Hunting Renormalons

Gunnar Bali
University of Regensburg

Mainz, 7.9.15

Based on

PoS (Lattice08) 215 [0812.1680][1]
PoS (Lattice10) 221 [1011.1165]
PRL 108 (12) 242002 [1111.3946]
PoS (Lattice11) 222 [1111.6158]
PRD 87 (13) 094517 [1303.3279][1]
PoS (Lattice13) 371 [1311.0114]
PRD 89 (14) 054505 [1401.7999]
PRL 113 (14) 092001 [1403.6477]
Confinement XI [1502.00086]

In collaboration with

- Clemens Bauer (Regensburg → Barcelona → Regensburg→ Münchener Rück)
- Antonio Pineda (UA Barcelona)
- [1]Christian Torrero (Regensburg→ Pisa→ Parma→ Marseille)

Outline

- Asymptotic behaviour of perturbative series and the OPE
- The heavy quark pole mass and the static energy
- Numerical stochastic perturbation theory (NSPT)
- Renormalon analysis
- The gluon condensate
- Summary

Anharmonic oscillator

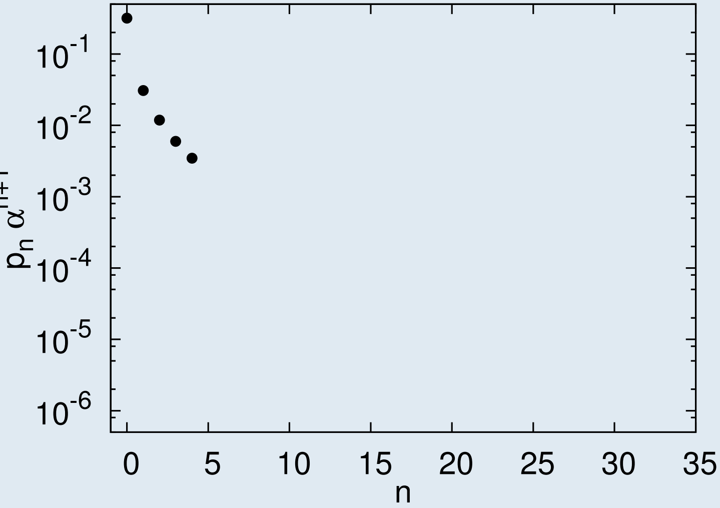
$$L = \frac{1}{2}(d_t\phi)^2 - V(\phi)\,, \qquad V(\phi) = \frac{1}{2}m^2\phi^2 + \alpha m^3\phi^4 \quad ([\phi] = \mathrm{mass}^{-1/2})$$

Ground state energy: $E_0(\alpha) = m\left(\frac{1}{2} + \sum_{n\geq 0} e_n\alpha^{n+1}\right)$

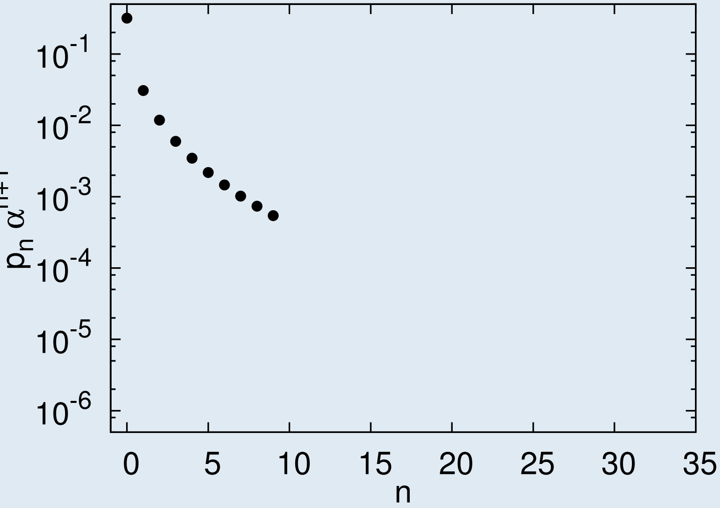
$$e_0 = \frac{3}{4}\,, \qquad e_4 = \frac{916731}{256}\,, \qquad e_8 = \frac{54626982511455}{65536}\,,$$
$$e_1 = -\frac{21}{8}\,, \qquad e_5 = -\frac{65518402}{1024}\,, \qquad e_9 \approx -0.24478940703\cdot 10^{11}\,,$$
$$e_2 = \frac{333}{16}\,, \qquad e_6 = \frac{2723294673}{2048}\,, \qquad e_{10} \approx 0.78933331600\cdot 10^{12}\,,$$
$$e_3 = -\frac{30885}{128}\,, \qquad e_7 = -\frac{1030495099053}{32768}\,, \qquad e_n \sim \frac{3\sqrt{6}}{\pi^{3/2}}(-3)^n\Gamma\left(n+\frac{3}{2}\right)\,.$$

The e_n diverge: the radius of convergence in α is zero!
However, $|e_{n_0}|\alpha^{n_0+1}$ is minimal for $n_0 \sim 1/(3\alpha)$.

