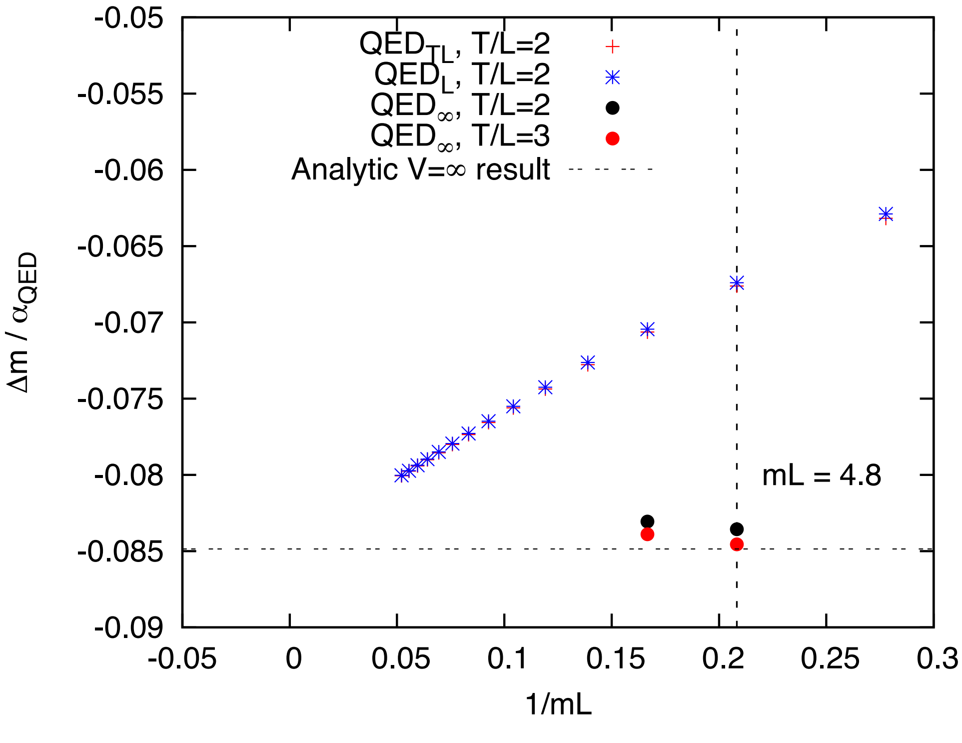
- Short separations, Min[|x|,|y|,|x-y|] < R ~ O(1) fm, which has a large contribution due to confinement, are summed for all points
- longer separations, Min[|x|,|y|,|x-y|] >= R, are stochastically with a probability shown above (Adaptive Monte Carlo sampling)

- All lepton and photon part produce no noise for given x,y (Ls = ∞ DWF muon)

We could examine different lepton/photon e.g. QED_L (Hayakwa-Uno 2008) with larger box, Twisting Averaging [Lehner TI LATTICE14] or Infinite Vol. Photon propagators [C. Lehner, L.Jin, TI LATTICE15]

Infinite volume photon propagator

Christoph Lehner



$$
\begin{aligned}
\hat{k}_\mu &= 2\sin k_\mu/2 \\
G(x) &= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{\hat{k}^2} \\
G_I(k') &= \sum_{x\in V} G(x)e^{-ik'x} \\
k' &= 2\pi n_\mu/L_\mu
\end{aligned}
$$

Current conservation & "-1" trick

- For HVP, conservation => transverse tensor

$$\Pi^{\mu\nu}(q) = (\hat{q}^2\delta^{\mu\nu} - \hat{q}^\mu\hat{q}^\nu)\Pi(\hat{q}^2)$$

- In infinite volume, q=0, $\Pi_{\mu\nu}(q) = 0$
- For finite volume, $\Pi_{\mu\nu}(0)$ is exponentially small
  e.g. DWF L=2, 3, 5 fm $\Pi_{\mu\nu}(0)$ = 8(3)e-4, 2(13)e-5, -1(5)e-8
- For general case with mass gap M, (L.Jin)

$$\int_V dx^4 \langle V_\mu(x)\mathcal{O}(0)\rangle = \int_V dx^4\,\partial_x\left(x\langle V_\mu(x)\mathcal{O}(0)\rangle\right)$$

$$= \int_{\partial V} dx^3\, x\langle V_\mu(x)\mathcal{O}(0)\rangle \propto L^4\exp(-ML/2) \to 0$$

- Subtract $\Pi_{\mu\nu}(0)$ alternates FVE, and reduce stat error
  "-1" subtraction trick :

$$\Pi^{\mu\nu}(q) - \Pi^{\mu\nu}(0) = \int d^4x(e^{iqx} - 1)\langle J^\mu(x)J^\nu(0)\rangle$$

Label	size	$m_\pi L$	m_π/GeV	#qcdtraj	t_{sep}	$\frac{F_2 \pm \mathrm{Err}}{(\alpha/\pi)^3}$	$\frac{\mathrm{Cost}}{\mathrm{BG/Q\ rack\ days}}$
16I	$16^3 \times 32$	3.87	0.423	16	16	0.1235 ± 0.0026	0.63
24I	$24^3 \times 64$	5.81	0.423	17	32	0.2186 ± 0.0083	3.0
24IL	$24^3 \times 64$	4.57	0.333	18	32	0.1570 ± 0.0069	3.2
32ID	$32^3 \times 64$	4.00	0.171	47	32	0.0693 ± 0.0218	10

Table 2. Central values and errors. $a^{-1} = 1.747\mathrm{GeV}$ except for 32ID where $a^{-1} = 1.371\mathrm{GeV}$. Muon mass and pion mass ratio is fixed at physical value. For comparison, at physical point, model estimation is 0.08 ± 0.02.

