QED corrections to hadronic processes in Lattice QCD

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QED corrections to hadronic processes in Lattice QCD

In collaboration with:

N. Carrasco, VL, G. Martinelli, C.T. Sachrajda,
N. Tantalo, C. Tarantino, M. Testa

PRD 91 (2015) 074506, arXiv: 1502.00257

Outline:

1) Phenomenological motivations

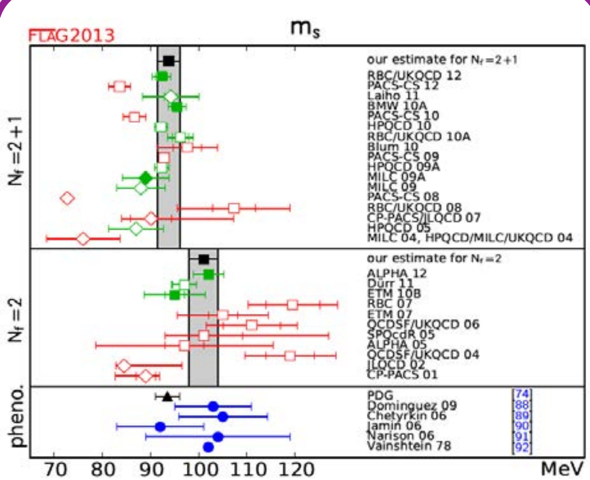
2) The method

3) Results of a preliminary numerical study (not in the paper)

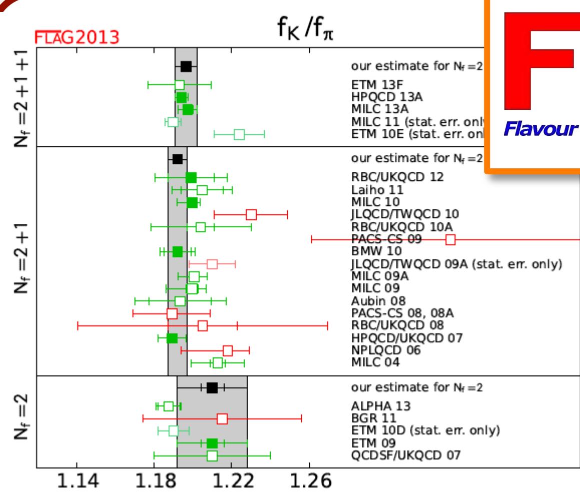
Motivations

The accuracy of lattice calculations of hadron spectrum (and hence of the quark masses) and of the decay constants and form factors is such that

electromagnetic corrections and isospin breaking effects cannot be neglected anymore

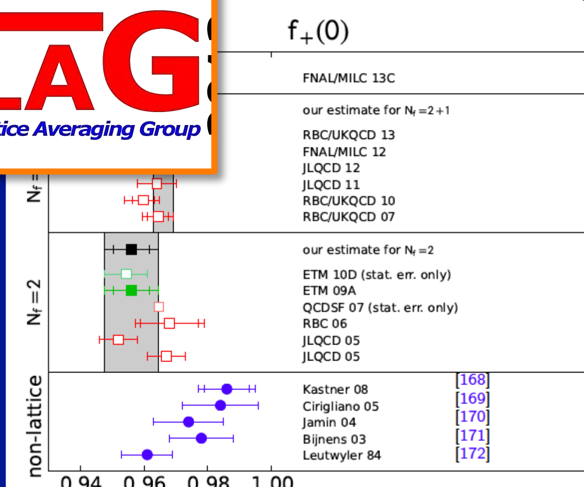


$\overline{m}_s = 93.8(2.4)$ MeV **2.6%**



FLAG
Flavour Lattice Averaging Group

$f_K / f_\pi = 1.192(5)$ **0.4%**

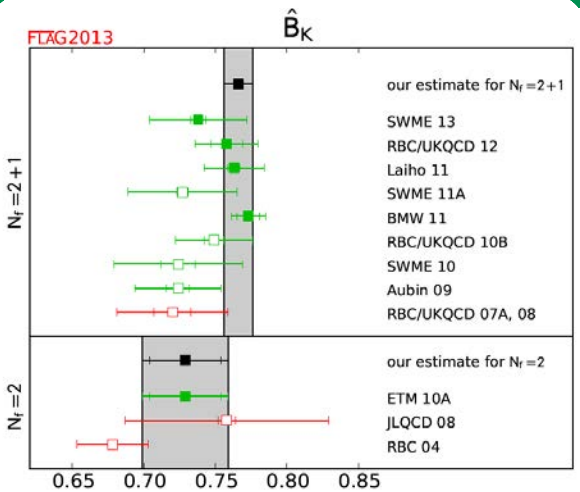


$f_+^{K\pi}(0) = 0.966(3)$ **0.3%**

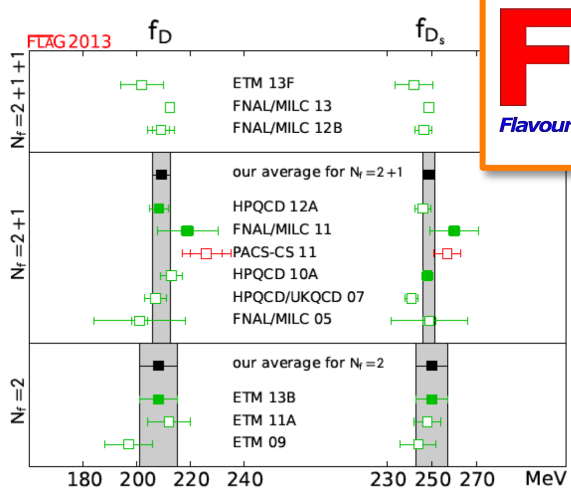
Motivations

The accuracy of lattice calculations of hadron spectrum (and hence of the quark masses) and of the decay constants and form factors is such that

electromagnetic corrections and isospin breaking effects cannot be neglected anymore

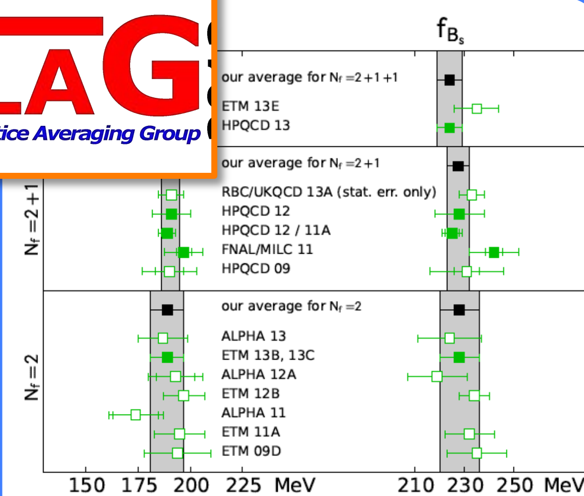


$\hat{B}_K = 0.7661(99)$ 1.2%



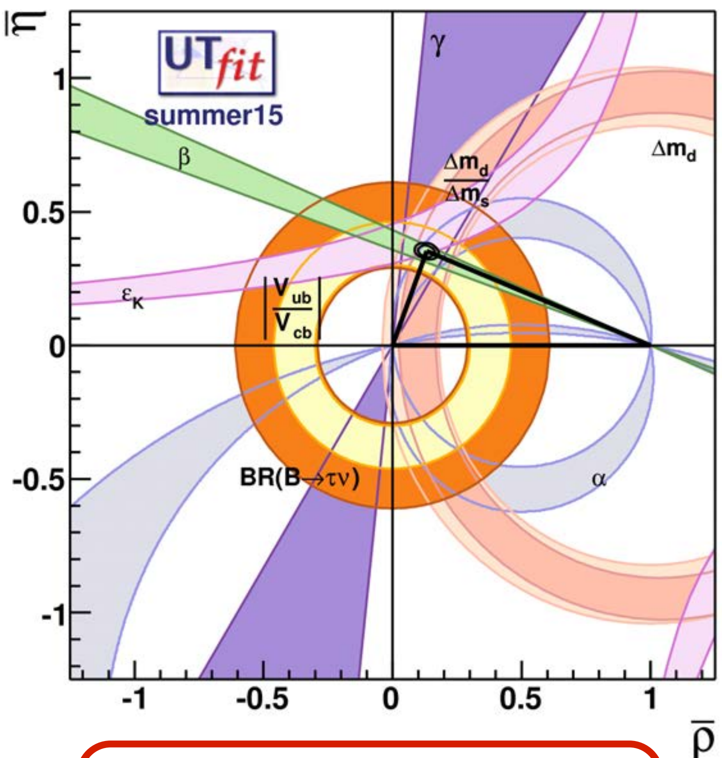
$f_{Ds} = 250(7)$ MeV 2.8%

FLAG Flavour Lattice Averaging Group



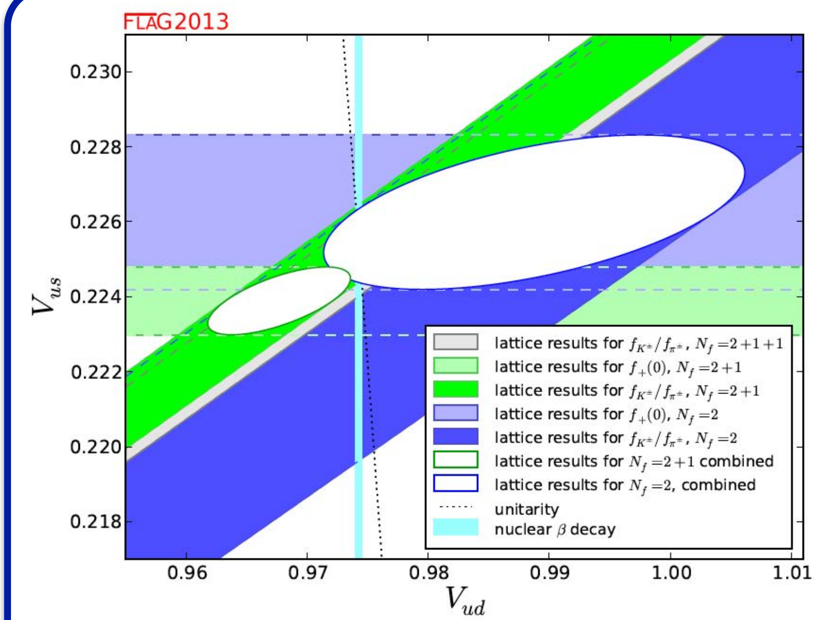
$f_{Bs} = 227.7(4.5)$ Me 2.0%

Motivations



$\bar{\rho}$ = 0.142 ± 0.018 **13%**

$\bar{\eta}$ = 0.340± 0.012 **3.5%**



$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1$

$= (-7 \pm 6) \times 10^{-4}$ from Kl3

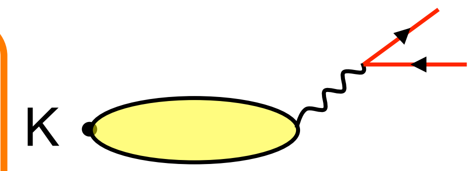
$= (\ 0 \pm 6) \times 10^{-4}$ from Kl2

+ nuclear β decay

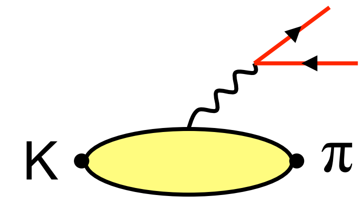
The determination of Vus and Vud

- The relevant processes are the leptonic and semileptonic K and π decays

$$\frac{\Gamma\left(K^+ \to \ell^+\nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+\nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{m_K\left(1-m_\ell^2/m_K^2\right)^2}{m_\pi\left(1-m_\ell^2/m_\pi^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$



$$\Gamma\left(K\to\pi\ell\nu(\gamma)\right) = \frac{G_F^2 m_K^5}{192\pi^3} C_K^2 S_{EW}\left(|V_{us}|\, f_+^{K^0\pi^-}(0)\right)^2 I_{K\ell}\left(1+\delta_{EM}^{K\ell}+\delta_{SU(2)}^{K\pi}\right)^2$$



- From the experimental measurements of the decay rates

$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi} = 0.2758(5)$$

$$|V_{us}|\, f_+^{K^0\pi^-}(0) = 0.2163(5)$$

FlaviA net Kaon WG

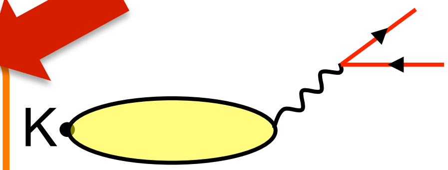
The accuracy is at the level of 0.2% for both determinations

M.Antonelli *et al.*, EPJ C69 (2010) 399

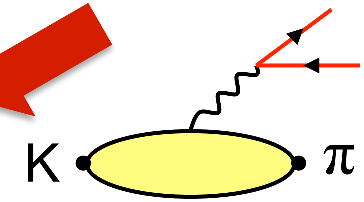
Electromagnetic and isospin breaking effects

- An important source of uncertainty are long distance electromagnetic and SU(2) breaking corrections

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{m_K\left(1-m_\ell^2/m_K^2\right)^2}{m_\pi\left(1-m_\ell^2/m_\pi^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$



$$\Gamma\left(K \to \pi \ell \nu(\gamma)\right) = \frac{G_F^2 m_K^5}{192\pi^3} C_K^2 S_{EW} \left(|V_{us}|\, f_+^{K^0\pi^-}(0)\right)^2 I_{K\ell} \left(1+\delta_{EM}^{K\ell}+\delta_{SU(2)}^{K\pi}\right)^2$$



For $\Gamma_{Kl2}/\Gamma_{\pi l2}$

At leading order in ChPT both δ_{EM} and $\delta_{SU(2)}$ can be expressed in terms of physical quantities (e.m. pion mass splitting, f_K/f_π, …)

- $\delta_{EM} = -0.0069\,(17)$ 25% of error due to higher orders ➡ 0.2% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

M.Knecht *et al.*, EPJ C12 (2000) 469; V.Cirigliano, H.Neufeld, PLB 700 (2011) 7

- $\delta_{SU(2)} = \left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}\right)^2 - 1 = -0.0044\,(12)$ 25% of error due to higher orders ➡ 0.1% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

J.Gasser, H.Leutwyler, NPB 250 (1985) 465; V.Cirigliano, H.Neufeld, PLB 700 (2011) 7

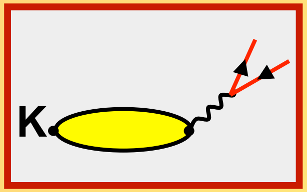
ChPT is not applicable to D and B decays. Estimates are model dependent.