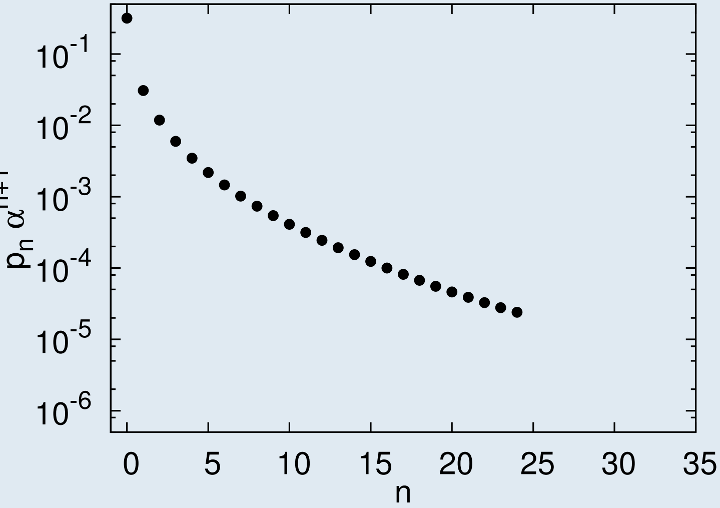
A more complex theory: α fixed



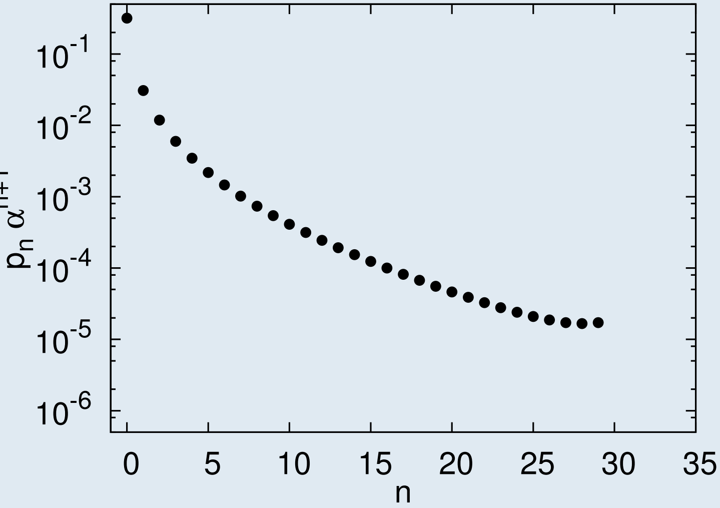
A more complex theory: α fixed



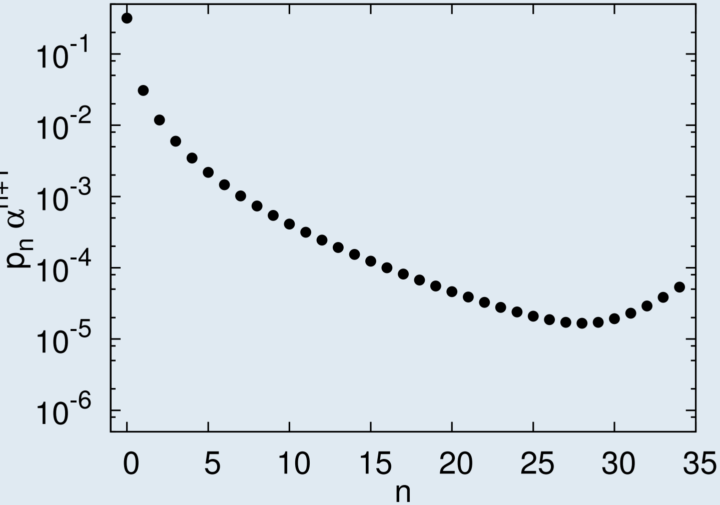
A more complex theory: α fixed



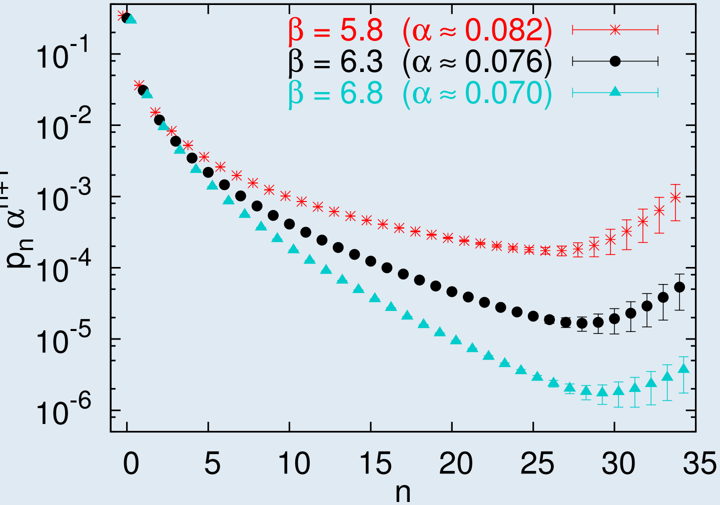
A more complex theory: α fixed



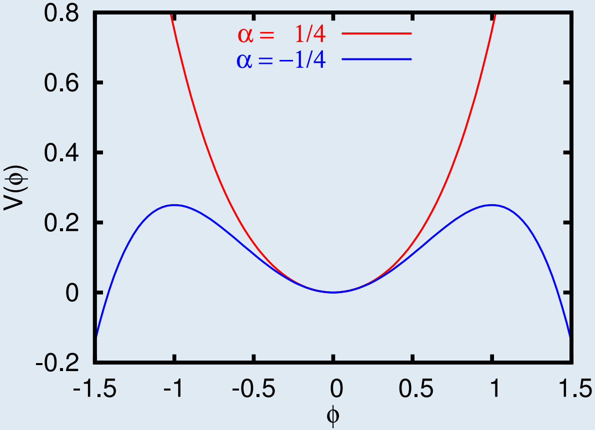
A more complex theory: α fixed



$\alpha = g^2/(4\pi) = 3/(2\pi\beta)$



Why does PT diverge for the anharmonic oscillator?