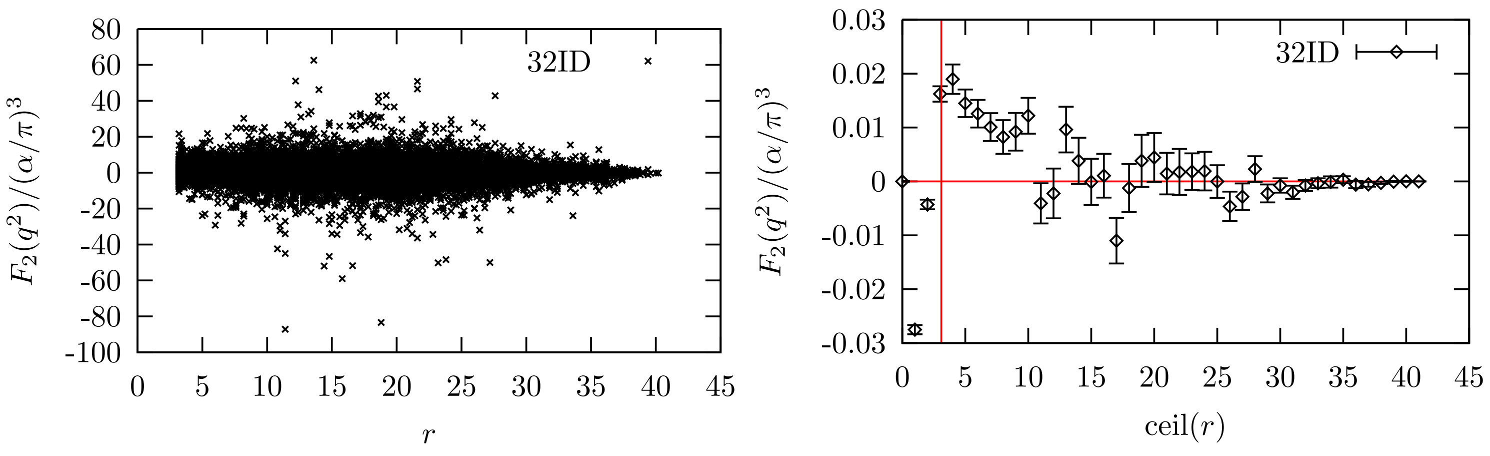
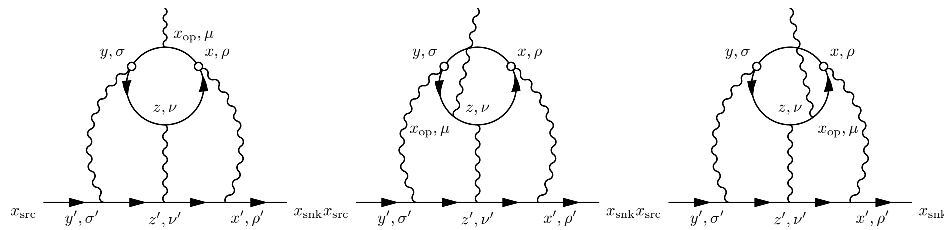


Figure 13. $32^3 \times 64$ lattice, with $a^{-1} = 1.371\mathrm{GeV}$, $m_\pi = 171\mathrm{MeV}$, $m_\mu = 134\mathrm{MeV}$.

Conserved current & moment method

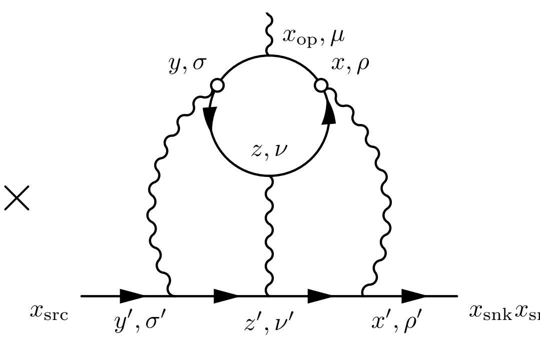
- To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external curents.



- This allows the "-1" subtraction trick seen in HVP. Now external current is also local x Z_V

- By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q->0 limit value is directly computed via the first moment of the relative coordinate, xop – (x+y)/2, by using "-1" trick, one could show

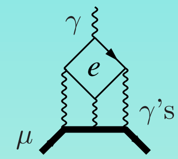
$$\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{\rm op}} (x_{\rm op} - (x+y)/2)_i \times$$



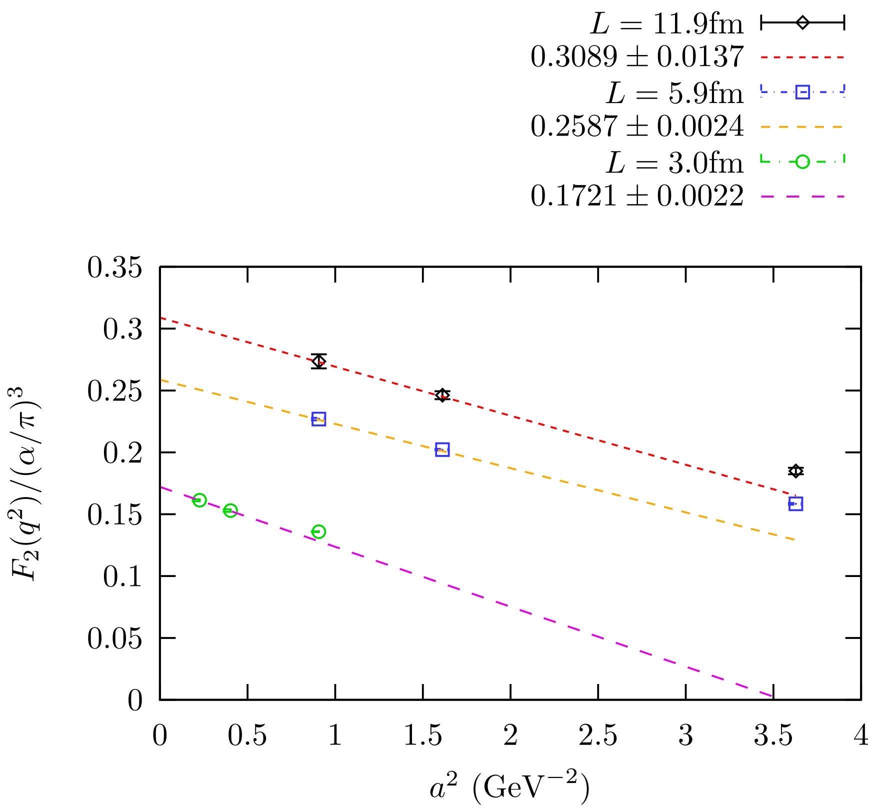
to directly get $F_2(q^2)$ without $q^2 \to 0$ extrapolation.