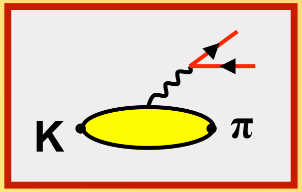
$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi} = 0.2758(5)$$ 0.2%

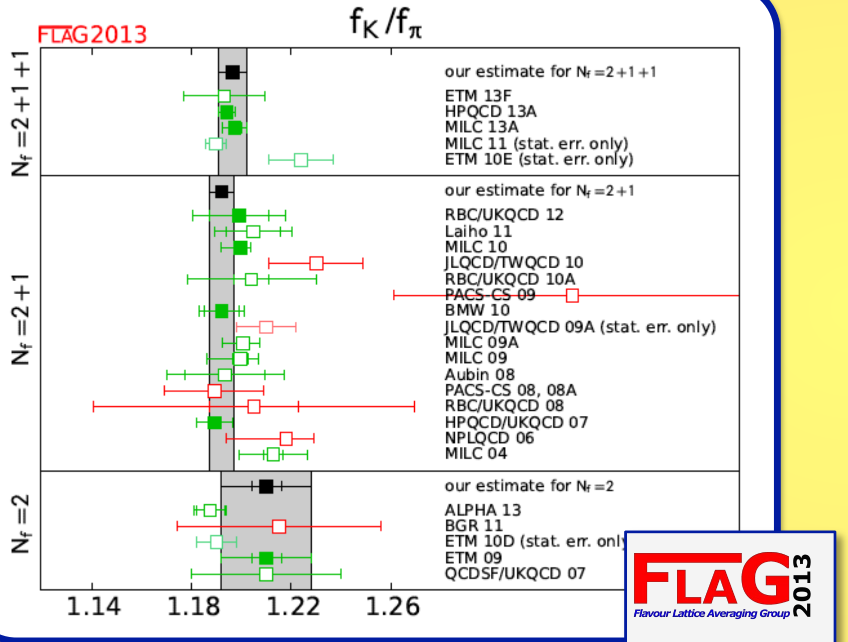
FlaviA net Kaon WG

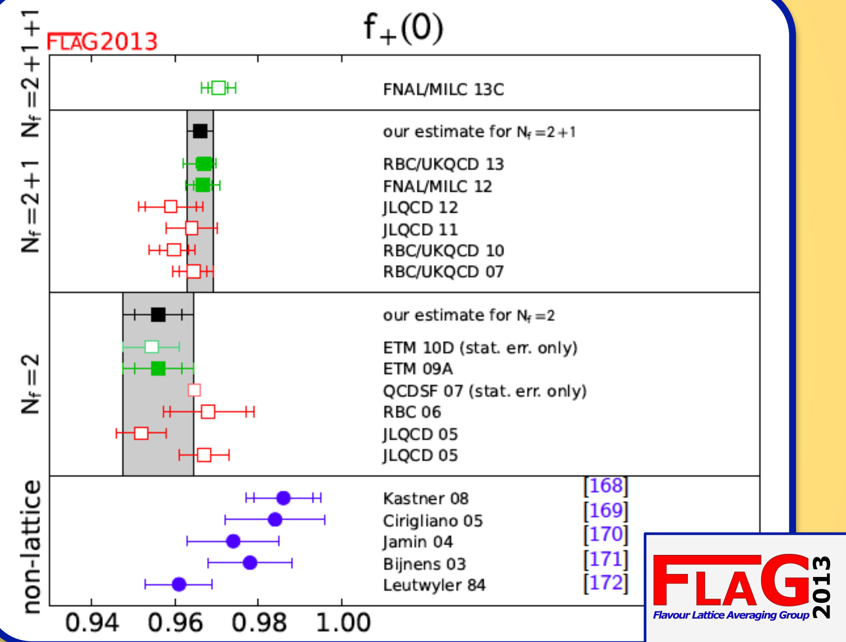
$$|V_{us}|\, f_+^{K^0\pi^-}(0) = 0.2163(5)$$ 0.2%

Lattice results for f_K/f_π and $f_+(0)$









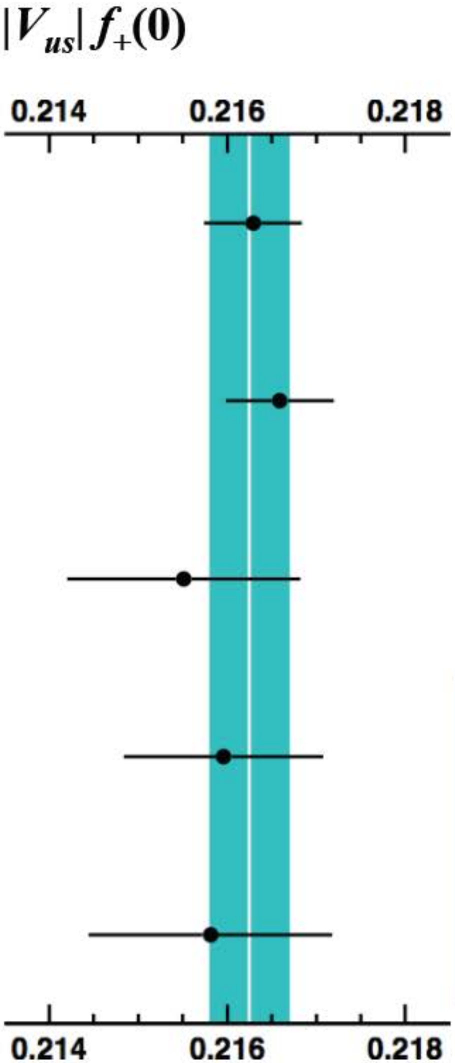
$f_{K^+}/f_{\pi^+} = 1.194(5)$ Nf=2+1+1 0.4%

$f_{K^+}/f_{\pi^+} = 1.192(5)$ Nf=2+1

$f_+(0) = 0.970(3)$ Nf=2+1+1 0.3%

$f_+(0) = 0.966(3)$ Nf=2+1

| Vus | f+(0) from world data: 2012



Approx. contrib. to % err from:

		% err	BR	τ	$\delta_{SU,EM}$	Int
$K_L e3$	**0.2163(5)**	**0.26**	0.09	0.20	0.11	0.05
$K_L \mu 3$	**0.2166(6)**	**0.28**	0.15	0.18	0.11	0.06
$K_S e3$	**0.2155(13)**	**0.61**	0.60	0.02	0.11	0.05
$K^{\pm} e3$	**0.2160(11)**	**0.52**	0.31	0.09	0.41	0.04
$K^{\pm} \mu 3$	**0.2158(13)**	**0.63**	0.47	0.08	0.41	0.06

Average: $|V_{us}| f_+(0)$ = 0.2163(5) χ^2/ndf = 0.84/4 (93%)

QED and Isospin corrections to hadron masses

Two approaches

1. QCD + QED montecarlo simulation

VOLUME 76, NUMBER 21 PHYSICAL REVIEW LETTERS 20 MAY 1996

Electromagnetic Splittings and Light Quark Masses in Lattice QCD

A. Duncan,[1] E. Eichten,[2] and H. Thacker[3]
[1]Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15620
[2]Fermilab, P.O. Box 500, Batavia, Illinois 60510
[3]Department of Physics, University of Virginia, Charlottesville, Virginia 22901
(Received 5 February 1996)

A method for computing electromagnetic properties of hadrons in lattice QCD is described and

The strategy of the calculation is as follows. Quark propagators are generated in the presence of background SU(3) × U(1) fields where the SU(3) component repre-

noncompact form of the pure photon action $S_{\rm em}$

$$S_{\rm em} = \frac{1}{4e^2} \sum_{n\mu\nu} (\nabla_\mu A_{n\nu} - \nabla_\nu A_{n\mu})^2,$$

mass values. The gauge configurations were generated at $\beta = 5.7$ on a $12^3 \times 24$ lattice. 200 configurations each

Coulomb gauge field $A_{n\mu}$ is then promoted to a compact link variable $U^{\rm em}_{n\mu} = e^{\pm iqA_{n\mu}}$ coupled to the quark field in

the results reported here, we have used four different values of charge given by $e_q = 0, -0.4, +0.8,$ and -1.2 in units in which the electron charge is $e = \sqrt{4\pi/137} = 0.3028\ldots$. For each quark charge we calculate propaga-

$$m_{\pi^+} - m_{\pi^0} = 4.9 \pm 0.3\ \text{MeV}$$

o the experimental value of 4.6 MeV

QED and Isospin corrections to hadron masses

Two approaches

1. QCD + QED montecarlo simulation

from A. Portelli @ Lattice 2014

Full QCD + QED projects

	RBC-UKQCD	PACS-CS	QCDSF-UKQCD	BMWc
arXiv	1006.1311	1205.2961	1311.4554 and Lat. 2014	1406.4088
fermions	DWF	clover	clover	clover
N_f	2+1	1+1+1	1+1+1	1+1+1+1
method	reweighting	reweighting	RHMC	RHMC
$\min(M_\pi)$ (MeV)	420	135	250	195
a (fm)	0.11	0.09	0.08	0.06 — 0.10
$\#a$	1	1	1	4
L (fm)	1.8	2.9	1.9 — 2.6	2.1 — 8.3
$\#L$	1	1	2	11

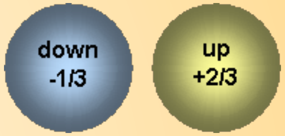
Calculations at several values of α_{em}. Not really “full”: linear extrapolation to 1/137 without the renormalization of α_{em}

QED and Isospin corrections to hadron masses

2. The RM123 approach

QED and isospin breaking effects are small because:

$m_u \neq m_d$: $O[(m_d - m_u)/\Lambda_{QCD}] \approx 1/100$ "Strong"



$Q_u \neq Q_d \neq 0$: $O(\alpha_{e.m.}) \approx 1/100$ "Electromagnetic"

The electromagnetic and isospin breaking part of the Lagrangian can be treated as a perturbation

Expand in: $m_d - m_u$ + α_{em}

JHEP

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Published: April 26, 2012

arXiv:1110.6294

Isospin breaking effects due to the up-down mass difference in lattice QCD

RM123 collaboration

PHYSICAL REVIEW D 87, 114505 (2013)

Leading isospin breaking effects on the lattice

M. de Divitiis,[1,2] R. Frezzotti,[1,2] V. Lubicz,[3,4] G. Martinelli,[5,6] R. Petronzio,[1,2] G. C. Rossi,[1,2] F. Sanfilippo,[7] S. Simula,[4] and N. Tantalo[1,2]

(RM123 Collaboration) arXiv:1303.4896

RM123 Collaboration

The (md-mu) expansion

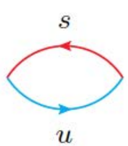
G.M.de Divitiis *et al.*, RM123 collaboration, JHEP 04 (2012) 124

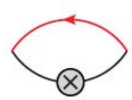
- Identify the isospin breaking term in the action and expand in $\Delta m = (m_d - m_u)/2$

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) - \frac{1}{2}(m_d - m_u)(\bar{u}u - \bar{d}d) \right] = S_0 - \Delta m \hat{S}$$

$$\langle O \rangle = \frac{\int D\phi\, O\, e^{-S_0 + \Delta m \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O\, e^{-S_0}\left(1 + \Delta m\, \hat{S}\right)}{\int D\phi\, e^{-S_0}\left(1 + \Delta m\, \hat{S}\right)} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O\, \hat{S} \rangle_0}{1 + \Delta m \cancel{\langle \hat{S} \rangle_0}} = \langle O \rangle_0 + \Delta m \langle O\, \hat{S} \rangle_0$$

- For the kaon decay constant:

$C_{K^+K^-}(t) = -$ 

$= -$  $-$  $+ \mathcal{O}(\Delta m_{ud})^2$

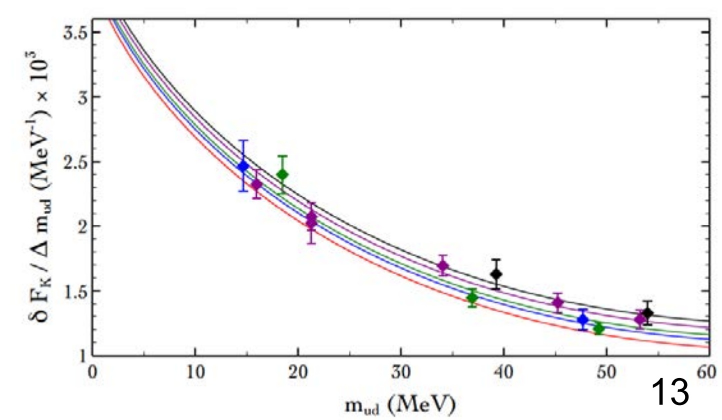
$$\delta_{SU(2)} = -0.0080\,(7)$$

Lattice - Nf=2
Confirmed with Nf=2+1+1
RM123 collab. (2012, 2015)

which is ~2.6 σ larger than

$$\delta_{SU(2)} = -0.0044\,(12)$$

ChPT
Cirigliano, Neufeld (2011)