Vacuum is unstable for $\alpha < 0$ $\Rightarrow$ series cannot be analytic at $\alpha = 0$.
Similar to Dyson-instability of QED.
Such divergences also exist in QCD. This talk is about something else.

OPE and Renormalons (heuristic)

Starting point: $q > \nu\ (> \Lambda)$, $\langle O_d(\mu,\Lambda)\rangle \sim \Lambda^d$

$$\langle R(q)\rangle = C_0(q)\langle \mathbb{1}\rangle + \frac{C_d(q,\nu)}{q^d}\langle O_d(\nu,\Lambda)\rangle + \cdots$$

$$C_0(q) = \sum_n c_n \alpha^{n+1}(q)$$

Diagrammatic analysis: $c_n \overset{n\to\infty}{\sim} Na^{-n} n! n^{db} \simeq Na^{-n}\Gamma(n+1+db)$

$$\left| C_0 - \sum_{n=0}^{n_0} c_n \alpha^{n+1} \right| \lesssim \sqrt{n_0}\, c_{n_0}\alpha^{n_0+1}$$

$c_n\alpha^{n+1} \propto n! n^{db}(\alpha/a)^{n+1} \overset{n\,\mathrm{large}}{\sim} \exp\{(n+db)[\ln[(\alpha/a)(n+db)]-1]+\cdots\}$

Minimal for $n_0 \sim \frac{|a|}{\alpha}$. Minimal term: $c_{n_0}\alpha^{n_0+1} \sim \exp\left(-\frac{|a|}{\alpha}\right)$

OPE and Renormalons (heuristic) II

QCD β-function:

$$\frac{d\alpha}{d\ln\nu} = -2\alpha\left[\beta_0\frac{\alpha}{4\pi} + \beta_1\left(\frac{\alpha}{4\pi}\right)^2 + \beta_2\left(\frac{\alpha}{4\pi}\right)^3 + \cdots\right]$$

$$\Rightarrow \left(\frac{\Lambda}{q}\right)^d \approx \exp\left(-\frac{|a|}{\alpha}\right) \quad \text{with} \quad |a| = \frac{2\pi d}{\beta_0}$$

$\langle R\rangle$ does not depend on ν: the so-called *infrared* **renormalon** of the perturbative expansion C_0 is related to the *ultraviolet* behaviour of $\langle O_d\rangle$.

The ambiguity is due to the arbitrariness of the factorization between short-distance and long-distance contributions.

“Worst case”: $d = 1$! Ideal to detect the leading renormalon.

NB: we use the conventions ($N_f = 0$ QCD):

$$\beta_0 = 11\,,\ \beta_1 = 102\,,\ \beta_2^{\overline{\mathrm{MS}}} = \frac{2857}{2}\,,\ \beta_2^{\mathrm{latt}} = -6299.8999(6)\,.$$

Borel transform

Borel transform of series $C(\alpha) = \sum_n c_n \alpha^{n+1}$:

$$B[C](u) = \sum_n \frac{c_n}{n!} \left(\frac{4\pi}{\beta_0} u\right)^n$$

Borel integral of the series (Laplace transform in $1/\alpha$):

$$\tilde{C}(\alpha) = \frac{4\pi}{\beta_0} \int_0^\infty du \exp\left[-\frac{4\pi}{\beta_0 \alpha} u\right] B[C](u)$$

In general: $\nexists\, C$ and $\nexists\, \tilde{C}$.

$c_n = N a^{-n} \Gamma(n + 1 + db)/\Gamma(1 + db)$. Then (assuming $-db \notin \mathbb{N}$):

$$B[C](t) = \frac{N}{(1 - 2u/d)^{1+db}}$$

$d = a\beta_0/(2\pi) < 0$: ultraviolet renormalon at $u = d/2$ (alternating sign).

$d > 0$: infrared renormalon at $u = d/2$.

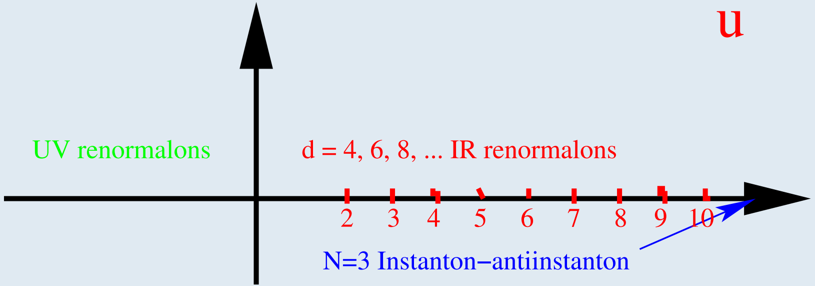
Also instanton-antiinstanton contributions at $u = \beta_0, 2\beta_0, \ldots$.

Borel transform II: Borel plane and large N_c

Large N_c: 't Hooft coupling $\lambda = g^2 N_c$, $\beta_0 = 11N_c/3$.
Instanton contribution vs dimension d renormalon:

$$\exp\left(-\frac{8\pi^2}{g^2}\right) = \exp\left(-\frac{8\pi^2}{\lambda}N_c\right) \quad \text{vs} \quad \exp\left(-\frac{8\pi^2}{\lambda}\frac{3}{11}d\right)$$

Instanton-antiinstantons move even further away at large N_c to $u = 11N_c/3$. Renormalons remain at the same positions $u = d/2$.