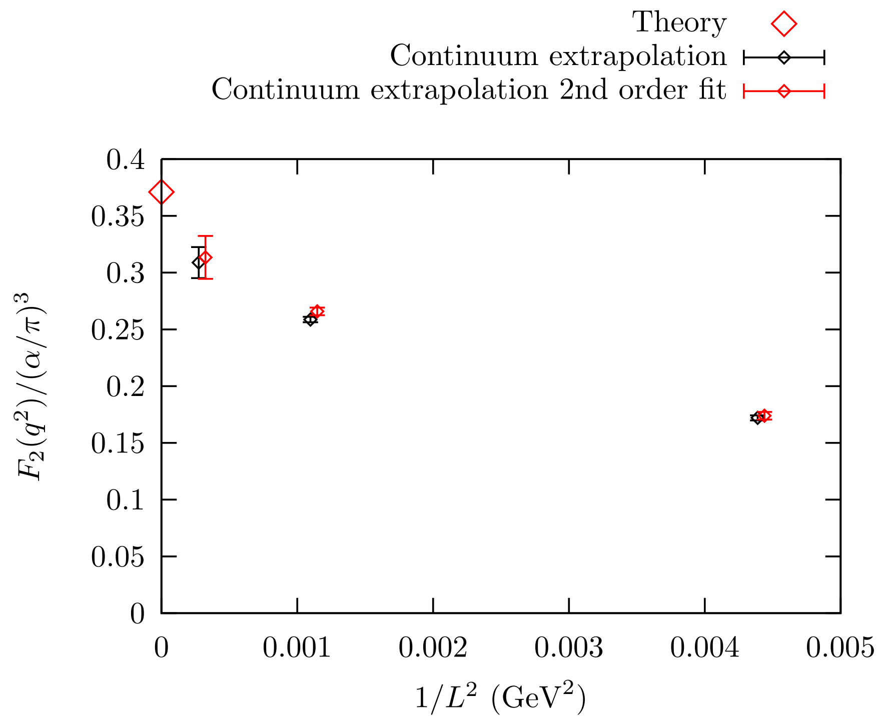
QED only study

$a = a\, m_\mu\ /\ 106\ \mathrm{MeV}$



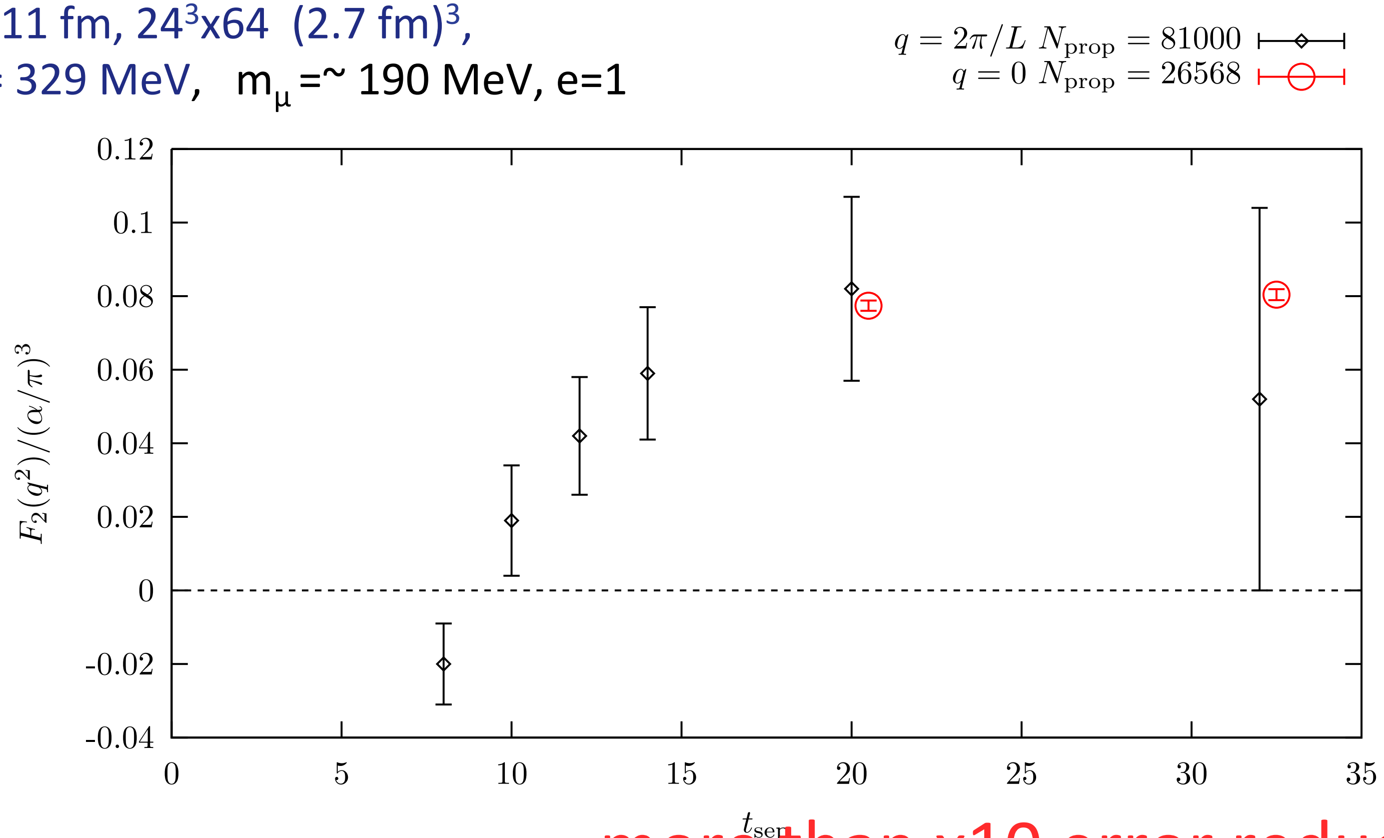
$$a_\mu^{(6)}(\mathrm{lbl}, e) = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \frac{59}{270}\pi^4 - 3\,\zeta(3) - \frac{10}{3}\pi^2 + \frac{2}{3} + O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^3$$





Dramatic Improvement !
Luchang Jin et al. , paper in prep.

a=0.11 fm, 24^3x64 (2.7 fm)^3,
m_π = 329 MeV, m_μ =~ 190 MeV, e=1

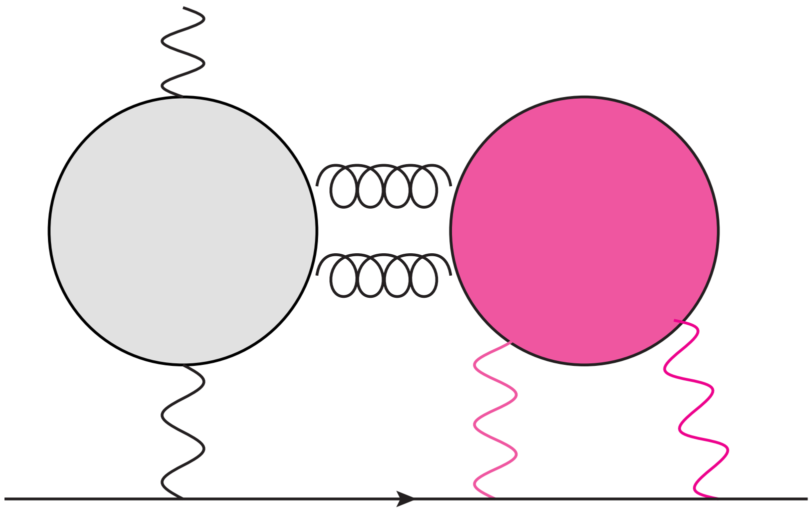


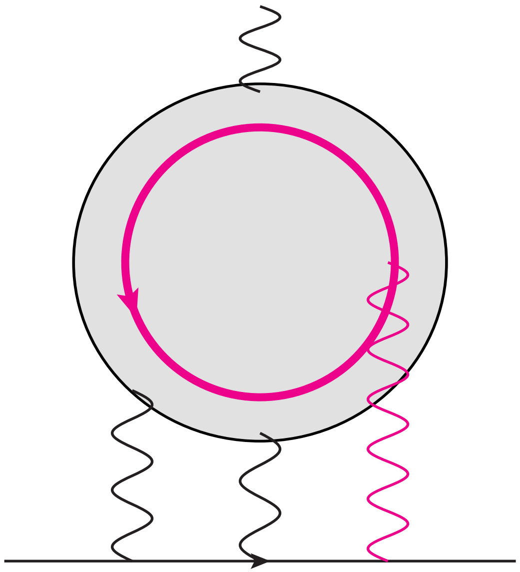
more than x10 error reduced !