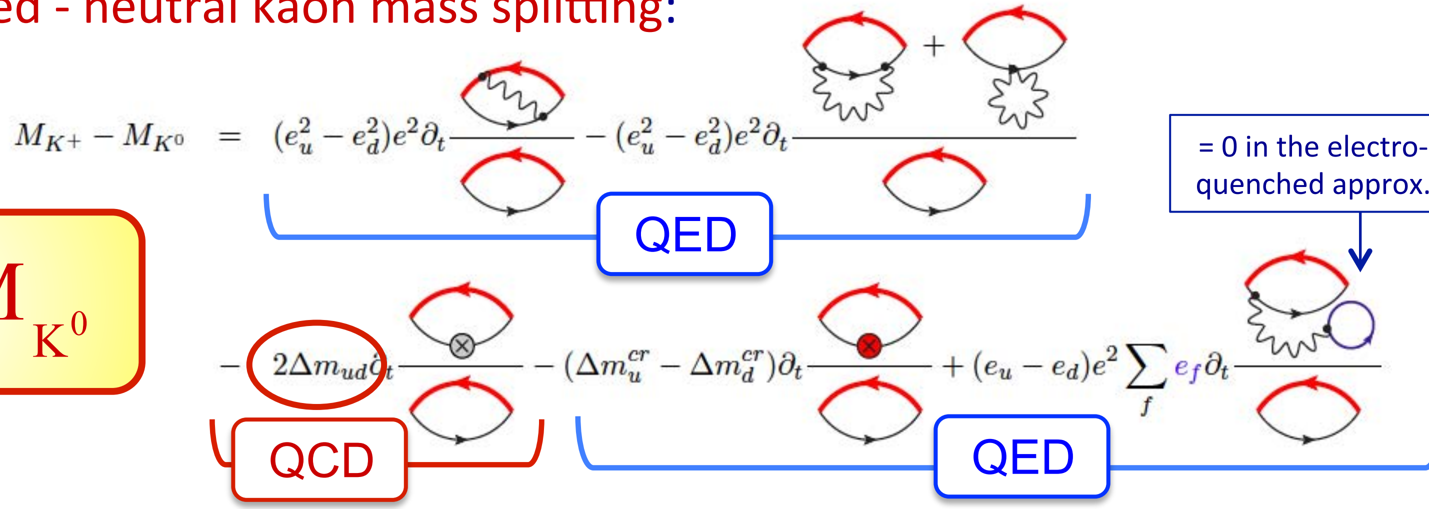


The QED expansion

G.M.de Divitis *et al.*, RM123 collaboration, PRD 87 (2013) 114505

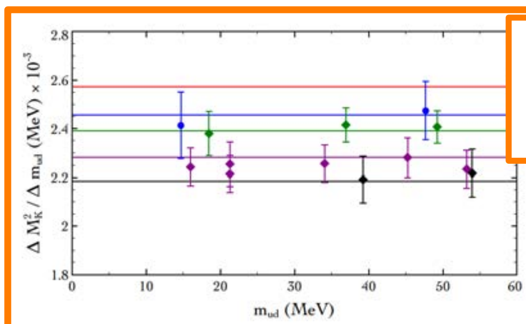
- The expansion can be generalized to include the electromagnetic corrections. For the charged - neutral kaon mass splitting:

$$M_{K^+} - M_{K^0}$$



- $$\left[M_{K^+} - M_{K^0}\right]^{QED} = 2.3(2)(2)\ \text{MeV}$$

  $$\left[M_{K^+} - M_{K^0}\right]^{QCD} = -6.2(2)(2)\ \text{MeV}$$



$$\frac{\left[M_{K^0}^2 - M_{K^+}^2\right]^{QCD}}{m_d - m_u}$$

$$(\bar{m}_d - \bar{m}_u) = 2.39(8)(17)\ \text{MeV}$$

$$\bar{m}_u / \bar{m}_d = 0.50(2)(3)$$

The QED expansion

G.M.de Divitis *et al.*, RM123 collaboration, PRD 87 (2013) 114505

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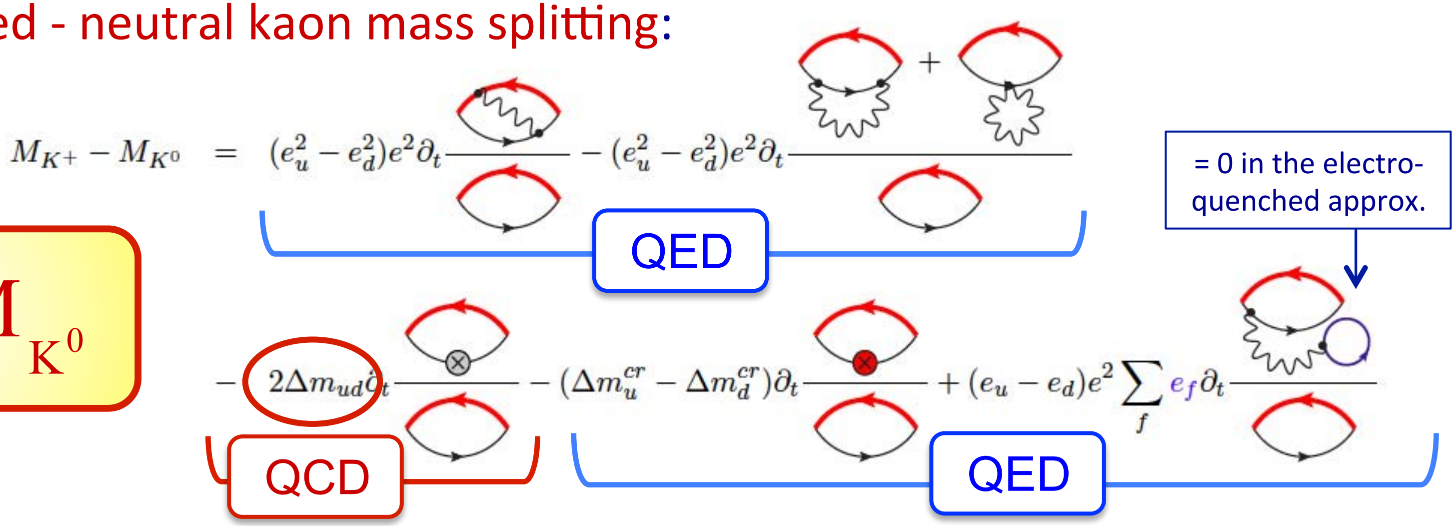
$$M_{K^+} - M_{K^0}$$



- Advantage: we compute the insertion of operators of O(1) and no extrapolation $\alpha_{em} \to 1/137$ is required. No need to generate new gauge confs.
- Disadvantage: More vertices and correlations functions to be computed, including disconnected diagrams
- Unavoidable in electromagnetic corrections to hadronic amplitudes

QED corrections to hadronic processes

After the renormalization of the QCD+QED Lagrangian you still need:

1) The renormalization of the operators mediating the physical process of interest (e.g. the weak effective Hamiltonian). But this is not a novelty.

2) A complex procedure to remove the infrared cutoff because in general the amplitudes, contrary to the masses, are infrared divergent.

A method to solve this problem is presented

N.Carrasco, VL, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino, M. Testa

PRD 91 (2015) 074506, arXiv: 1502.00257

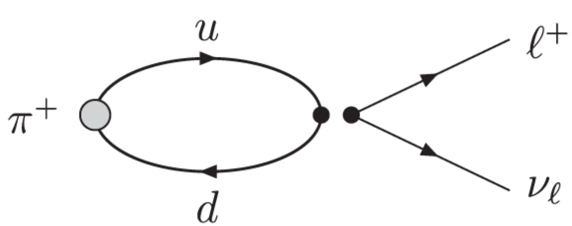
The strategy

- To be specific, we consider the leptonic decay of a charged pion, but the method is general (it can be applied for example to semileptonic decays).

- The rate at $O(\alpha^0)$ is:

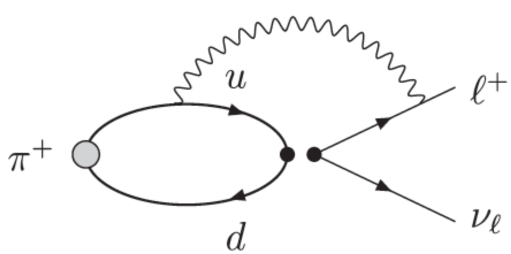
$$\Gamma_0^{\text{tree}}\left(\pi^+ \to \ell^+\nu_\ell\right) = \frac{G_F\left|V_{ud}\right|^2 f_\pi^2}{8\pi} m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

In the absence of electromagnetism, the nonperturbative QCD effects are contained in a single number, the decay constant:



$$\langle 0|\bar{d}\gamma_\mu(1-\gamma_5)u|\pi^+(p)\rangle = i p_\mu f_\pi$$

- In the presence of electromagnetism, because of the contributions of diagrams like this one, it is not even possible to give a physical definition of f_π.

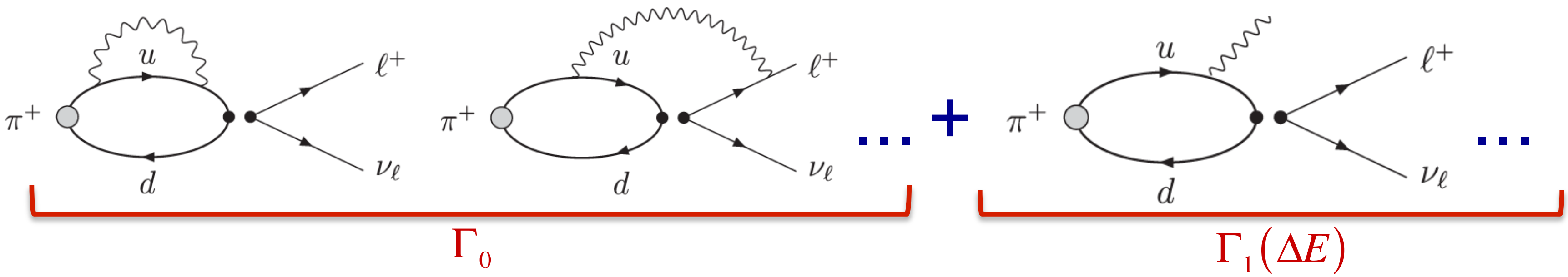


For a discussion on this point based on ChPT see

J. Gasser and G.R.S. Zarnauskas, PLB 693 (2010) 122.

The strategy

- At O(α), the rate Γ_0 contains infrared divergences. One has to consider:



$$\Gamma(\Delta E) = \Gamma\left(\pi^+ \to \ell^+ \nu_\ell\right) + \Gamma\left(\pi^+ \to \ell^+ \nu_\ell \gamma(\Delta E)\right) \equiv \Gamma_0 + \Gamma_1(\Delta E)$$

with $0 \le E_\gamma \le \Delta E$. The sum is infrared finite

F. Bloch and A. Nordsieck, PR 52 (1937) 54

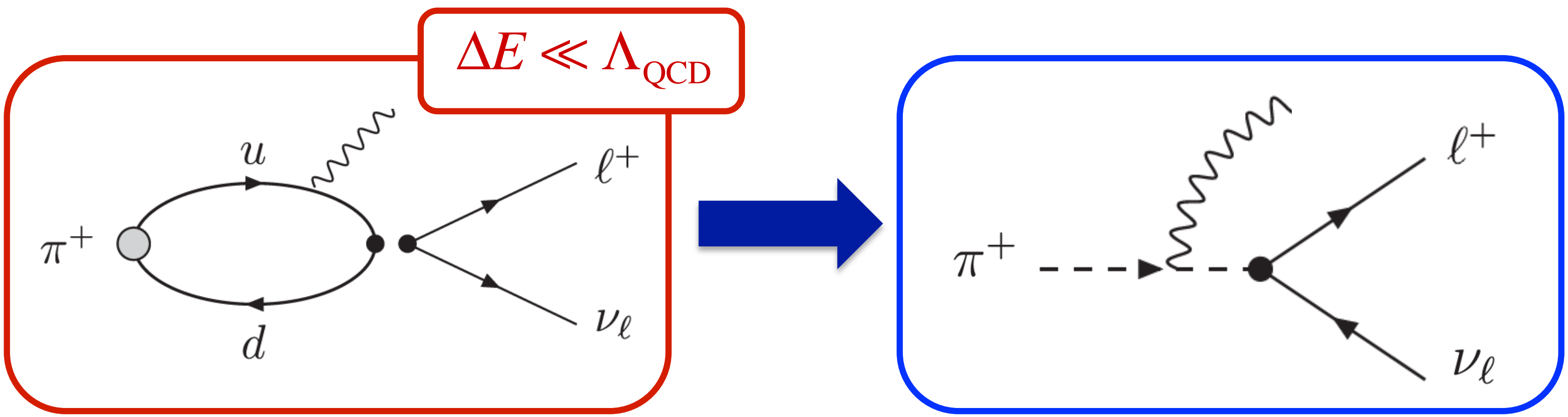
- In principle, both Γ_0 and $\Gamma_1(\Delta E)$ can be evaluated in lattice simulations.

But $\Gamma_1(\Delta E)$ is very challenging, due to discretized photon momenta: the integral up to ΔE replaced by a finite sum, many correlation functions…

We thus propose a different strategy ➡

The strategy

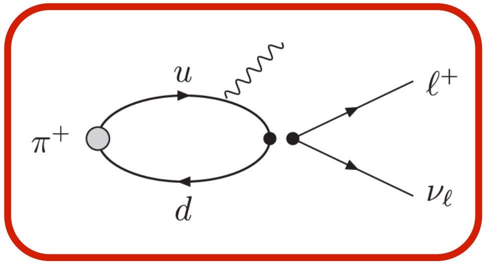
- We propose to consider sufficiently soft photons such that they do not resolve the internal structure of the pion. Then the pointlike approximation can be used to compute $\Gamma_1(\Delta E)$ in perturbation theory.