Borel transform III

- The Borel integral is dominated by the singularities closest to the origin.
- The Borel integral for $u < 0$ singularities can usually be defined.
- The Borel integral for IR renormalons depends on the integration path in the complex plane:

$$\mathrm{Im}\,\tilde{C} = \pm N \frac{2\pi^2 d^{1+db}}{\beta_0 \Gamma(1+db)} \exp\left(-\frac{2\pi d}{\beta_0 \alpha}\right) \left(\frac{\beta_0 \alpha}{4\pi}\right)^{-bd}$$

The ambiguity is of the magnitude of the minimal term in the asymptotic series (and of power correction type).

Heavy quark masses

$\exists$ various definitions:

- $m_{\overline{\rm MS}}(\nu) \to$ scale-dependent mass used in dimensional regularization.
- $m_{\rm OS} \to$ on-shell (pole) mass. "Natural" definition for heavy quark physics.
- $\hat{m} = m(\mu) \exp\left[-\int^{\alpha(\mu)} d\alpha' \frac{\gamma_m(\alpha')}{\beta(\alpha')}\right] \to$ renormalization group invariant.
- $m^{\rm latt}(a^{-1}) \to$ mass in a lattice scheme at a scale $a^{-1} \sim \nu$.
- $\ldots$

$$m_{\rm OS} = m_{\overline{\rm MS}}(\nu) + \underbrace{\sum_{n\geq 0} r_n \alpha^{n+1}(\nu)}_{\delta m(\nu)}$$

"OPE" predicts $r_n / m_{\overline{\rm MS}} \equiv c_n \sim n!$ for n large.

Relation to heavy-light mesons and glueballinos

$$\begin{aligned} M_B &= m_{\mathrm{OS}} + \overline{\Lambda}_B + \mathcal{O}(1/m) \\ &= m_{\overline{\mathrm{MS}}}(\nu) + \delta m_{\overline{\mathrm{MS}}}(\nu) + \overline{\Lambda}_B + \mathcal{O}(1/m) \\ &= m_{\mathrm{latt}}(a^{-1}) + \delta m_{\mathrm{latt}}(a^{-1}) + \overline{\Lambda}_B + \mathcal{O}(1/m) \end{aligned}$$

$E_{B,\mathrm{latt}}(a^{-1}) = \delta m_{\mathrm{latt}}(a^{-1}) + \overline{\Lambda}_B$ is the static-light meson mass in the lattice scheme. We will perturbatively expand $a\delta m_{\mathrm{latt}}$.

Analogously for the octet representation:

$$M_{\tilde{G}} = m_{\tilde{g},\mathrm{OS}} + \overline{\Lambda}_H + \mathcal{O}(1/m_{\tilde{g}})$$

M_B and $M_{\tilde{G}}$ are free of renormalons or power terms.
m_{OS} (short distance) will have an IR ambiguity and $\overline{\Lambda}$ (long distance) an UV ambiguity.

Above is not really an OPE since $\overline{\Lambda}$ is no expectation value of a local operator. To emphasize the similarity with the OPE we rewrite ($m_{\overline{\rm MS}} \sim q$)

$$\frac{M_B}{m_{\overline{\rm MS}}(\nu)} - 1 = \underbrace{\frac{\delta m_{\overline{\rm MS}}(\nu)}{m_{\overline{\rm MS}}(\nu)}}_{C_0(m)\langle \mathbb{1}\rangle} + \underbrace{\frac{\overline{\Lambda}_B}{m_{\overline{\rm MS}}(\nu)}}_{(C_1(\nu)/m)\,\langle O_1(\nu)\rangle} + \mathcal{O}\left[\left(\frac{\Lambda}{m}\right)^2\right].$$

$m_{\overline{\rm MS}}(\nu)$ is renormalon-free and so is $m_{\rm latt}(a^{-1})$.

$$m_{\rm OS} = Z(\nu) m_{\overline{\rm MS}}(\nu) = \left(1 + \frac{\delta m(\nu)}{m_{\overline{\rm MS}}(\nu)}\right) m_{\overline{\rm MS}}(\nu)$$

$$Z(\nu) \approx 1 + \sum_{n\geq 0} \underbrace{\frac{r_n(\nu)}{m_{\overline{\rm MS}}(\nu)}}_{:=c_n^{(3)}} \alpha^{n+1}(\nu) \quad \text{up to } \mathcal{O}(\Lambda^2/m^2)$$

Same for the lattice scheme. Things simplify for $\nu = m_{\overline{\rm MS}} \sim 1/a$.
The perturbative series of $\delta m(\nu)$ and $m_{\rm OS}$ share the leading renormalon.

$$m_{\mathrm{OS}} = m_{\overline{\mathrm{MS}}}(\nu) + \sum_{n\geq 0} r_n \alpha^{n+1}(\nu)$$

Leading renormalon in Borel plane at $u = 1/2$ ($d = 1$)

$$B[m_{\mathrm{OS}}](u) = \frac{N_m \nu}{(1-2u/d)^{1+db}}\left[1 + b_1\left(1-\frac{2u}{d}\right) + b_2\frac{db}{db-1}\left(1-\frac{2u}{d}\right)^2 + \cdots\right]$$

plus terms analytic around $u = d/2 = 1/2$. The next renormalon is at $u = 1$.