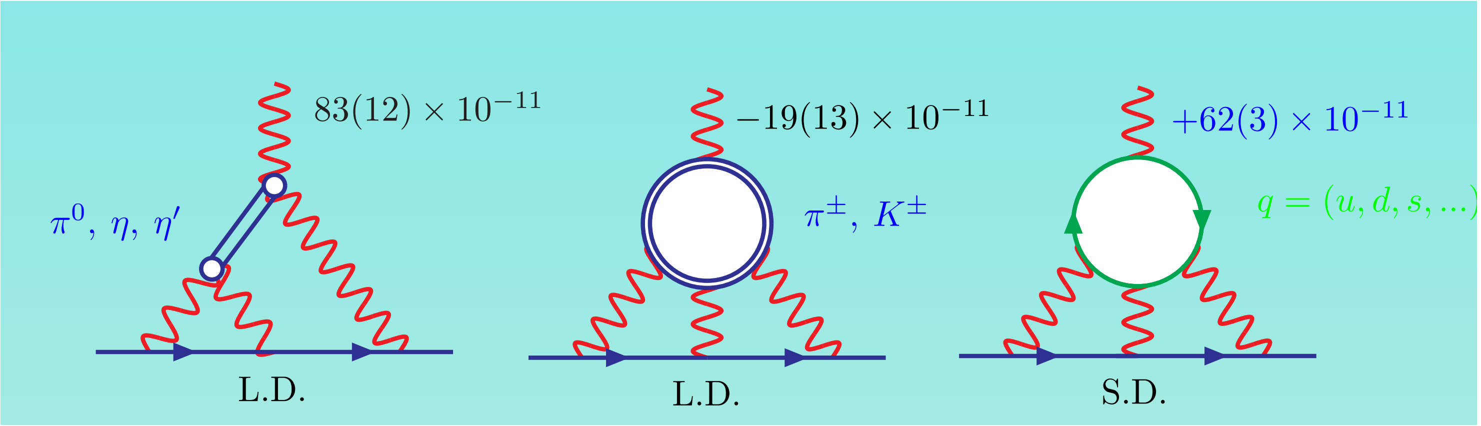
Also calculation speed up, x 200, using AMA, zMobius (Ls=10)
compared to traditional CG (Ls=24)

Disconnected diagrams in HLbL

- Missing disconnected diagrams

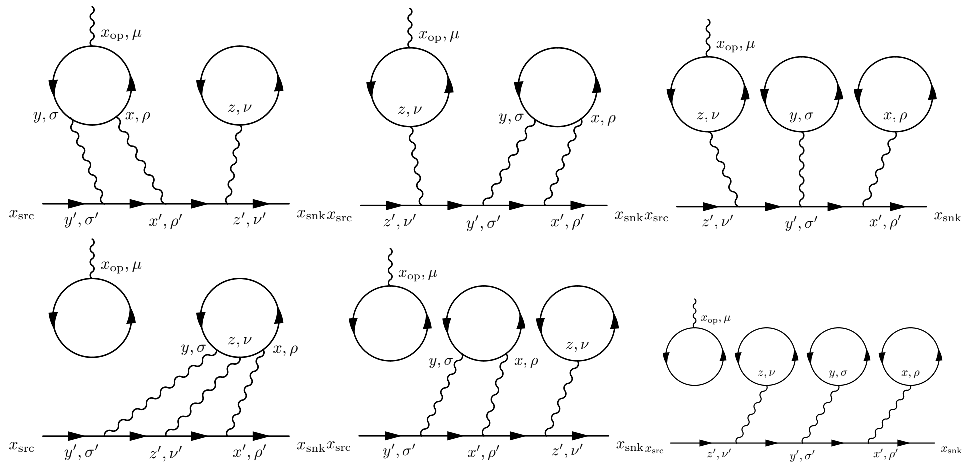






Two strategies for disconnected

- Compute disconnected loops made of valence quark



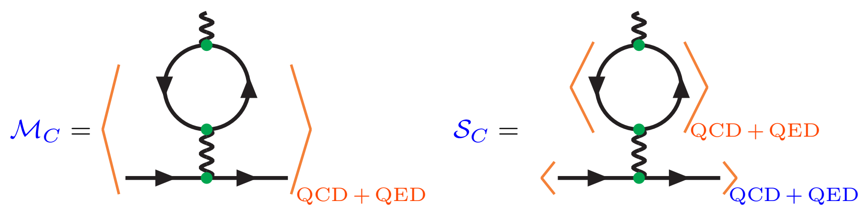
- Using dynamical QED+QED

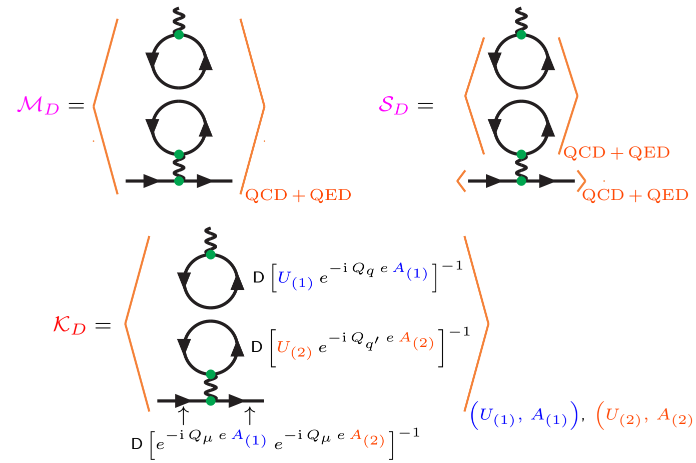
  Hayakawa, Latticer2015 talk

Disconnected quark loop using Dynamical QED + QCD

- M. Hayakawa Lattice2015 talk

$$\frac{1}{3}\{(\mathcal{M}_C - \mathcal{S}_C) + (\mathcal{M}_{C'} - \mathcal{S}_{C'}) + (\mathcal{M}_D - \mathcal{S}_D) - \mathcal{K}_D\}$$





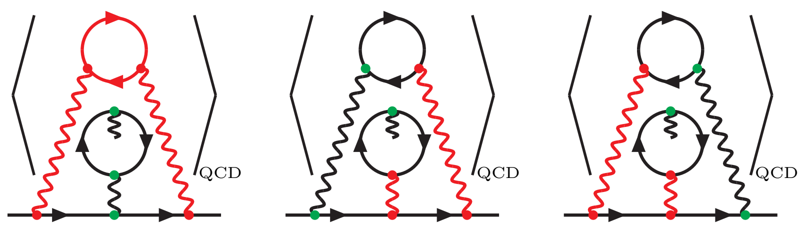


Figure: The same diagram of $(2_E, 2)$-type is generated in three ways from $\mathcal{M}_C$ (left) and $\mathcal{M}_D$ (middle, right). The red stuffs are generated by the ensemble average of (QCD + QED).