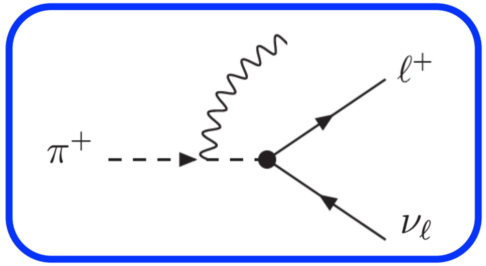


➡ A cut-off $\Delta E \sim O(20\ \mathrm{MeV})$ appears to be appropriate, both experimentally and theoretically

↳ F. Ambrosino et al., KLOE Collaboration, PLB 632 (2006) 76; EPJC 64 (2009) 627; 65 (2010) 703(E)

The strategy



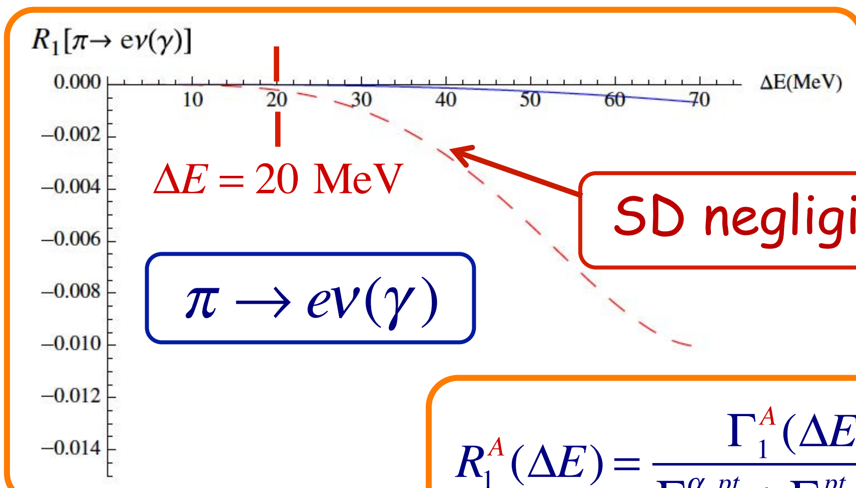


$\Delta E \sim O(20\ \text{MeV})$

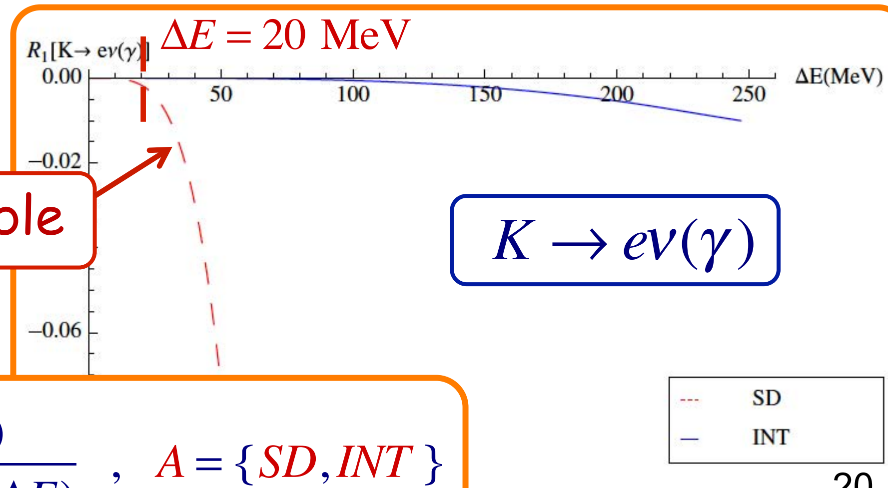
- For π and K decays, the size of the neglected structure-dependent contributions can be estimated, as a function of ΔE, using ChPT

J. Bijnens, G. Ecker, J. Gasser, NPB 396 (1993) 81; V.Cirigliano, I.Rosell, JHEP 0710 (2007) 005

The structure dependent component can be parametrized in terms of two independent form factors. At O(p[4]) in ChPT: $F_V = m_P/(4\pi^2 f_\pi)$, $F_A = (8m_P)/f_\pi \cdot \left(L_9^r + L_{10}^r\right)$



SD negligible



$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,pt} + \Gamma_1^{pt}(\Delta E)} \quad , \quad A = \{SD, INT\}$$

The strategy

- $$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1^{pt}(\Delta E) \qquad \Delta E \sim O(20\ \mathrm{MeV})$$

$\Gamma(\pi^+ \to \ell^+ \nu_\ell)$
Montecarlo simulation
Lattice QCD

$\Gamma(\pi^+ \to \ell^+ \nu_\ell \gamma(\Delta E))$
Perturbation theory
pointlike pion

- In order to ensure the cancellation of IR divergences with good numerical precision, an intermediate step is required. We then rewrite:

$$\Gamma(\Delta E) = \underbrace{\lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{pt}\right)} + \underbrace{\lim_{V\to\infty}\left(\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E)\right)}$$

$\Gamma_0^{pt} = \Gamma(\pi^+ \to \ell^+ \nu_\ell)^{pt}$ is an unphysical quantity

The strategy

$$\Gamma(\Delta E) = \underbrace{\lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\mathrm{pt}}\right)} + \underbrace{\lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)}$$

- The second term is calculated in perturbation theory directly in infinite volume. The sum $\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)$ is IR finite.
- $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$ in the first term is calculated, in the intermediate step, in the finite volume. The contributions from small virtual photon momenta to Γ_0 and Γ_0^{pt} are the same, and the first term is IR finite.
- ⟹ IR divergences cancel separately in each of the two terms, and so we can calculate each of these terms separately. We also use different IR regulators: the finite volume for the first term and a photon mass for the second term.
- The two terms are also separately gauge invariant.

Outline

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\text{pt}}(L)\right) + \lim_{V\to\infty}\left(\Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)\right)$$

- 1. General strategy ✓
- 2. Calculation of $\Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)$ ←
- 3. Calculation of Γ_0
  - G_F and the UV matching
  - Lattice calculation
- 4. Calculation of $\Gamma_0^{\text{pt}}(L)$
- 5. Conclusions

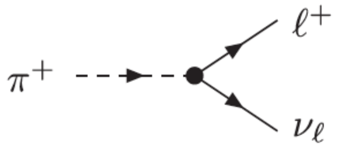
Calculation of $\Gamma^{pt}(\Delta E)$

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{pt}\right) + \lim_{V\to\infty}\left(\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E)\right)$$

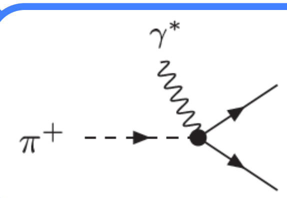
- $\Gamma^{pt}(\Delta E)$ is calculated in perturbation theory with a pointlike pion

$$\mathcal{L}_{\pi-\ell-\nu_\ell} = iG_F f_\pi V_{ud}^*\{(\partial_\mu - ieA_\mu)\pi\}\left\{\bar{\psi}_{\nu_\ell}\frac{1+\gamma_5}{2}\gamma^\mu\psi_\ell\right\}$$

+ QED for π and l^+

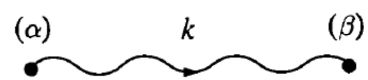


$$= -iG_F f_\pi V_{ud}^* p_\pi^\mu \frac{1+\gamma^5}{2}\gamma_\mu$$



$$= ie\, G_F f_\pi V_{ud}^*\, g^{\mu\nu}\frac{1+\gamma^5}{2}\gamma_\mu$$

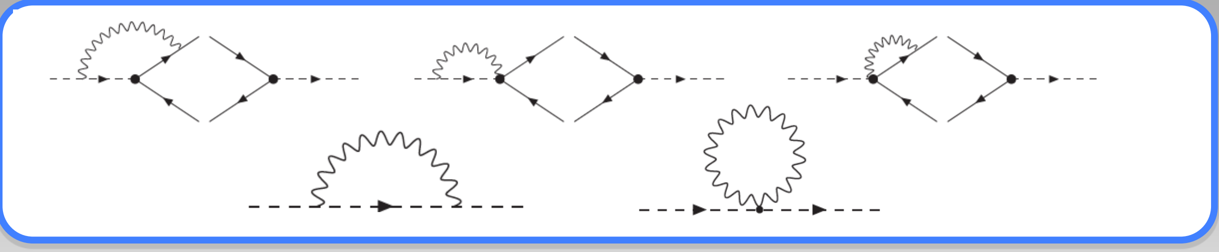
- UV divergences in Γ_0^{pt} are regularized with the "W-regularization" (more on this point later)

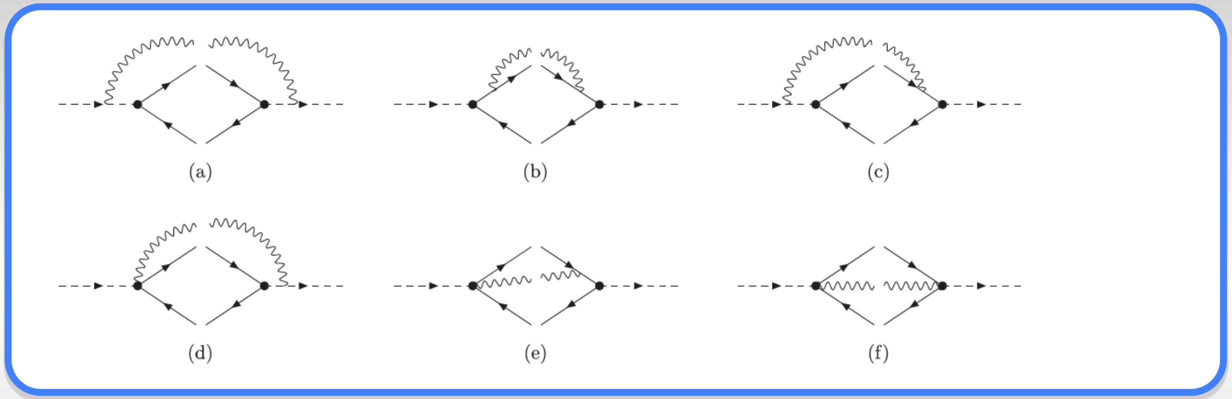


$$\frac{1}{k^2} \to \frac{M_W^2}{M_W^2 - k^2}\frac{1}{k^2}$$

- IR divergences are regularized with the a photon mass

Calculation of $\Gamma^{\rm pt}(\Delta E)$





$\Gamma_0^{\rm pt}$ + $\Gamma_1^{\rm pt}(\Delta E)$ = $\Gamma^{\rm pt}(\Delta E)$

NEW

$$\Gamma^{\rm pt}(\Delta E) = \Gamma_0^{\rm tree} \times \Bigg(1 + \frac{\alpha}{4\pi}\Bigg\{\underbrace{3\log\left(\frac{m_\pi^2}{M_W^2}\right)} + \log(r_\ell^2) - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2) - 2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2)$$

$$- 4\frac{1+r_\ell^2}{1-r_\ell^2}{\rm Li}_2(1-r_\ell^2) - 3 + \Bigg[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) + \frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2)$$

$$- \frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2} - 4\frac{1+r_\ell^2}{1-r_\ell^2}{\rm Li}_2(r_E)\Bigg]\Bigg\}\Bigg).$$

$r_\ell = m_\ell / m_\pi$

$r_E = 2\Delta E / m_\pi$

$0 \le r_E \le 1 - r_\ell^2$

- IMPORTANT CHECK: For $\Delta E = \Delta E_{MAX}$ we obtain the well known result for the total rate as in S. M. Berman, PRL 1 (1958) 468 and T. Kinoshita, PRL 2 (1959) 477

Outline

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)$$

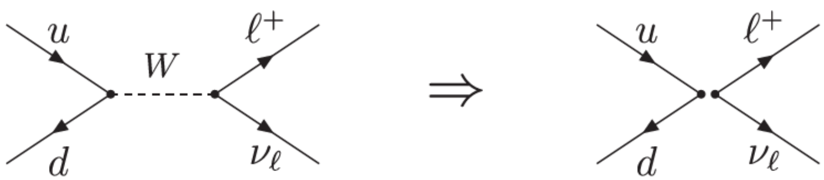
- 1. General strategy ✓
- 2. Calculation of $\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)$ ✓
- 3. Calculation of Γ_0 ←
  - G_F and the UV matching
  - Lattice calculation
- 4. Calculation of $\Gamma_0^{\mathrm{pt}}(L)$
- 5. Conclusions

Calculation of $\Gamma_0(L)$

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\mathrm{pt}}\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)$$

- Γ_0 is calculated in the finite volume with a lattice simulation

- At lowest order in electromagnetic (and strong) perturbation theory the amplitude can be rewritten in terms of a four-fermion local interaction



$$H_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} V_{ud}^{*}\left(\bar{d}\gamma^{\mu}(1-\gamma^{5})u\right)\left(\bar{\nu}_{\ell}\gamma_{\mu}(1-\gamma^{5})\ell\right)$$

This replacement is necessary in a lattice calculation, since $1/a \ll M_W$

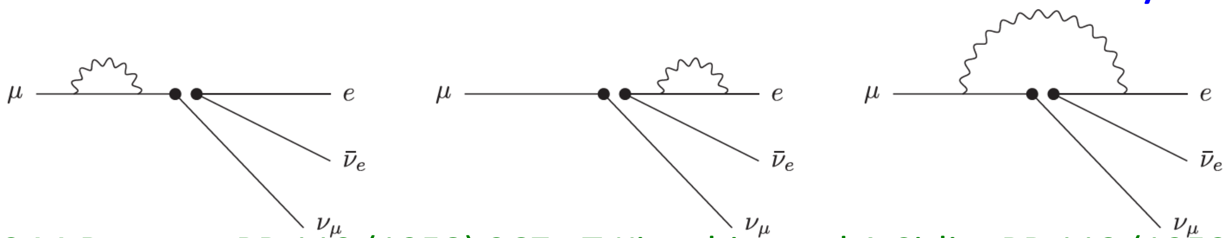
- When including the O(α) corrections, the UV contributions to the matrix element of the local operator are different from those in the Standard Model:
  ➡ a matching between the two theories is required

SM electroweak corrections to muon decay

- The Fermi constant G_F is conventionally defined from the muon lifetime using

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1-\frac{8m_e^2}{m_\mu^2}\right)\left[1+\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right] \Rightarrow G_F = 1.16634\times10^{-5}\ \mathrm{GeV}^{-2}$$

Many EW corrections are absorbed into the definition of GF; the explicit O(α) corrections are those obtained in the effective theory: [UV finite for muon decay]