$$r_n \overset{n\to\infty}{=} N_m\,\nu\left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)}\left(1 + \frac{db}{(n+db)}b_1 + \frac{(db)^2}{(n+db)(n+db-1)}b_2 + \cdots\right)$$

$$b = \frac{\beta_1}{2\beta_0^2}\,,\quad s_1 = \frac{\beta_1^2 - \beta_0\beta_2}{4b\beta_0^4}\,,\quad s_2 = \frac{\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3}{16b^2\beta_0^6}\,.$$

$$\Lambda = \nu\exp\left\{-\left[\frac{2\pi}{\beta_0\alpha(\nu)} + b\ln\left(\frac{1}{2}\frac{\beta_0\alpha(\nu)}{2\pi}\right) + \sum_{j\geq 1} s_j\,(-b)^j\left(\frac{\beta_0\alpha(\nu)}{2\pi}\right)^j\right]\right\},$$

$$b_1 = ds_1\,,\quad b_2 = \frac{(ds_1)^2}{2} - ds_2 \quad \text{(for a trivial Wilson coefficient } C_d = 1\text{)}.$$

Why bother?

- The normalization N_m is of phenomenological relevance: e.g. top and bottom quark masses.
- Only the lowest few orders are diagrammatically accessible.
- Beyond this approximations are used, e.g., large-β_0 or models.
- Searches for $d = 4$ renormalon in $\mathcal{O}(\alpha^{\leq 20})$ expansions of the plaquette gave ambiguous results.
- Renormalon existence doubted by some (Suslov, wikipedia.org), renormalon dominance doubted by others (Zakharov and followers).

In the absence of a proof, why not attempt numerical clarification regarding the renormalon structure predicted by the OPE?

Buzzword: transseries

OPE is an example of a "transseries" [Écalle 80]: Wilson coeffs contain powers of α while $(\Lambda/q)^d \sim \exp[-2\pi/(\beta_0\alpha)]^d$ are essential singularities.

$$R(\alpha) = \sum_{n=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\ell=0}^{\ell_{\max}(d)} c_{n,d,\ell}\,[\alpha]^n\left[\exp\left(-\frac{a}{\alpha}\right)\right]^d[\ln(1/\alpha)]^{\ell}$$

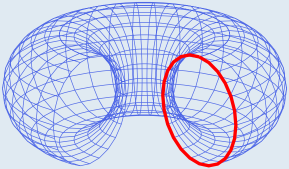
- Set of functions with these "trans-monomial" elements is closed under Borel transform, analytic continuation and Laplace transform.
- Any "reasonable" function admits transseries expansions.
- Existence of transseries implies consistency relations ("conspiracy") between the coefficients $c_{n,d,\ell}$ (resurgence). Example: Bogomolnyi–Zinn-Justin cancellations. Other example: this talk.
- Intricate connection between perturbative (α^n) and non-perturbative ($[\exp(-a/\alpha)]^d$) physics. ($c_{n,d,\ell}$ contain NP matrix elements.)

The Polyakov loop

Hypercubic $N_S^3 \times N_T$ lattice (periodic in time), gauge field $U_\mu(n) \in \mathrm{SU}(3)$

Polyakov Loop

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \prod_{n_4=0}^{N_T-1} U_4^R(n)$$



We implement triplet and octet representations $d_R = 3, 8$ as well as two discretizations of the static action (smeared/un-smeared).

$$P^{(R)}(N_S, N_T) = \frac{\ln\langle L^{(R)}(N_S, N_T)\rangle}{aN_T} = \sum_{n\geq 0} c_n^{(R)}(N_S, N_T)\alpha^{n+1}$$

$$\delta m = \lim_{N_S, N_T \to \infty} P^{(3)}(N_S, N_T)\,, \quad \delta m_{\tilde{g}} = \lim_{N_S, N_T \to \infty} P^{(8)}(N_S, N_T)$$

$$\delta m(a^{-1}) = \frac{1}{a}\sum_{n\geq 0} c_n^{(3)}\alpha^{n+1}(a^{-1})$$

$$c_n^{(3)} = r_n(\nu)/m(\nu) = \lim_{N_s,N_T\to\infty} c_n^{(3)}(N_S,N_T).$$

We directly compute the $c_n^{(R)}(N_S,N_T)$ using numerical stochastic perturbation theory.

Some cross-checks are made using diagrammatic methods.

Stochastic Quantization

alternative way of calculating expectation values in Euclidian FT
Parisi, Wu (1981)

1. additional, fictitious time coordinate t:

$$\phi(x) \to \phi(x,t)$$

2. Langevin equation

$$\frac{d}{dt}\phi(x,t) = -\frac{\delta S}{\delta\phi(x,t)} + \eta(x,t)$$

3. expectation values via

$$\overline{\mathcal{O}[\phi]} = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \int [d\eta]\,\mu[\eta]\,\mathcal{O}\,[\phi] = \langle\mathcal{O}[\phi]\rangle$$

Numerical Stochastic Perturbation Theory (NSPT)

Perturbative expansion via Stochastic Quantization
Di Renzo, Marchesini, Marenzoni, Onofri (1994)
Good review: Di Renzo, Scorzato (2004)

1. Langevin equation for gauge fields:

$$\frac{d}{dt}U_\mu(x,t) = -i\left(\nabla_{x,\mu}S[U] + t^c\eta^c_\mu(x,t)\right)U_\mu(x,t)$$

2. Perturbative expansion of gauge fields

$$U = \mathbb{1} + \beta^{-\frac{1}{2}}U^{(1)} + \beta^{-1}U^{(2)} + \cdots + \beta^{-\frac{M}{2}}U^{(M)}; \quad \beta^{-1} = \frac{2\pi\alpha}{N_c}$$

Langevin update $\rightarrow$ hierarchical system of differential equations
$\Rightarrow$ discretize stochastic time t and integrate numerically.
We use a new $\mathcal{O}(\epsilon^2)$ integrator. Needs to be extrapolated to zero!