Summary of HLBL

- Connected part is in a good shape

- Finite Volume effect from photon : Twisting Averaging, Infinite volume photon propagator
- Current plans (2015-16) :

  - Nf=2+1 DWF/ Mobius ensemble at physical point, L=5.5 fm, a=0.11fm at ALCC @Argonne
  - using AMA with zMobius fermion [C. Jung.
    (more than x100 generic cost reduction to traditional CG)

  - Also HVP, EM splitting, EW 2-loop VVA [M. Knect] ... as byproducts.
- Disconnected quark loop. Can we come up an estimation how large would it be ?

Backup slides

[Blum, TI, Shintani PRD 88 (2013) 094503]

Covariant Approximation Averaging (CAA) a new class of Error reduction techniques

Original

$$\mathcal{O} = \mathcal{O}^{(\mathrm{appx})} + \mathcal{O}^{(\mathrm{rest})}$$

unbiased imporved

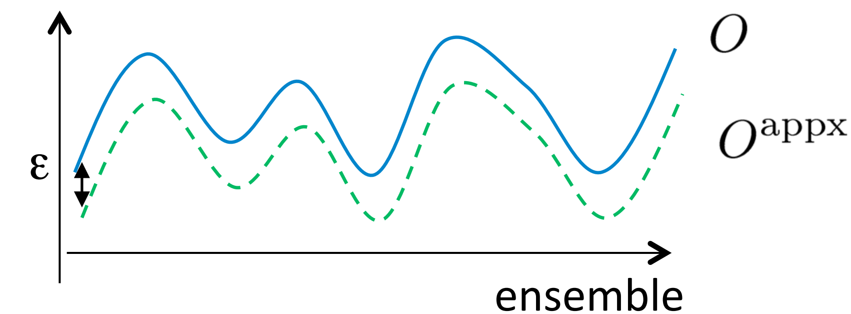
$$\mathcal{O}^{(\mathrm{imp})} = \mathcal{O}^{(\mathrm{rest})} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{(\mathrm{appx}),g}$$

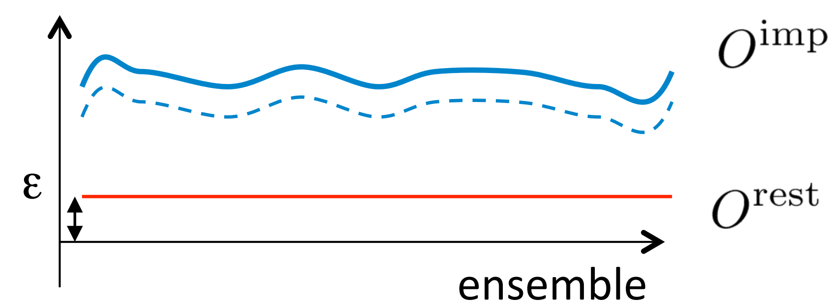
Lattice Symmetry

Expensive : infrequently measured

Cheap : frequently measured

- $O^{(\mathrm{imp})}$ has smaller error
  $O^{(\mathrm{appx})}$ need to be cheap & not to be too accurate
  N_G suppresses the bulk part of noise cheaply





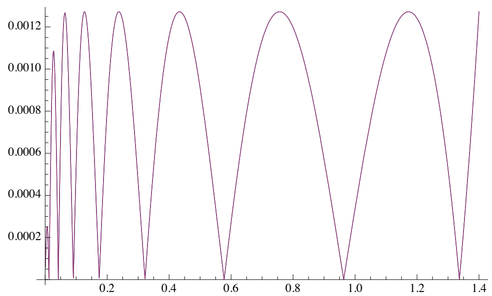
New bias-free estimator even without covariant approximation by a stochastic choice of source location for the exact/rest computation is now available : Appendix D of arXiv:1402.0244

AMA+MADWF(fastPV)+zMobius accelerations

- We utilize complexified 5d hopping term of Mobius action [Brower, Neff, Orginos], zMobius, for a better approximation of the sign function.

$$\epsilon_L(h_M) = \frac{\prod_s^L(1+\omega_s^{-1}h_M) - \prod_s^L(1-\omega_s^{-1}h_M)}{\prod_s^L(1+\omega_s^{-1}h_M) + \prod_s^L(1-\omega_s^{-1}h_M)}, \qquad \omega_s^{-1} = b + c \in \mathbb{C}$$

- 1/a~2 GeV, Ls=48 Shamir ~ Ls=24 Mobius (b=1.5, c=0.5) ~ Ls=10 zMobius (b_s, c_s complex varying) ~5 times saving for cost AND memory



Ls	|eps(48cube) – eps(zMobius)|
6	0.0124
8	0.00127
10	0.000110
12	8.05e-6

- The even/odd preconditioning is optimized (sym2 precondition) to suppress the growth of condition number due to order of magnitudes hierarchy of b_s, c_s [also Neff found this]

$$\mathtt{sym2}: \quad 1 - \kappa_b M_4 M_5^{-1} \kappa_b M_4 M_5^{-1}$$

- Fast Pauli Villars (mf=1) solve, needed for the exact solve of AMA via MADWF (Yin, Mawhinney) is speed up by a factor of 4 or more by Fourier acceleration in 5D [Edward, Heller]
- All in all, sloppy solve compared to the traditional CG is 160 times faster on the physical point 48 cube case. And ~100 and 200 times for the 32 cube, Mpi=170 MeV, 140, in this proposal (1,200 eigenV for 32cube) .

$$\underbrace{\frac{20,000}{600}}_{\text{MADWF+zMobius+deflation}} \times \underbrace{\frac{600*32/10}{300}}_{\text{AMA+zMobius}} = 33.3 \times 6.4 = \underline{210 \text{ times faster}}$$

Examples of Covariant Approximations (contd.)

- **All Mode Averaging AMA**

  Sloppy CG or Polynomial approximations

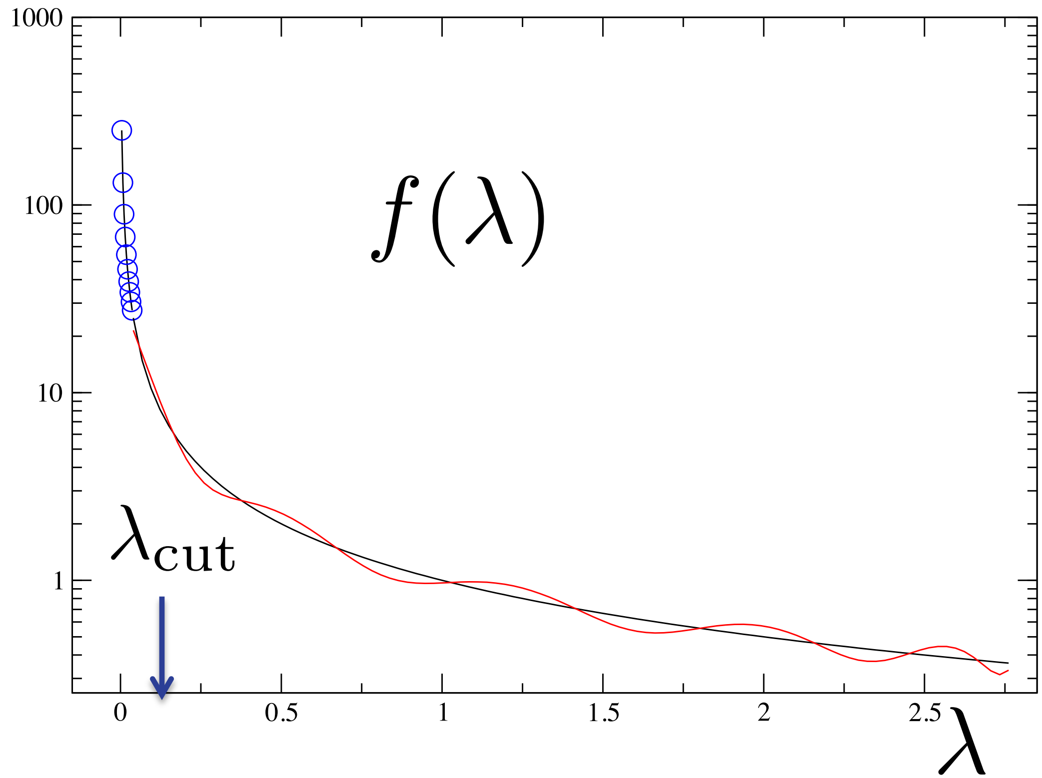
$$\mathcal{O}^{(\mathrm{appx})} = \mathcal{O}[S_l],$$