\+ diagrams with the real photon

S.M.Berman, PR 112 (1958) 267; T.Kinoshita and A.Sirlin, PR 113 (1959) 1652

- In the Standard Model, in order to separate out the traditional photonic corrections, it is convenient to write the (Feynman gauge) photon propagator as:

A. Sirlin, RMP 50 (1978) 573, PRD 22 (1980) 971

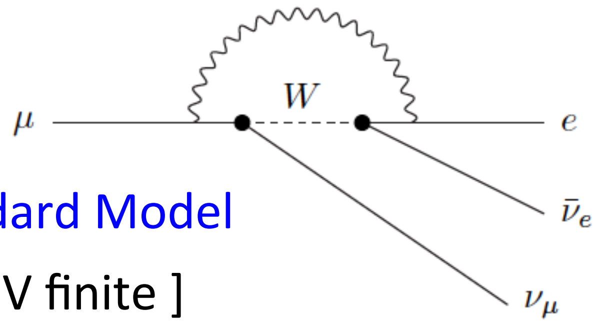
$$\frac{1}{k^2} = \frac{1}{k^2 - M_W^2} + \frac{M_W^2}{M_W^2 - k^2}\frac{1}{k^2}$$

UV divergent. Absorbed in the definition of G_F together with the other EW corrections

UV finite. Not included in GF. Equal to the corresponding diagram in the effective theory with the "W-regularization" (up to negligible terms of $O(q^2/M_W^2)$)

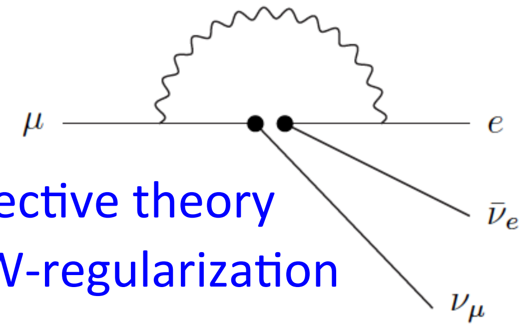
SM electroweak corrections to muon decay

- The γ-W box diagram, which is finite in the Standard Model, is also equal to the corresponding diagram of the effective theory with W-regularization:



Standard Model

[UV finite]



Effective theory with W-regularization

$$\frac{1}{k^2} \rightarrow \frac{M_W^2}{M_W^2 - k^2}\frac{1}{k^2}$$

- The Standard Model diagram contains both the γ and W propagators

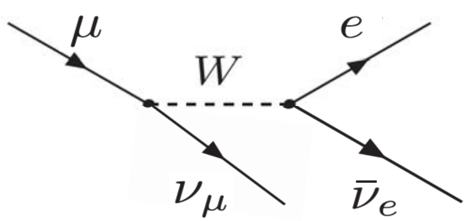
- In the effective theory this is preserved with the W-regularization, and the two diagrams are equal up to negligible terms of $O(q^2/M_W^2)$

➡ The W-regularization provides a convenient way to separate out the traditional photonic corrections. For muon decay, all other EW corrections are absorbed in G_F .

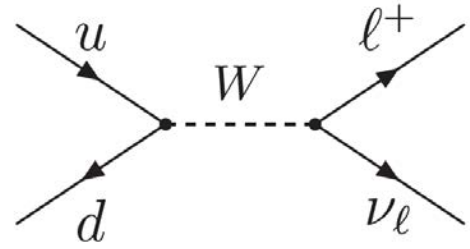
SM electroweak corrections to pion decay

- Most of the terms which are absorbed into the definition of G_F are common to other processes, including the leptonic decays of pseudoscalar mesons

muon decay



pion decay



Same weak isospin charges

- Some short-distance contributions, however, do depend on the electric charges of the fields in the four-fermion operators. These lead to a correction factor of

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud}^* \left(1 + \frac{3\alpha}{4\pi}(1+2\bar{Q}) \log\frac{M_Z}{M_W}\right)\left(\bar{d}\gamma^\mu(1-\gamma^5)u\right)\left(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell\right)$$

W-regularization

$$2\bar{Q} = (Q_{\nu_\mu} + Q_\mu) = -1$$

$$1 + 2\bar{Q} = 0$$

muon decay

pion decay

$$2\bar{Q} = (Q_u + Q_d) = 1/3$$

$$1 + 2\bar{Q} = 4/3$$

A.Sirlin, NP B196 (1982) 83; E.Braaten & C.S.Li, PRD 42 (1990) 3888

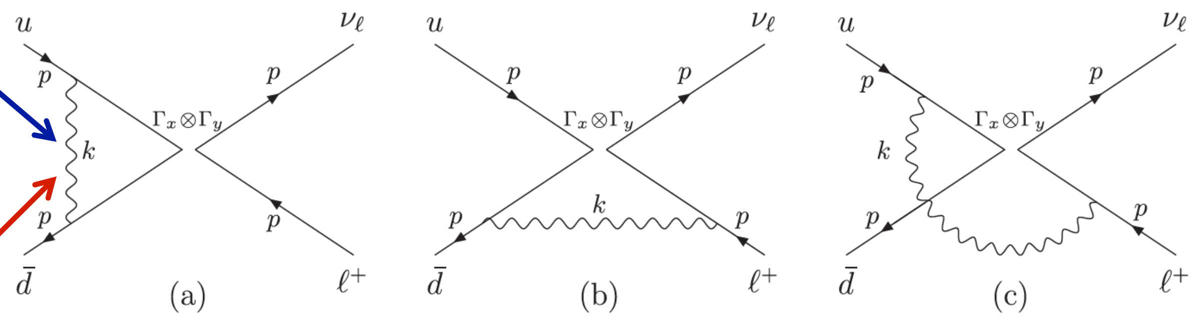
This provides the matching between the Standard Model and the local Fermi theory

Matching the W and lattice regularizations

- The W regularization cannot be implemented directly in present day lattice simulations since $1/a \ll M_W$
- The relation between the Fermi effective Hamiltonian in the lattice and W regularizations can be computed in perturbation theory:

W regul. $\left(\frac{M_W^2}{M_W^2 - k^2}\right)\frac{1}{k^2}$

Lattice $1/\left[\sum_\rho \frac{4}{a^2}\sin^2\left(\frac{ak_\rho}{2}\right)\right]$



- The result, with the lattice Wilson action for both gluons and fermions, is:

$$O_1^{\text{W-reg}} = \left(1 + \frac{\alpha}{4\pi}(2\log(a^2M_W^2) - 15.539)\right)O_1^{\text{bare}} + \frac{\alpha}{4\pi}(0.536\,O_2^{\text{bare}} + 1.607\,O_3^{\text{bare}} - 3.214\,O_4^{\text{bare}} - 0.804\,O_5^{\text{bare}}),$$

$$O_1 = (\bar{d}\gamma^\mu(1-\gamma^5)u)(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell),$$
$$O_2 = (\bar{d}\gamma^\mu(1+\gamma^5)u)(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell),$$
$$O_3 = (\bar{d}(1-\gamma^5)u)(\bar{\nu}_\ell(1+\gamma^5)\ell),$$
$$O_4 = (\bar{d}(1+\gamma^5)u)(\bar{\nu}_\ell(1+\gamma^5)\ell),$$
$$O_5 = (\bar{d}\sigma^{\mu\nu}(1+\gamma^5)u)(\bar{\nu}_\ell\sigma_{\mu\nu}(1+\gamma^5)\ell)$$

Outline

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\text{pt}}(L)\right) + \lim_{V\to\infty}\left(\Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)\right)$$

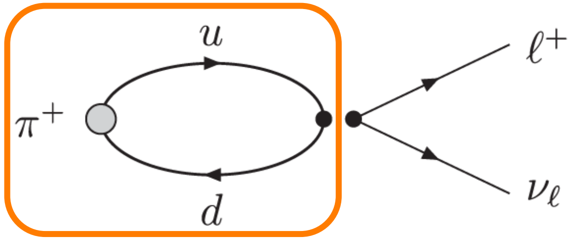
- 1. General strategy ✓
- 2. Calculation of $\Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)$ ✓
- 3. Calculation of Γ_0 ✓
  - G_F and the UV matching ✓
  - Lattice calculation ←
- 4. Calculation of $\Gamma_0^{\text{pt}}(L)$
- 5. Conclusions

Lattice calculation of $\Gamma_0(L)$

- The lattice calculation at $O(\alpha^0)$, i.e. without electromagnetism, is standard

$$M_0 = \frac{G_F}{\sqrt{2}} V_{ud}^* \langle 0|\bar{d}\gamma^\mu\gamma^5 u|\pi^+(p_\pi)\rangle \left[u_{\nu_\ell}(p_{\nu_\ell})\gamma_\mu(1-\gamma^5)v_\ell(p_\ell)\right] =$$

$$= \frac{iG_F}{\sqrt{2}} V_{ud}^* f_\pi p_\pi^\mu \left[u_{\nu_\ell}(p_{\nu_\ell})\gamma_\mu(1-\gamma^5)v_\ell(p_\ell)\right]$$



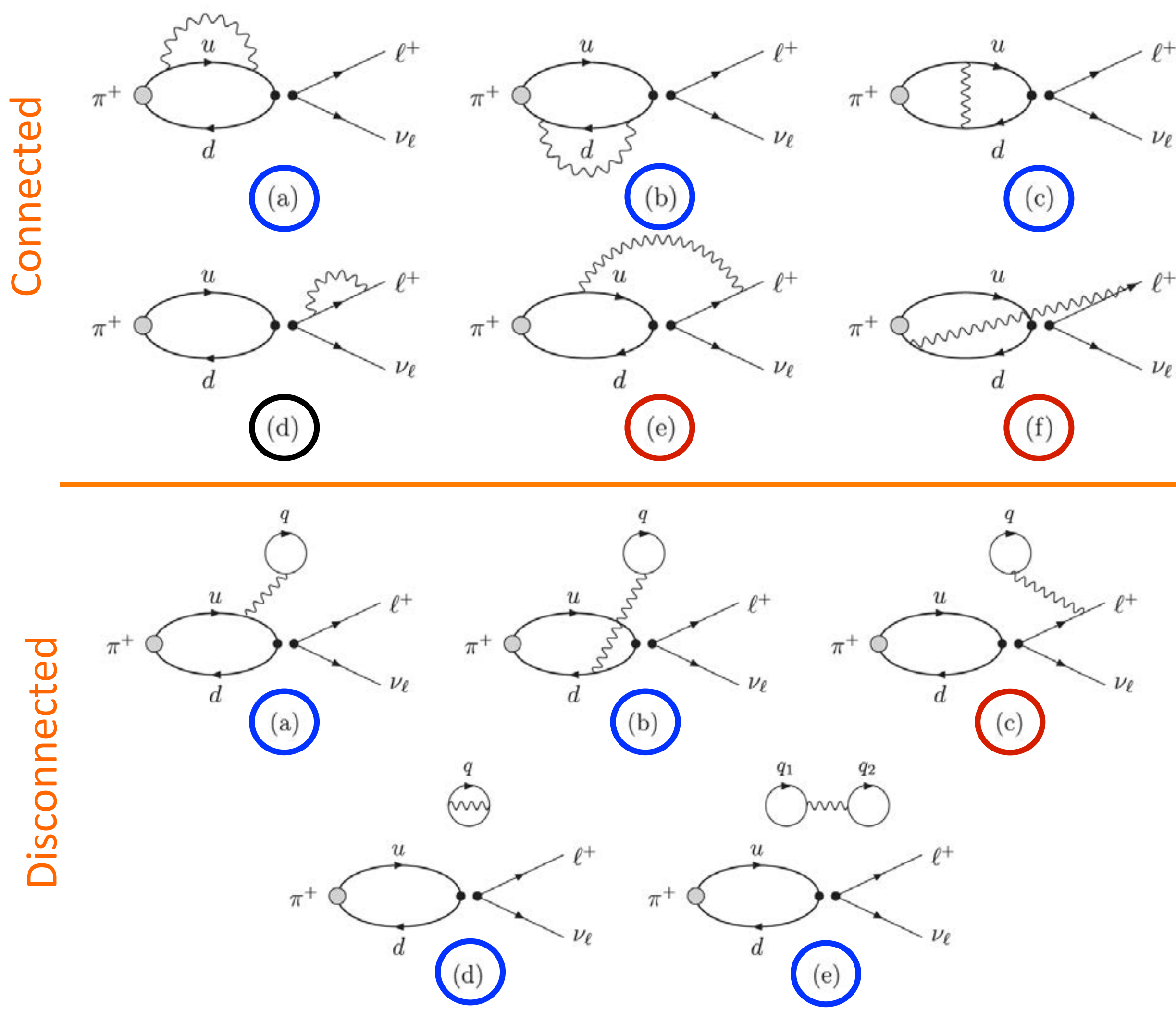
The amplitude is obtained from the large t behavior of the 2-point correlator

$$C_0(t) \equiv \sum_{\vec{x}} \langle 0|\left(\bar{d}(\vec{0},0)\gamma^\mu\gamma^5 u(\vec{0},0)\right)\phi^\dagger(\vec{x},-t)|0\rangle \simeq \frac{e^{-m_\pi^0 t}}{2m_\pi^0} Z_0^\phi \langle 0|\bar{d}\gamma^\mu\gamma^5 u|\pi^+(\vec{0})\rangle_0$$

$$C_0^{\phi\phi}(t) \equiv \sum_{\vec{x}} \langle 0|\phi(\vec{0},0)\phi^\dagger(\vec{x},-t)|0\rangle \simeq \frac{e^{-m_\pi^0 t}}{2m_\pi^0} \left(Z_0^\phi\right)^2 \qquad Z_0^\phi \equiv \langle \pi^+(\vec{0})|\phi^\dagger(\vec{0},0)|0\rangle$$

Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$

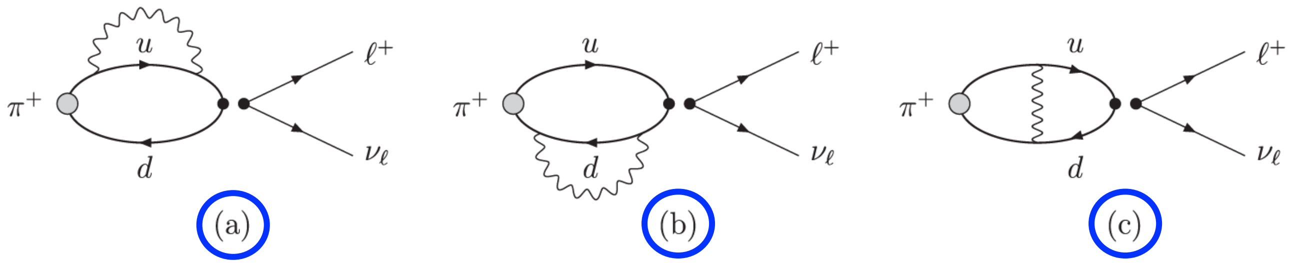
- The Feynman diagrams at $O(\alpha)$ can be divided in 3 classes



1. The photon connects two quark lines
2. The photon connects one quark and one charged lepton line
3. Leptonic wave function renormalization. It cancels in $\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)$

Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [I]

- Let us consider the Feynman diagrams of the 1st class:



+ disconnected diagrams

The leptonic contribution to the amplitude is factorized and we need to compute

$$C_1(t) = -\frac{1}{2}\int d^3\vec{x}\, d^4x_1\, d^4x_2 \sum_{\vec{x}} \langle 0 | T\left\{ \underbrace{J_W^0(0)}_{\text{weak}} \underbrace{j_\mu(x_1) j_\mu(x_2)}_{\text{e.m.}} \underbrace{\phi^\dagger(\vec{x},-t)}_{\text{pion}} \right\} | 0 \rangle \underbrace{\Delta(x_1,x_2)}_{\text{photon}}$$

- Combining $C_1(t)$ with the lowest order correlator

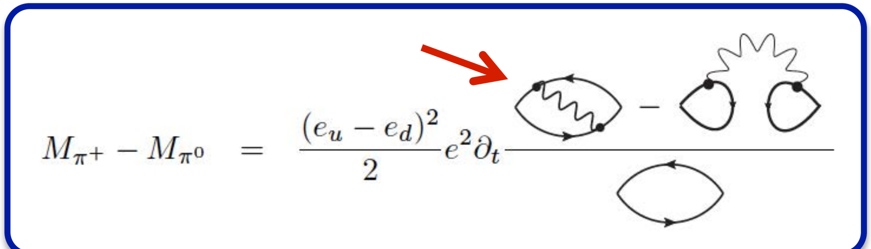
$$C_0(t) + C_1(t) \simeq \frac{e^{-m_\pi t}}{2m_\pi} Z^\phi \langle 0 | J_W^0(0) | \pi^+ \rangle \simeq \frac{e^{-m_\pi^0 t}\left(1-\delta m_\pi t\right)}{2\left(m_\pi^0 + \delta m_\pi\right)} Z_0^\phi \left(1+\delta Z^\phi\right) \langle 0 | J_W^0(0) | \pi^+ \rangle_0 \left(1+\delta\mathcal{A}\right)$$

- δm_π is infrared finite and gauge invariant

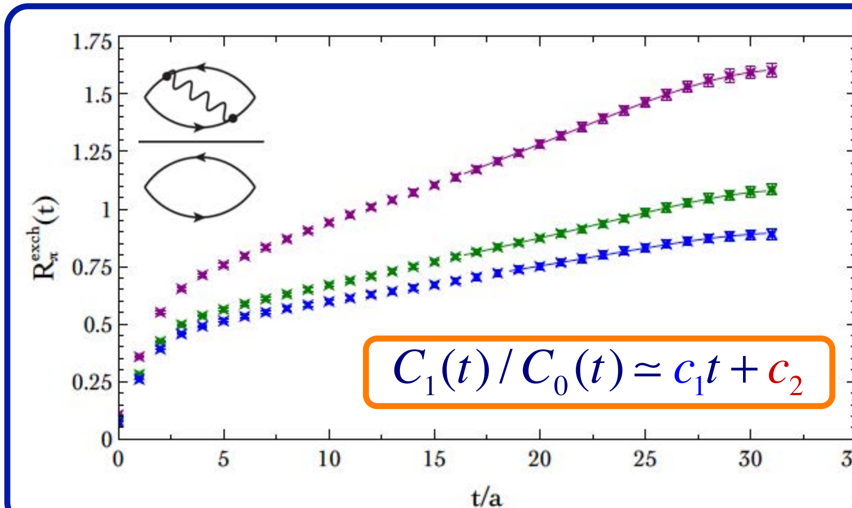
- δZ^ϕ and $\delta\langle 0 | J_W^0(0) | \pi^+ \rangle$ however are infrared divergent and gauge dependent and cannot be interpreted as a correction to f_π

Results for δm$_\pi$

- Only 2 diagrams contribute to the pion mass splitting



- The 1st diagram also contributes to the π^+ decay rate
- The 2nd diagram comes from the neutral pion. It is $O(\alpha\, m_{ud})$ and it has been neglected



- From the linear slope in time we find

$$M_{\pi^+} - M_{\pi^0} = 5.33(76)\ \text{MeV}$$

RM123c, PRD 87 (2013) 114505 [Nf=2]

$$M_{\pi^+} - M_{\pi^0} = 4.28(39)\ \text{MeV}$$

RM123c '15, preliminary [Nf=2+1+1]

Experimental value:

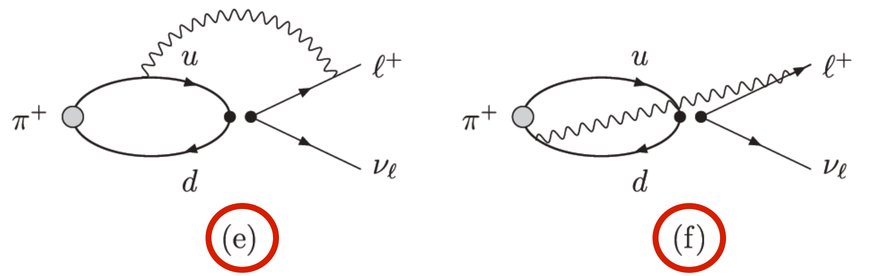
$$M_{\pi^+} - M_{\pi^0} = 4.59\ \text{MeV}$$

Finite volume correction to the mass up to $O(1/L^2)$ are now known and found to be universal

BMWc '14, Davoudi et al. '14

Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [II]

- For these diagrams the leptonic and hadronic contributions do not factorize



+ disconnected diagrams

They are not simply generalization of the evaluation of f_π

- The amplitude is obtained from the following Euclidean space correlation function

weak | e.m. | pion

$$C_1(t)_{\alpha\beta} = -\int d^3\vec{x}\, d^4x_1\, d^4x_2\, \langle 0|T\left\{J_W^\nu(0)\, j_\mu(x_1)\, \phi^\dagger(\vec{x},-t)\right\}|0\rangle$$
$$\times\, \Delta(x_1,x_2)\left(\gamma_\nu(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta} e^{E_\ell t_2 - i\,\vec{p}_\ell\cdot\vec{x}_2}$$

photon | weak | lepton | e.m.

- The large t behavior of the correlator is given by:

$$C_1(t)_{\alpha\beta} \simeq \frac{e^{-m_\pi^0 t}}{2m_\pi^0} Z_0^\phi \left(\bar{M}_1\right)_{\alpha\beta}$$

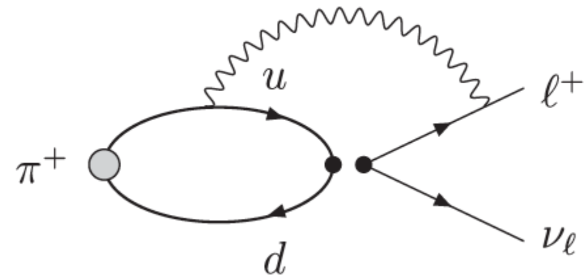
The corresponding contribution to the amplitude is:

$$\bar{u}_\alpha(p_\nu)\left(\bar{M}_1\right)_{\alpha\beta} v_\beta(p_\mu)$$

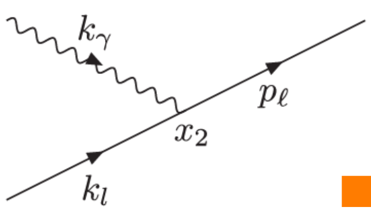
Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [II]

- A technical but important point:

$$C_1(t)_{\alpha\beta} = -\int d^3\vec{x}\, d^4x_1\, d^4x_2 \langle 0|T\left\{J_W^\nu(0)\, j_\mu(x_1)\, \phi^\dagger(\vec{x},-t)\right\}|0\rangle$$
$$\times\, \Delta(x_1,x_2)\left(\gamma_\nu(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta} e^{E_\ell t_2 - i\,\vec{p}_\ell\cdot\vec{x}_2}$$



We need to ensure that the t_2 integration converges as $t_2 \to \infty$. The large t_2 behavior is given by the factor $\exp\left[\left(E_\ell - \omega_\ell - \omega_\gamma\right)t_2\right]$



$$E_\ell = \sqrt{\vec{p}_\ell^{\,2} + m_\ell^2} \qquad \omega_\ell = \sqrt{\vec{k}_\ell^{\,2} + m_\ell^2} \qquad \omega_\gamma = \sqrt{\vec{k}_\gamma^{\,2} + m_\gamma^2} \qquad \vec{k}_\ell + \vec{k}_\gamma = \vec{p}_\ell$$

$$\Rightarrow \quad \left(\omega_\ell + \omega_\gamma\right)_{\min} = \sqrt{\left(m_\ell^2 + m_\gamma^2\right) + \vec{p}_\ell^{\,2}} > E_\ell \quad \Rightarrow$$

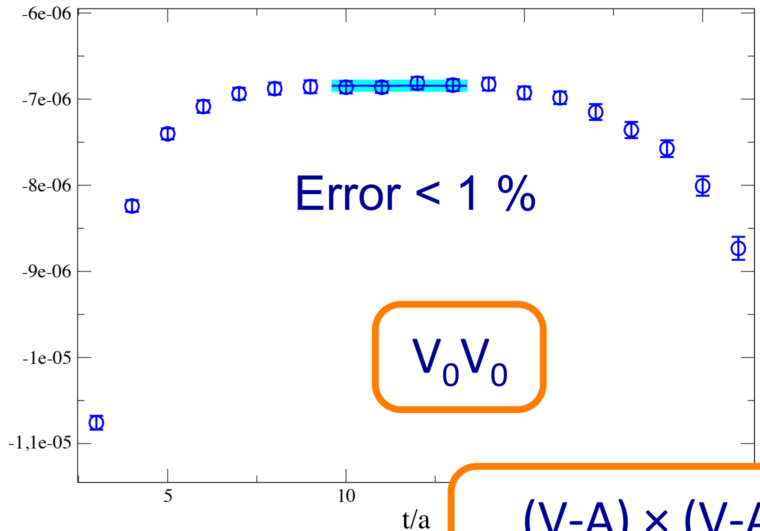
The integral is convergent and the continuation from Minkowski to Euclidean space can be performed (same if we set m_γ=0 but remove the photon zero mode in FV).

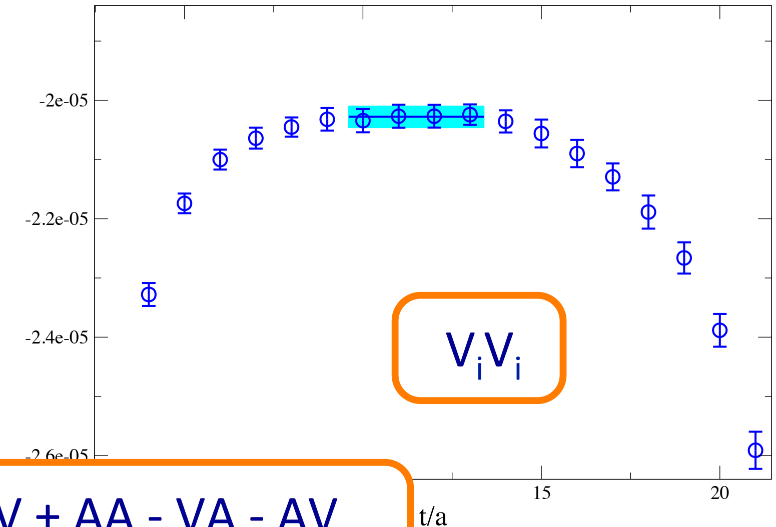
CONDITIONS: - mass gap between the decaying particle and the intermediate states
- absence of lighter intermediate states

Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [II]

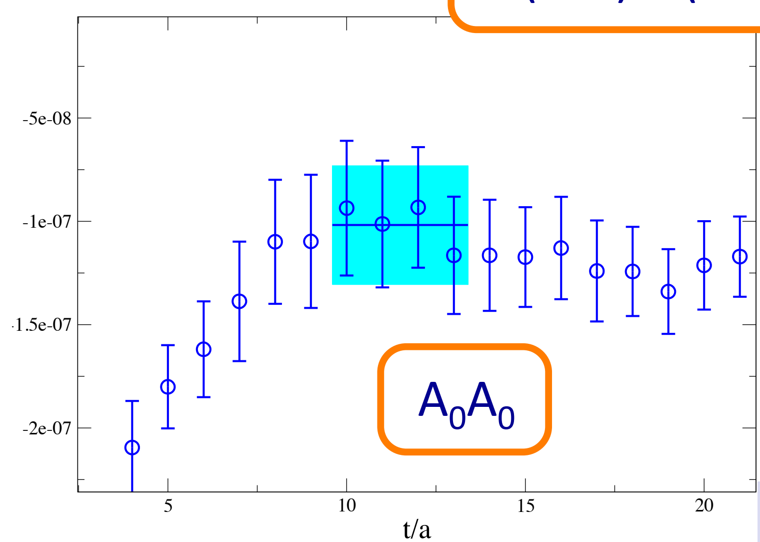
Preliminary results for crossed diagrams