Expansion up to $X^{(M)}$

$$cX \longrightarrow (cX)^{(m)} = cX^{(m)}, \qquad m = 1, \ldots, M;$$

$$X = Y + Z \longrightarrow X^{(m)} = Y^{(m)} + Z^{(m)}, \qquad m = 1, \ldots, M;$$

$$X = Y \cdot Z \longrightarrow X^{(m)} = \sum_{j=0}^{m} Y^{(j)} \cdot Z^{(m-j)}, \qquad m = 1, \ldots, M.$$

Calculation costs

⇒ bad: memory need $\propto M$

⇒ good: simulation time $\propto M^2$

diagrammatic perturbation theory: $\propto M!$

⇒ NSPT great for high-order calculations!!!

Other technicality: twist

In a finite periodic box naive perturbation theory involves sums over momenta (we assume even N_μ):

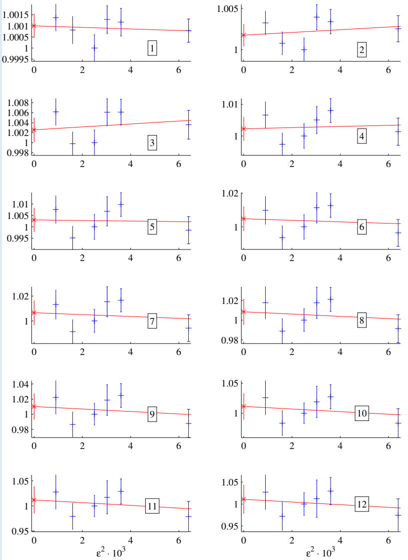
$$p_\mu \in \left\{ n_\mu \frac{2\pi}{N_\mu} : n_\mu = -\frac{N_\mu}{2} + 1, -\frac{N_\mu}{2} + 2, \ldots, \frac{N_\mu}{2} \right\} ,$$

where $N_1 = N_2 = N_3 = N_S$, $N_4 = N_T$. This includes $p = 0$.

Standard to discard this contribution, resulting in an error $\propto 1/(N_S^3 N_T)$.

We employ so-called twisted boundary conditions (TBC) in all three spatial directions.
This eliminates zero modes and strongly reduces finite size effects, at low orders.
(We don't know how to include zero modes. Their presence results in a non-perturbative scale $g^{1/2}/(Na) < 1/(Na) < 1/a$. Just subtracting/ignoring $p = 0$ terms messes up the Finite Size Effect OPE.)

Extrapolation $\epsilon^2 \to 0$



Unsmeared triplet representation

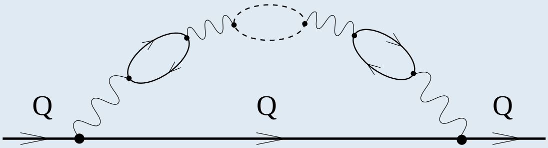
Coefficients $c^{(3)}_{n-1}(N_S, N_T)$ of α^n

$N_S = N_T = 16$

$\epsilon = 0.03, 0.03, 0.05, 0.055, 0.06, 0.08$

Large volume behaviour

"Bubble chain"



Receives dominant order n contribution from momenta $k \sim e^{-n}$.

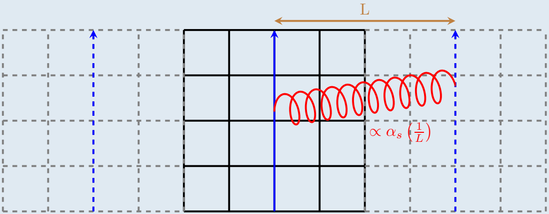
Finite box: spatial momentum cut-off $k \gtrsim 1/(aN_S)$.
Due to this IR cut-off, we cannot encounter infrared divergencies.

For a direct detection of the expected asymptotic behaviour of, e.g., the Polyakov line expansion at order n lattice sizes $N_S \propto e^n$ are necessary.

$\Rightarrow$ Understanding of finite size effects is of utmost importance.

Large volume behaviour II

Interactions with mirror images produce $1/L = 1/(aN_S)$ Coulomb terms.

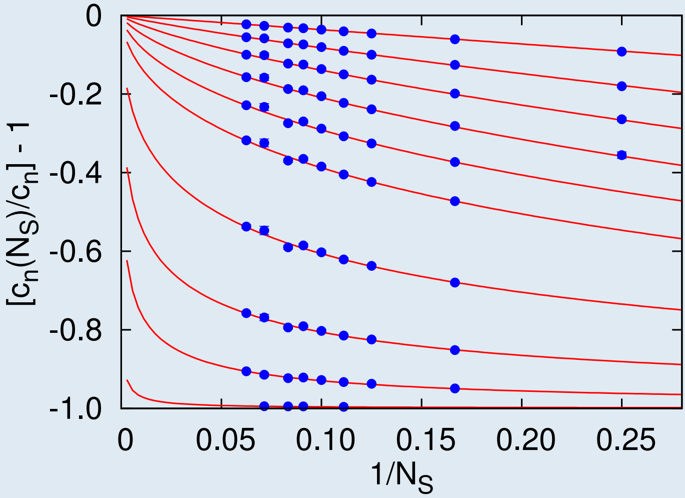


Large N_S: $$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right)$$

$$\delta m(N_S) = \underbrace{\frac{1}{a}\sum_n c_n \alpha^{n+1}(1/a)}_{\delta m} - \frac{1}{aN_S}\sum_n f_n \alpha^{n+1}\left(1/(aN_S)\right) + \mathcal{O}(1/N_S^2)$$