$$S_l = \sum_\lambda v_\lambda f(\lambda) v_\lambda^\dagger,$$

$$f(\lambda) = \begin{cases} \frac{1}{\lambda}, & |\lambda| < \lambda_{\mathrm{cut}} \\ P_n(\lambda) & |\lambda| > \lambda_{\mathrm{cut}} \end{cases}$$

$$P_n(\lambda) \approx \frac{1}{\lambda}$$

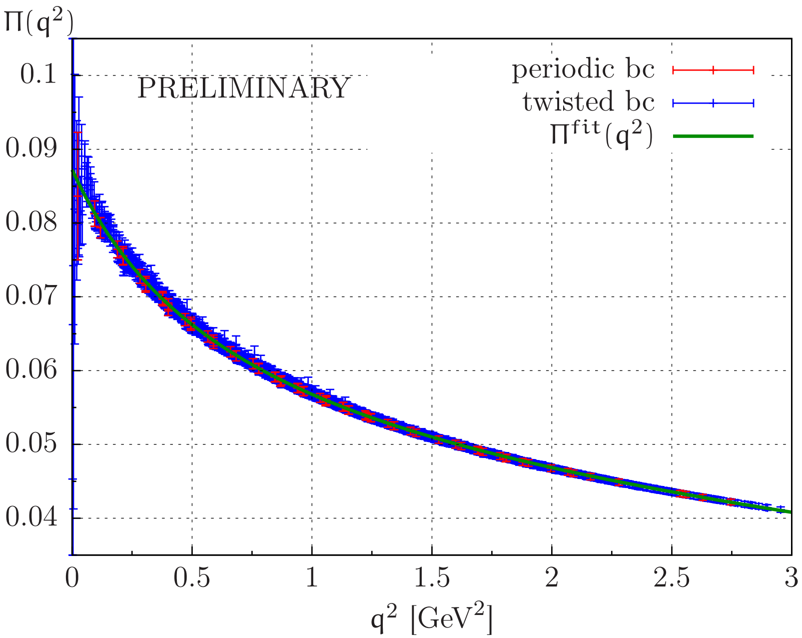
If quark mass is heavy, e.g. ~ strange, low mode isolation may be unneccesary



accuracy control :

- low mode part : # of eig-mode
- mid-high mode : degree of poly.

Twisted boundary condition



B. Jaeger [Mainz group] @ Lattice 2012

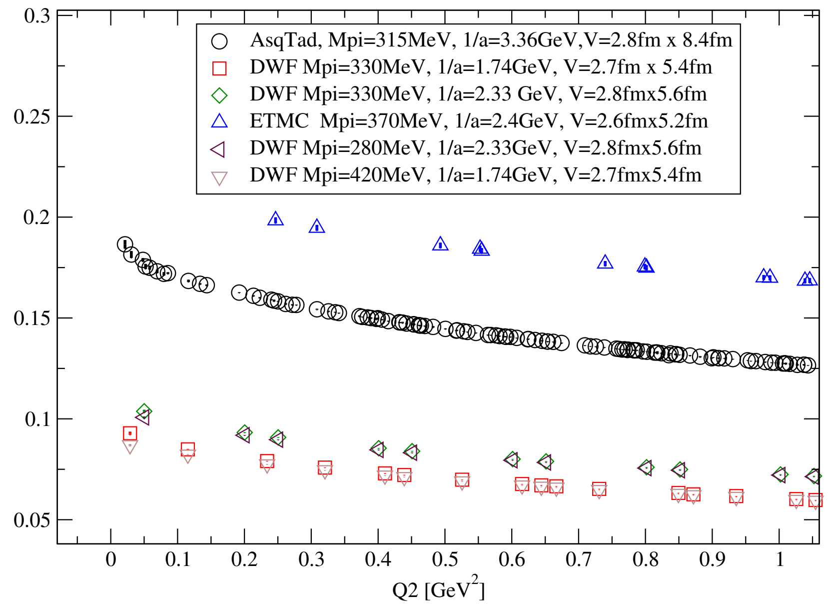
- On a torus, the action must be single-valued, while fields do not have to be.
- Impose the twisted boundary condition on quark fields.

$$q(x+L) = q(x)\exp(i\theta)$$
$$\rightarrow \quad p = (2\pi n + \theta)/L$$
$$(\theta\text{ :arbitrary input})$$

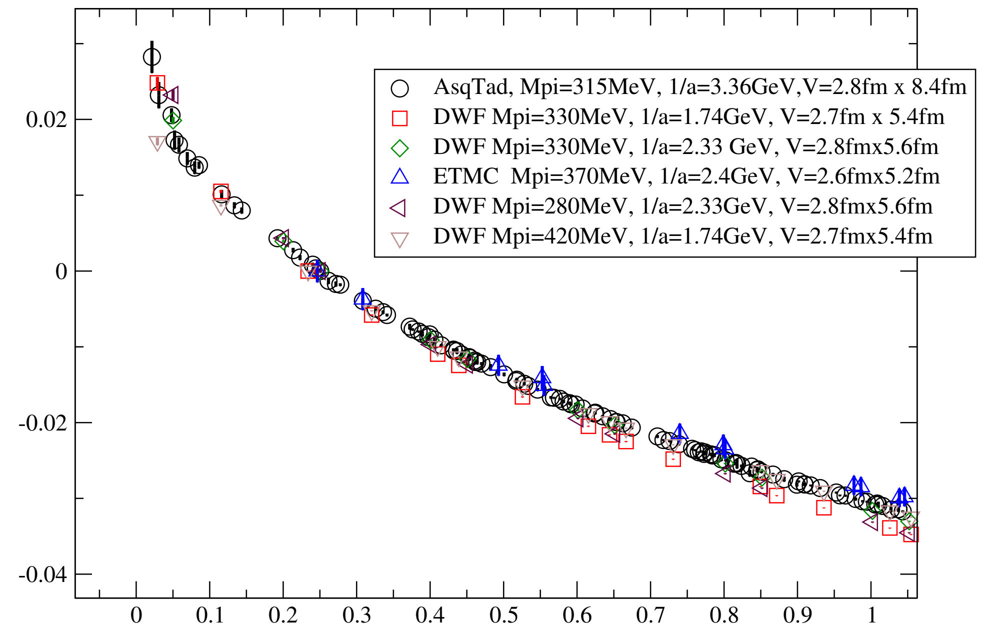
- q^2 can be arbitrary small.
- Breaking isospin, Vector ward identity is broken, could be exactly subtracted [Aubin et al 2012]
- Noise in small q2