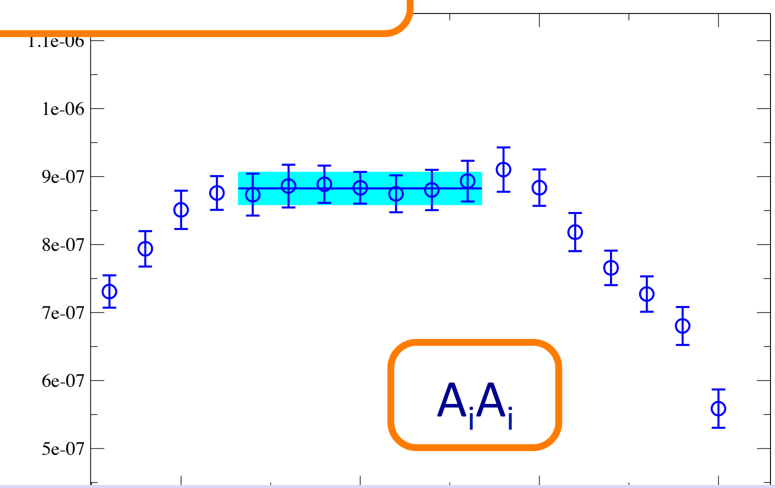
Twisted-mass, $V=24^3x48$, a=0.086 fm, m_π=475 MeV, 240 confs, 3 stochastic sources/conf.





$$(V\text{-}A) \times (V\text{-}A) = VV + AA - VA - AV$$





together with F. Sanfilippo and S. Simula

Outline

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)$$

- 1. General strategy ✓
- 2. Calculation of $\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)$ ✓
- 3. Calculation of Γ_0 ✓
  - G_F and the UV matching ✓
  - Lattice calculation ✓
- 4. Calculation of $\Gamma_0^{\mathrm{pt}}(L)$ ←
- 5. Conclusions

Calculation of $\Gamma_0^{\mathrm{pt}}(L)$

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\mathrm{pt}}\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)$$

- $\Gamma_0^{\mathrm{pt}}(L)$ is calculated in perturbation theory with a pointlike pion

- UV divergences are regularized with the W-regularization $\left(\frac{1}{q^2} \to \frac{M_W^2}{M_W^2 - q^2}\frac{1}{q^2}\right)$

- IR divergences are regularized by the finite volume (same of $\Gamma_0(L)$)

$$\int d^3q \ldots \to \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{q}} \ldots \quad \text{with} \quad \begin{cases} \vec{q} = \frac{2\pi}{L}(n_x, n_y, n_z) \\ \vec{q} \neq (0,0,0) \end{cases}$$

We have now derived an analytical expression for $\Gamma_0^{\mathrm{pt}}(L)$ and we

$$\Gamma_0^{\mathrm{pt}}(L) = 2V_{\pi\ell} + 2\,\delta m_\ell + \Sigma_\pi + \not{\Sigma}_\ell$$

[Cancels in $\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)$]

$$\vec{q} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$r_\ell = m_\ell / m_\pi$$

$$E_X = \sqrt{M_X^2 + q^2}$$

- Pion self-energy

$$\Sigma_\pi = \frac{\alpha_{em}}{4\pi^2}\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{q}} \left\{ \frac{1}{q^3} - \frac{1}{\left(M_W^4 - 4m_\pi^2 E_W^2\right)^2} \left[16 m_\pi^4 \left(E_W + \frac{M_W^2}{E_\pi} \right) + M_W^4 \left(\frac{4q^2 + M_W^2}{E_W} - \frac{4q^2 - M_W^2}{E_\pi} \right) - 4 M_W^2 m_\pi^2 \left(\frac{3q^2 + 2M_W^2}{E_W} - \frac{3q^2 - 2M_W^2}{E_\pi} \right) \right] \right\}$$

NEW

- Vertex

$$V_{\pi\ell} = \frac{\alpha_{em}}{4\pi^2}\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{q}} \left[\frac{1}{M_W^2}\left(\frac{1}{E_W} - \frac{1}{q} \right) - \frac{r_\ell^2}{(1-r_\ell^2) q^2 E_\ell} + \frac{1+r_\ell^2}{2(1-r_\ell^2) q^3} \left(\log r_\ell^2 + \frac{q}{E_\pi} - \log \frac{E_\pi + q}{E_\pi - q} + \log \frac{E_\ell + q}{E_\ell - q} \right) \right]$$

- Lepton mass correction

$$\delta m_\ell = \frac{\alpha_{em}}{4\pi^2}\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{q}} \left\{ \frac{1}{q^3} \left[-\frac{q}{E_\ell} + \frac{q^2}{m_\ell^2}\left(\frac{q}{E_\ell} - 1 \right) - \frac{2q^2}{m_W^2}\left(\frac{3q}{E_W} - 2 \right) - \frac{4q^4}{m_W^4}\left(\frac{q}{E_W} - 1 \right) \right] \right\}$$

Outline

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)$$

- 1. General strategy ✓
- 2. Calculation of $\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)$ ✓
- 3. Calculation of Γ_0 ✓
  - G_F and the UV matching ✓
  - Lattice calculation ✓
- 4. Calculation of $\Gamma_0^{\mathrm{pt}}(L)$ ✓
- 5. Conclusions ←

Conclusions

- We have presented a method to compute electromagnetic effects in hadronic processes with lattice QCD.

- We have discussed a specific case, namely the leptonic decay, but the method is general and can be applied to many other processes

- The implementation of the method is challenging but within reach of present lattice technology. The numerical study is in progress.

- Since the effects we are calculating are of O(1%), computing the electromagnetic corrections to a precision of 20% or so would already be more than sufficient.

Physical results expected soon !

Supplementary slides

QED ON THE LATTICE

- Non-compact QED: the dynamical variable is the gauge potential $A_\mu(x)$ in a fixed covariant gauge ($\nabla^-_\mu A_\mu(x) = 0$)

$$S_{QED} = \frac{1}{2}\sum_{x;\mu\nu} A_\nu(x)\left(-\nabla^-_\mu \nabla^+_\mu\right) A_\nu(x) \overset{(p.b.c.)}{=} \frac{1}{2}\sum_{k;\mu\nu} \tilde{A}^*_\nu(k)\left(2\sin(k_\mu/2)\right)^2 \tilde{A}_\nu(k)$$

- The photon propagator is IR divergent → subtract the zero momentum mode

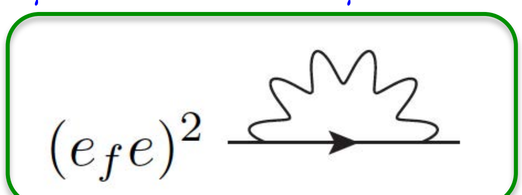
- Full covariant derivatives are defined introducing QED and QCD links:

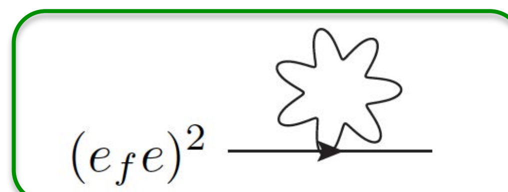
$$A_\mu(x) \to E_\mu(x) = e^{-iaeA_\mu(x)}$$

$$D^+_\mu q_f(x) = \left[E_\mu(x)\right]^{e_f} U_\mu(x)\, q_f(x+\hat{\mu}) - q_f(x)$$

$\left[E_\mu(x)\right]^{e_f}$ → QED; $U_\mu(x)$ → QCD

- Since $E_\mu(x) = e^{-ieA_\mu(x)} = 1 - ieA_\mu(x) - 1/2\, e^2 A^2_\mu(x) + \ldots$ the expansion of the lattice action up to $O(e^2)$ contains 2 contributions:





LATTICE QED

- The infrared problem is not specific of the lattice regularization but it is general for QED in a finite volume with periodic b.c.
  Already at the classical level, the Gauss' law for a charged particle is inconsistent for the zero mode:

$$\nabla_\mu^- F_{\mu\nu}(x) = j_\nu(x) \longrightarrow \nabla_i^- E_i(x) = \rho(x) \longrightarrow 0 = \sum_{\vec{x}} \nabla_i^- E_i(x) = e \sum_{\vec{x}} \delta^3(t,x) = e$$

- A solution to the infrared problem consists in removing the zero mode:

$$D_{\mu\nu}^{\perp}(x-y) = \sum_{k\neq 0} \frac{\delta_{\mu\nu}\; e^{ik(x-y)}}{\left[2\sin(k_\rho/2)\right]^2}$$

- We subtracted the zero mode in x-space and applied a stochastic technique

$$P^{\perp}\phi(x) \equiv \phi(x) - \frac{1}{V}\sum_y \phi(y)$$

$$\left[-\nabla_\rho^- \nabla_\rho^+\right]\phi_\mu(x) = P^{\perp}\eta_\mu(x) \longleftarrow \text{Real } Z_2 \text{ noise}$$

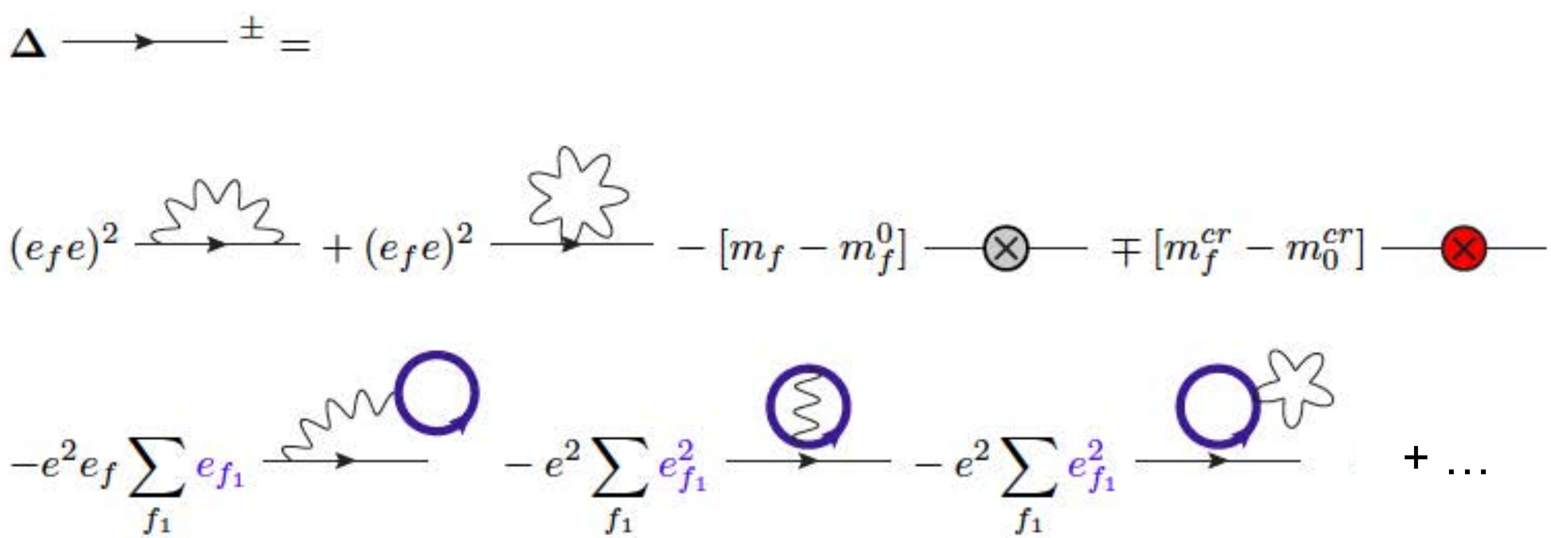
$$\phi_\mu(x) = \left[\frac{\delta_{\mu\nu}}{-\nabla_\rho^- \nabla_\rho^+} P^{\perp}\right]\eta_\nu(x) = \sum_y D_{\mu\nu}^{\perp}(x-y)\eta_\nu(y)$$

- Switching on the e.m. interactions requires the introduction of new counterterms which renormalize the couplings of the theory:

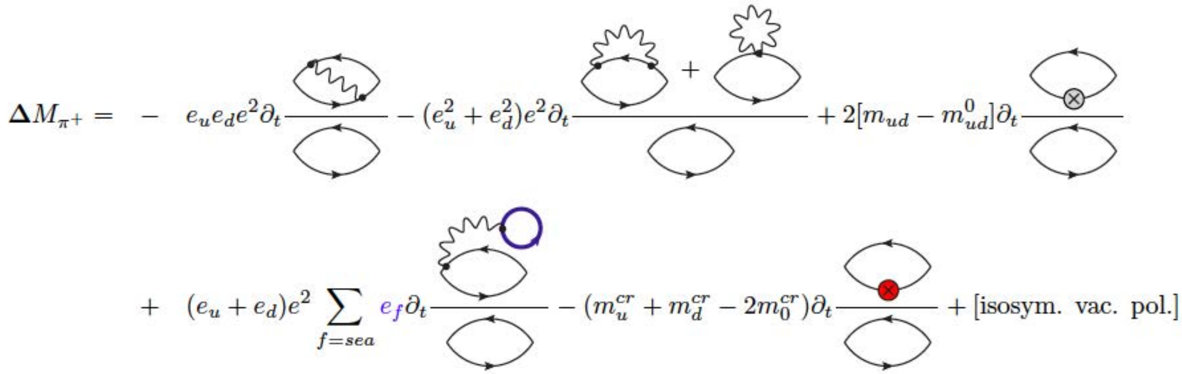
$$\vec{g}^0 = \left(0, g_s^0, m_u^0, m_d^0, m_s^0, \ldots\right) \rightarrow \vec{g} = \left(e^2, g_s, m_u, m_d, m_s, \ldots\right)$$