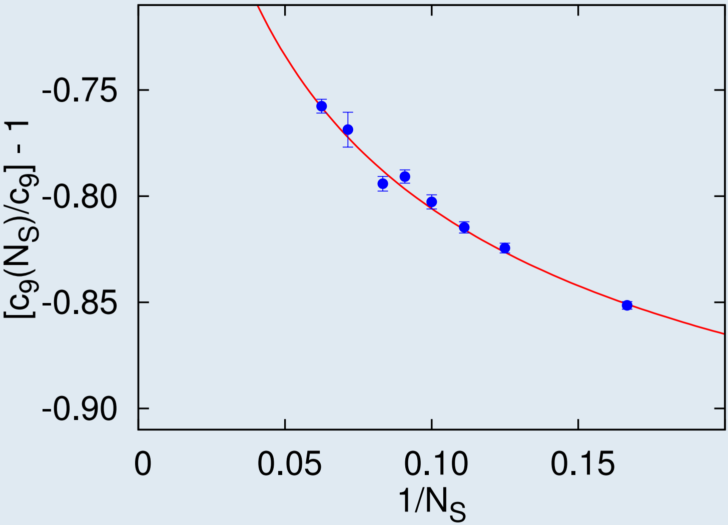
Finite size corrections are (up to lattice artefacts for $N_S \not\gg 1$) entirely dictated by the **perturbative** "OPE" and the renormalization group!

Large volume extrapolation



Global fit result: $n = 0, 1, 2, 3, 4, 5, 7, 9, 11, 15$. Two parameters per order.

Zoom for $n = 9$



Renormalon dominance

We fit using $\{\beta_0\}$, $\{\beta_0, \beta_1\}$ and $\{\beta_0, \beta_1, \beta_2^{\mathrm{latt}}\}$. Differences are included as a systematic error (the dominant one). Infinite volume triplet results (first two diagrammatic):

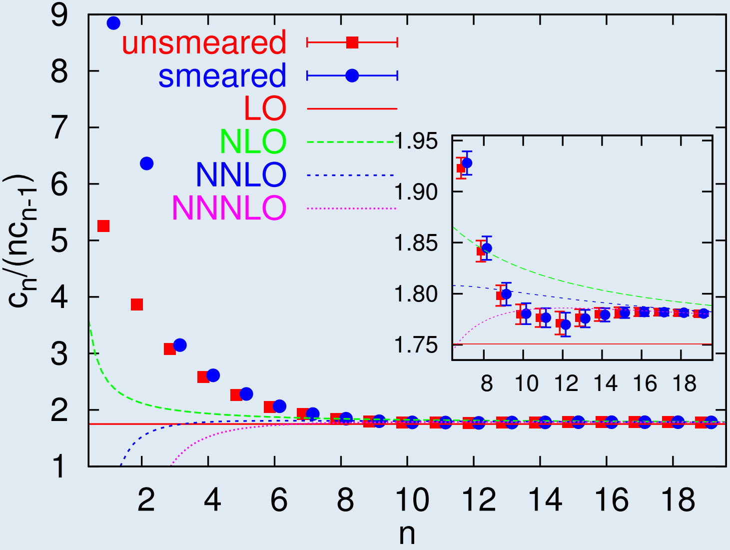
$$c_0 = 2.11727435708\ldots\,, \qquad c_1 = 11.1425(25)\,, \qquad c_2 = 86.10(13)\,,$$
$$c_3 = 794.5(1.6)\,, \qquad c_4 = 8215(34)\,, \qquad \ldots\,.$$

Below we again assume $C_d = 1$. In our case $d = 1$:

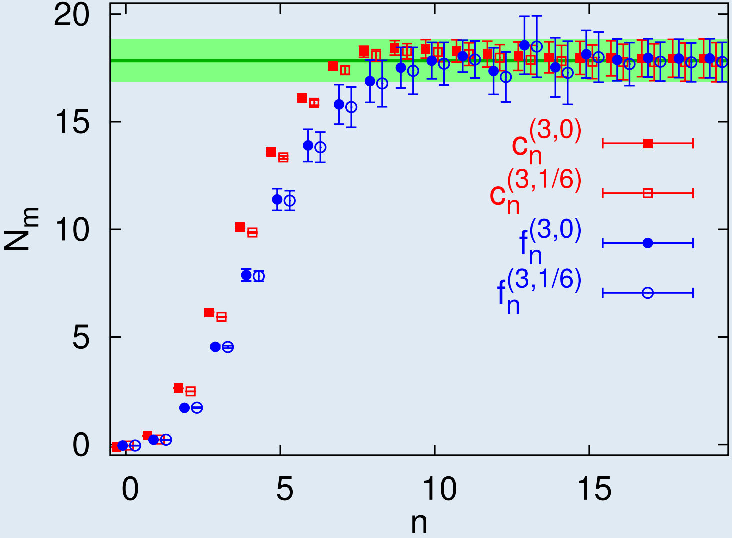
$$\frac{c_n^{(3)}}{c_{n-1}^{(3)}}\frac{1}{n} = \frac{c_n^{(8)}}{c_{n-1}^{(8)}}\frac{1}{n}$$
$$= \frac{\beta_0}{2\pi d}\left\{\underbrace{1}_{\mathrm{LO}} + \underbrace{\frac{db}{n}}_{\mathrm{NLO}} - \underbrace{\frac{d^2bs_1}{n^2}}_{\mathrm{NNLO}} + \underbrace{\frac{1}{n^3}\left[db(d^2b(s_1+2s_2) - ds_1)\right]}_{\mathrm{NNNLO}} + \mathcal{O}\left(\frac{1}{n^4}\right)\right\}$$

s_2 ($\mathrm{N}^3\mathrm{LO}$ in $1/n$) depends on $\beta_3^{\mathrm{latt}} = -1.16(12)\cdot 10^6$ (our determination).