HVP comparison

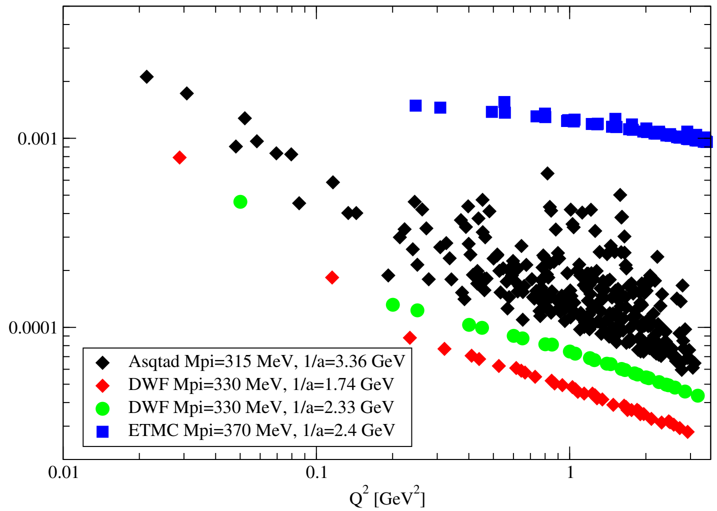
Pi(Q2) - Pi(Q2~0.25 GeV^2)



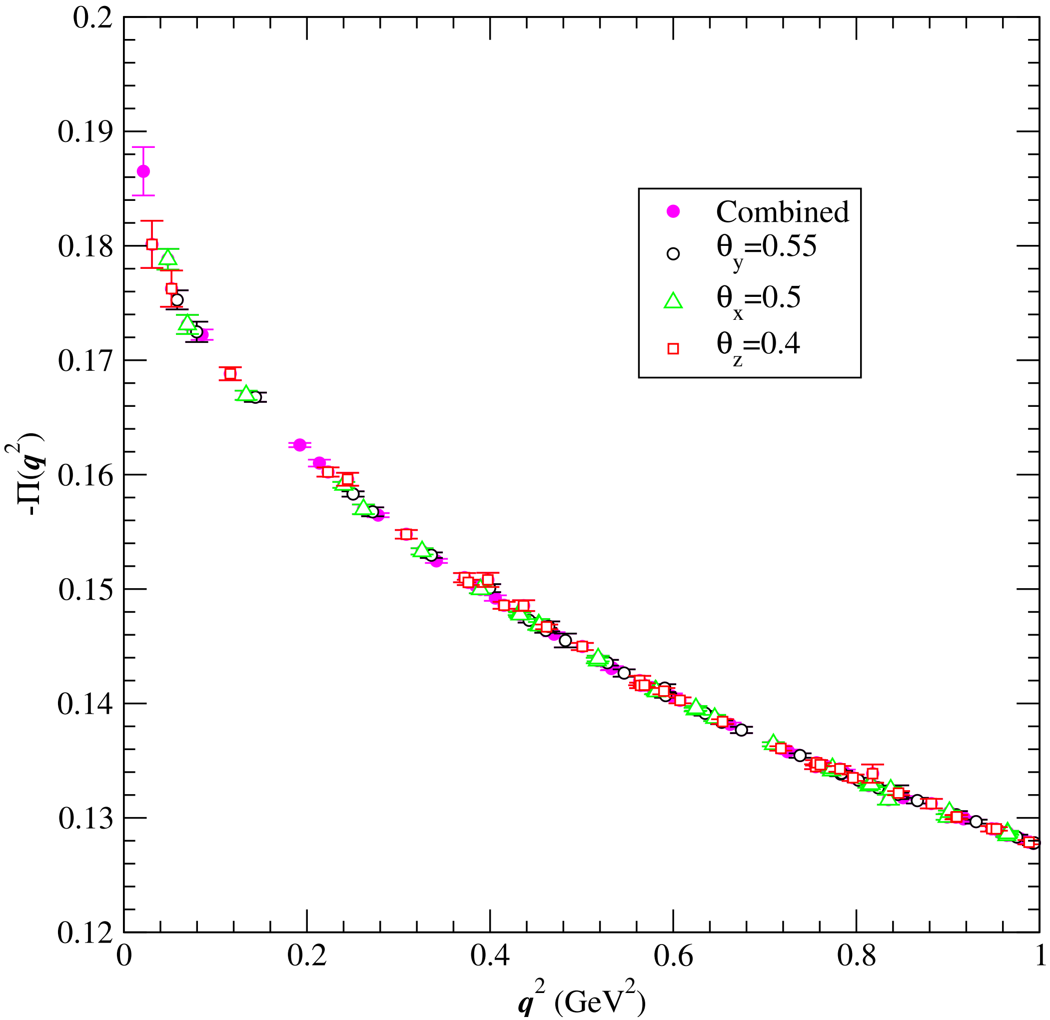
Pi(Q2) - Pi(Q2~0.25 GeV^2)



Absolute error Pi(Q2)



- AsqTad, DWF, ETM
- around
  Mpi ~ 300 MeV, L~3fm,
  1/a=1.7-3.3 GeV
- AsqTad : 48^3x144,
  theta(twist)=0.55, 0.5, 0.4



-Π(q^2)
0.12
0.13
0.14
0.15
0.16
0.17
0.18
0.19
0.2
0
0.2
0.4
0.6
0.8
1
q^2 (GeV^2)
Combined
θ_y=0.55
θ_x=0.5
θ_z=0.4

Fit functions

- Vector Meson Dominance

$$\Pi_V^{tree}(Q^2) = \frac{2}{3}\frac{f_V^2}{Q^2 + m_V^2}$$

- Multi point Pade fit [2012, Aubin et al.]