- For any observable, the leading isospin breaking expansion reads,

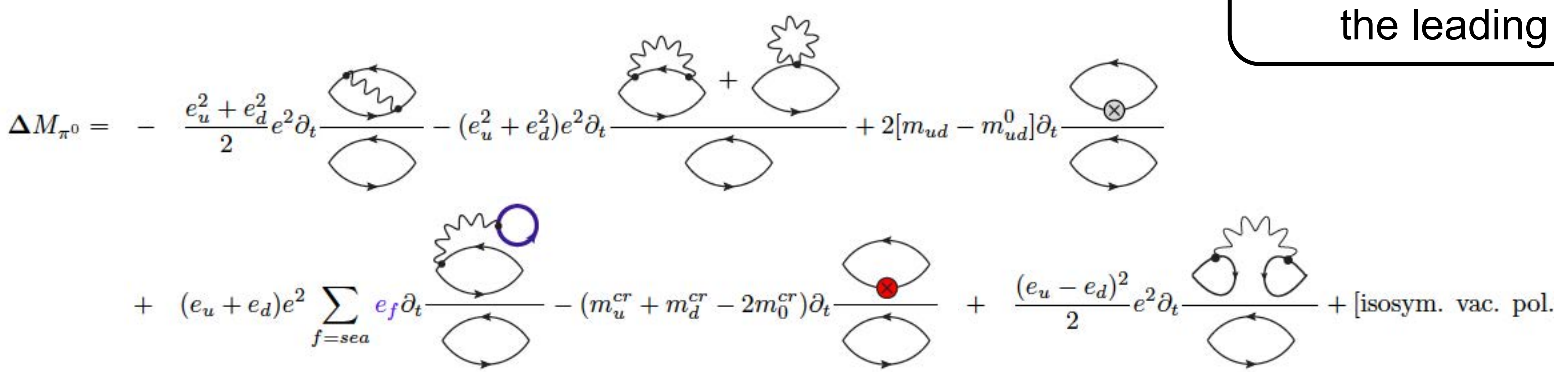
$$O(\vec{g}) = O(\vec{g}^0) + \left[e^2 \frac{\partial}{\partial e^2} + \left(g_s^2 - (g_s^0)^2\right)\frac{\partial}{\partial g_s^2} + \left(m_f - m_f^0\right)\frac{\partial}{\partial m_f} + \ldots \right] O(\vec{g})\Big|_{\vec{g}=\vec{g}^0}$$

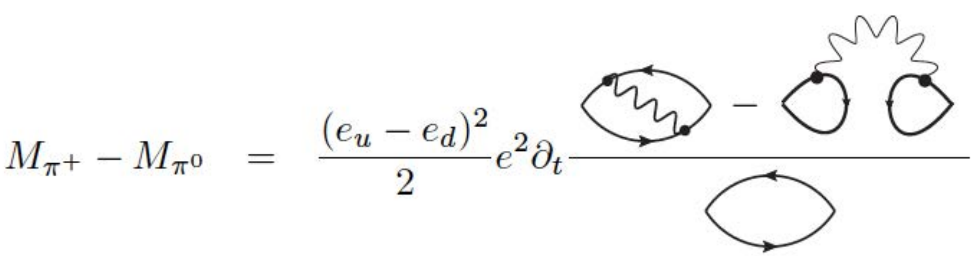


The charged-neutral pion mass splitting



Since $e_u \neq e_d$, sea quark contributions now enter at the leading order





Only 2 diagrams contribute to the pion mass splitting.

The disconnected diagram, of $O(\alpha_{em} m_{ud})$, has been neglected in the present calculation

Estimates of structure dependent contributions to $\Gamma_1(\Delta E)$

- We estimate the size of the neglected structure-dependent contributions to the decay $K^+/\pi^+ \to \ell \nu_\ell \gamma$ using chiral perturbation theory at $O(p^4)$

J. Bijnens, G. Ecker, J. Gasser, NPB 396 (1993) 81; V.Cirigliano, I.Rosell, JHEP 0710 (2007) 005

- Start with the decomposition in terms of Lorenz invariant form factors of the hadronic matrix element

$$H^{\mu\nu}(k,p_\pi) = \int d^4x\, e^{ikx} \langle 0 | T\left(j^\mu(x)\, J_W^\nu(0) \right) | \pi(p_\pi) \rangle$$

and separate the contribution corresponding to the approximation of a pointlike pion $H_{\rm pt}^{\mu\nu}$ from the structure dependent part $H_{\rm SD}^{\mu\nu}$

$$H^{\mu\nu} = H_{\rm pt}^{\mu\nu} + H_{\rm SD}^{\mu\nu}$$

- $H_{\rm pt}^{\mu\nu}$ is simply given by:

$$H_{\rm pt}^{\mu\nu} = f_\pi \left[g^{\mu\nu} - \frac{(2p_\pi - k)^\mu (p_\pi - k)^\nu}{(p_\pi - k)^2 - m_\pi^2} \right] \qquad \left(k_\mu H_{\rm pt}^{\mu\nu} = f_\pi p_\pi^\nu \right)$$

Estimates of structure dependent contributions to $\Gamma_1(\Delta E)$

- The structure dependent component $H_{\rm SD}^{\mu\nu}$ can be parametrized by four independent invariant form factors which we define as

$$H_{\rm SD}^{\mu\nu} = H_1\left[k^2 g^{\mu\nu} - k^\mu k^\nu\right] + H_2\left[\left(k\cdot p_\pi\right)k^\mu - k^2 p_\pi^\mu\right]\left(p_\pi - k\right)^\nu$$

$$-i\,\frac{F_V}{m_\pi}\varepsilon^{\mu\nu\alpha\beta}k_\alpha p_{\pi\beta} + \frac{F_A}{m_\pi}\left[\left(k\cdot p_\pi - k^2\right)g^{\mu\nu} - \left(p_\pi - k\right)^\mu k^\nu\right] \qquad \left(k_\mu H_{\rm SD}^{\mu\nu} = 0\right)$$

For the decay into a real photon, only F_V and F_A contribute

- At $O(p^4)$ in chiral perturbation theory F_V and F_A are constant:

$$F_V = \frac{m_P}{4\pi^2 f_\pi}\ , \qquad F_A = \frac{8 m_P}{f_\pi}\left(L_9^r + L_{10}^r\right)$$

J. Bijnens, G. Ecker, J. Gasser, NPB 396 (1993) 81

- For our estimates we use:

Direct measurement PDG 2014 →

$$F_V^{(\pi)} = 0.0254 \qquad F_A^{(\pi)} = 0.0119$$

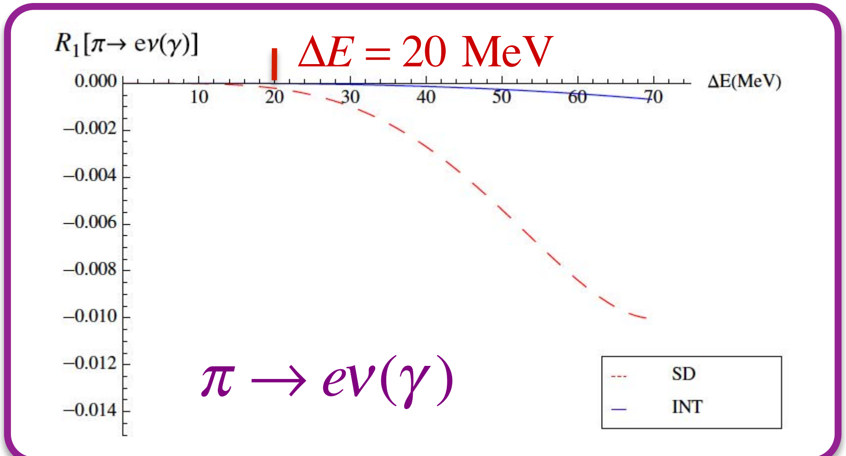
$$F_V^{(K)} = 0.096 \qquad F_A^{(K)} = 0.042$$

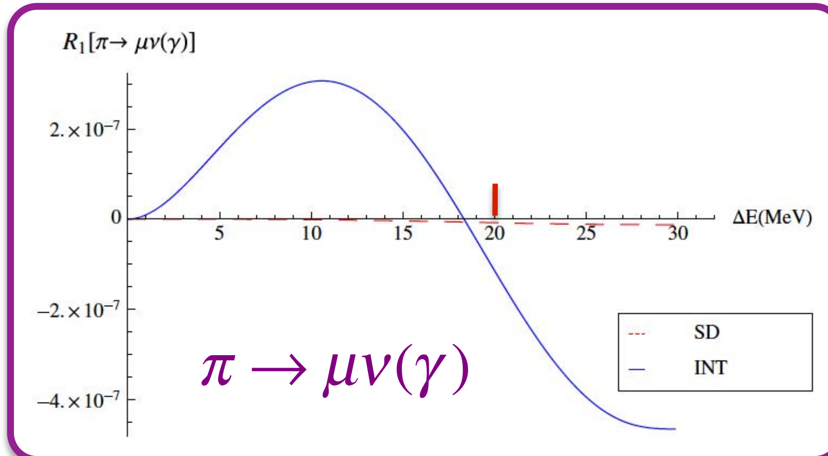
← ChPT

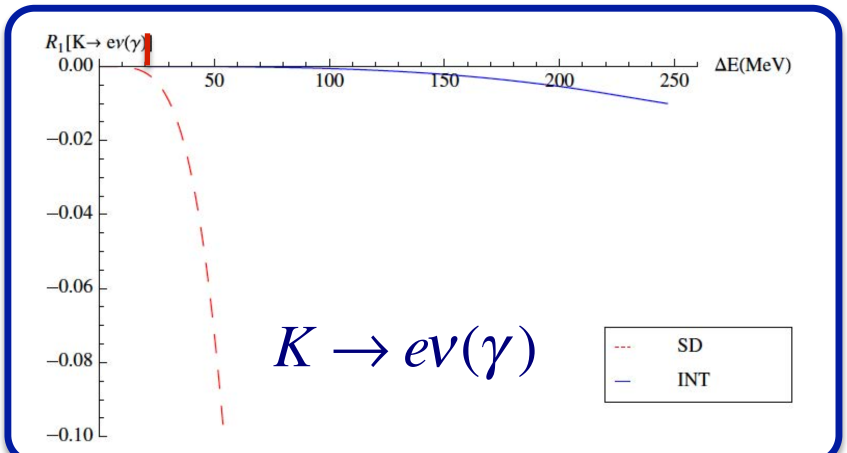
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)} \quad , \quad A = \{\mathrm{SD}, \mathrm{INT}\}$$

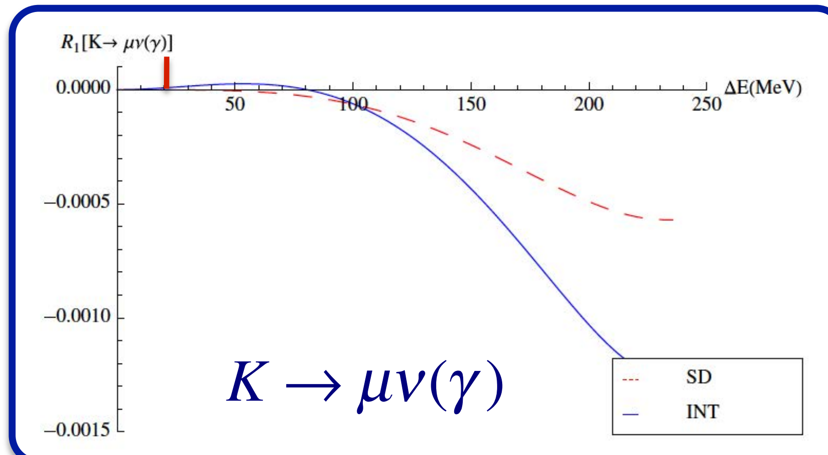
SD = structure dependent

INT = interference







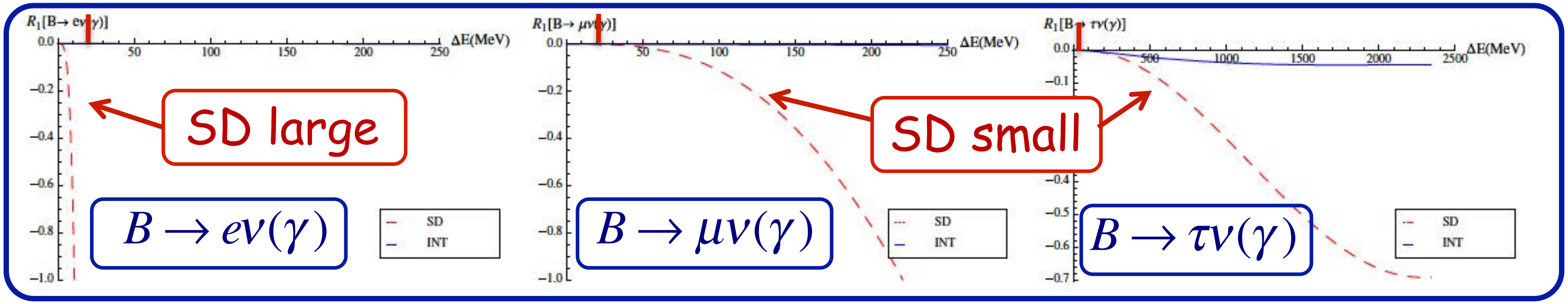


- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for $K \to e\nu(\gamma)$ but they are negligible for $\Delta E < 20\ \mathrm{MeV}$ (which is experimentally accessible)

Structure dependent contributions to decays of D and B mesons

- For the studies of D and B mesons decays we cannot apply ChPT
- For B mesons in particular we have another small scale, $m_{B^*} - m_B \simeq 45\ \text{MeV}$
  ➡ the radiation of a soft photon may still induce sizeable SD effects
- A phenomenological analysis based on a simple pole model for F_V and F_A confirms this picture

D. Becirevic, B. Haas, E. Kou, PLB 681 (2009) 257



$$F_V \simeq \frac{\tilde{C}_V}{1-(p_B-k)^2/m_{B^*}^2}$$

$$F_A \simeq \frac{\tilde{C}_A}{1-(p_B-k)^2/m_{B_1}^2}$$

Under this assumption the SD contributions to $B \to e\nu(\gamma)$ for $E_\gamma \simeq 20$ MeV can be very large, but are small for $B \to \mu\nu(\gamma)$ and $B \to \tau\nu(\gamma)$

- A lattice calculation of F_V and F_A would be very useful