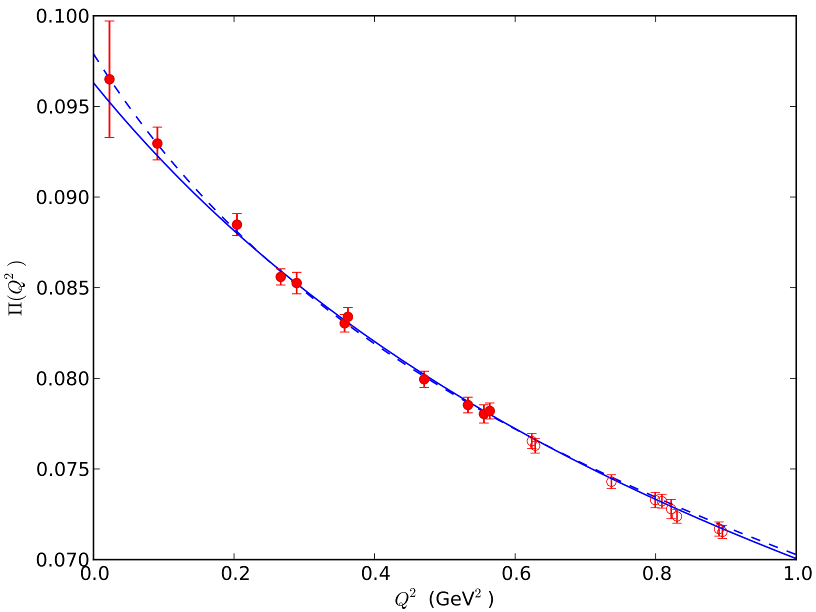
$$\Pi(Q^2) = \Pi(0) - Q^2\left(a_0 + \sum_{n=1}^{[P/2]} \frac{a_n}{b_n + Q^2}\right)$$

Conditions : $a_n > 0$, $\underline{b_n > 4\, m_\pi^2}$

- In principle, these are only true at the continuum limit (but not necessarily infinite volume limit)

Pade fit results

- solid: correlated fit (q2 <=0.6 GeV2) ,
  dash : uncorrelated fit (q2 <= 1 GeV2)



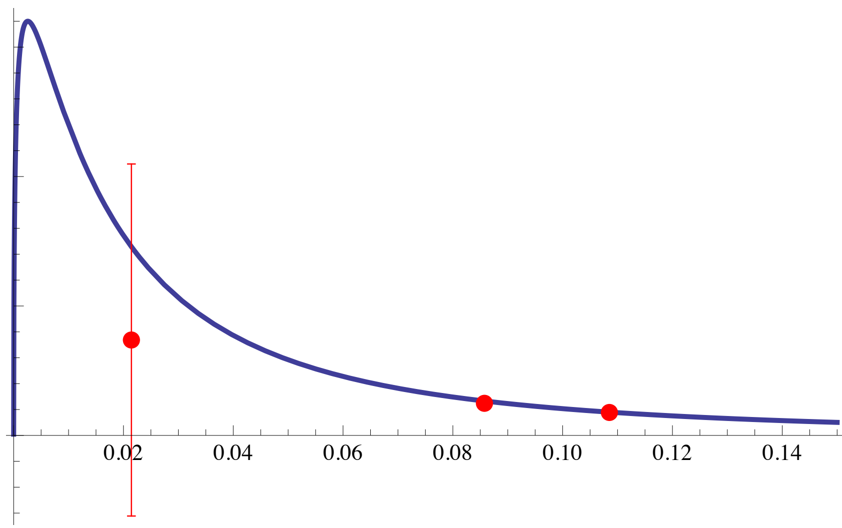
	χ^2/dof	$10^{10}a_\mu^{\mathrm{HLO},Q^2\le 1}$	$\Pi(0)$	a_i	b_i	a_0
VMD	38.6/18	646(8)	0.1222(6)	0.0595(8)	0.64 (fixed)	–
[0, 1]	14.3/17	550(20)	0.1203(7)	0.0646(16)	0.83(5)	–
[1, 1]	13.9/16	572(41)	0.1206(8)	0.052(16)	0.68(20)	0.005(7)
[1, 2]	13.9/15	572(37)	0.1206(8)	0.052(14)	0.68(19)	–
				1(6)	$0.3(1.0)\times 10^3$	
[2, 2]	13.9/14	572(38)	0.1206(8)	0.052(14)	0.68(18)	0.003(27)
				1(31)	$0.4(6.0)\times 10^3$	

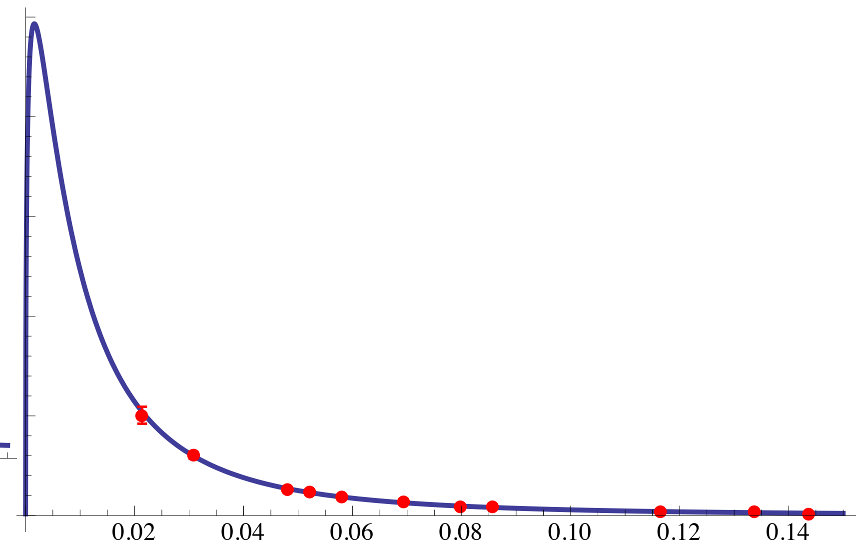
[Aubin et al. Phys. Rev. D 86 (2012) 054509]

- Pade approximation converges, results stable.

AMA + twisting

- AsqTad plot from Golterman (preliminary)



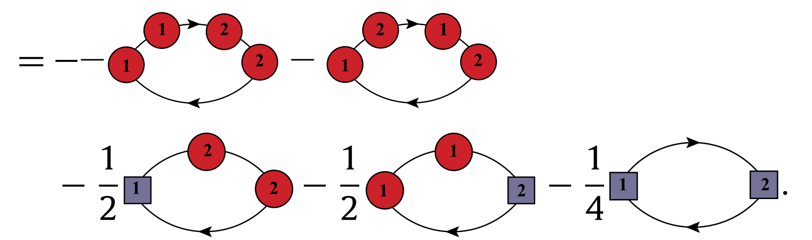


- Assumptions behind the Fit & discretization error ?

Subtraction Strategy: Derivative of Twisting Angle [Divitiis et al. PLB 718(2012) 589]

- $p_i = (2\pi n + \theta_i) / L$

$$\Pi(0) = -\left.\frac{\partial^2 \hat{C}^{12}(p)}{\partial p_1 \partial p_2}\right|_{p^2=0}$$



$$C^{\mu\nu}(p) = \frac{1}{(TL^3)^2} \sum_{x,y} e^{ip(y-x+\hat{\nu}/2-\hat{\mu}/2)} \langle V^{\mu}_{em}(x) V^{\nu}_{em}(y) \rangle$$
$$= \left(\delta^{\mu\nu} \hat{p}^2 - \hat{p}^{\mu} \hat{p}^{\nu}\right) \Pi(p^2),$$