Renormalon dominance II



Renormalon normalization: triplet



Asymptotic divergence

Minimal term $r_{n_0}\alpha^{n_0+1}$ for

$$(n_0+db)\frac{\beta_0\alpha}{2\pi d} = \exp\left\{-\frac{1}{2(n_0+db)} + \mathcal{O}\left[\frac{1}{(n_0+db)^2}\right]\right\}.$$

This gives ($\nu = a^{-1} = m$: $r_n(\nu) = a^{-1}c_n$)

$$r_{n_0}(\nu)\alpha^{n_0+1}(\nu) = \frac{2\pi d^{1/2+db}}{2^{db}\Gamma(1+db)}\sqrt{\frac{\alpha(\nu)}{\beta_0}}N\Lambda^d\left[1+\mathcal{O}(\alpha)\right].$$

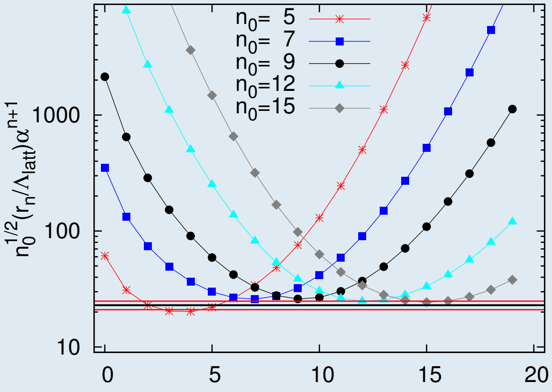
Uncertainty of the sum, truncated at order n_0:

$$\sqrt{n_0}\,|r_{n_0}(\nu)|\alpha^{n_0+1}(\nu) = \frac{(2\pi)^{3/2}d^{1+db}}{2^{db}\Gamma(1+db)}\frac{|N|\Lambda^d}{\beta_0} \overset{d=1}{\approx} 1.206|N|\Lambda \overset{N=N_m}{\approx} 180\,\text{MeV}$$

This is **exactly** $\sqrt{2/\pi}$ times the ambiguity of the Borel integral and scheme- and scale-independent up to β_2-corrections (i.e. $\mathcal{O}(1/n_0)$).

Asymptotic divergence II

$$(\sqrt{n_0}r_n/\Lambda)\alpha^{n+1} = c_n\alpha^{n+1}\sqrt{n_0}\exp[2\pi/(\beta_0\alpha(\nu)][\beta_0\alpha(\nu)/(4\pi)]^b + \cdots$$



$\beta = 3/(2\pi\alpha)$: $(n_0, \beta) \approx (5, 5.0), (6, 5.8), (7, 6.6), (15, 13.3)$, $\alpha(\nu) \in [0.036, 0.096]$

If the scale ν is higher (and α is smaller), you have to work harder!

Implications for dimensional regularization

The combination $N_m\Lambda$ is RG-invariant and scheme-independent.

$$\text{Exact relation:} \qquad N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}} \Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}$$

$$\Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}} \Lambda_{\rm latt} \approx 28.809338139488\, \Lambda_{\rm latt}$$

$$N_m^{\overline{\rm MS}} = 0.620(35)\,, \quad C_F/C_A\, N_{m_{\tilde g}}^{\overline{\rm MS}} = 0.610(41)$$

This is very similar to continuum-scheme extrapolation from $n \leq 3$. Possibly in the $\overline{\rm MS}$-scheme renormalon dominance already sets in at $n = 2, 3$. Assume:

$$c_3^{\overline{\rm MS}} \simeq N_m^{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^3 \frac{\Gamma(4+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(3+b)} b_1 + \frac{b^2}{(3+b)(2+b)} b_2 + \cdots \right).$$

Then

$$c_3^{\overline{\rm MS}} \simeq r_3/m_{\overline{\rm MS}} = 37.9(2.2)\,, \quad d_3 \simeq 352\,, \quad \beta_3^{\rm latt} = -1.16(12)\cdot 10^6$$

Gluon condensate

Plaquette

$$P = \frac{\pi\alpha}{36} a^4 G^c_{\mu\nu} G^c_{\mu\nu} + \mathcal{O}(a^6) = a^4 \frac{\alpha^2\pi}{9\beta(\alpha)} T^{\rm latt}_{\mu\mu}$$

$$\langle P \rangle = \sum_{n\geq 0} p_n \alpha^{n+1} \langle \mathbb{1} \rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha) \langle GG \rangle a^4 + C_6(\alpha) \langle O_6 \rangle a^6 + \cdots$$

with the RGI gluon condensate

$$\langle GG \rangle = -\frac{2}{\beta_0} \left\langle \Omega \left| \frac{\beta(\alpha)}{\alpha} G^c_{\mu\nu} G^c_{\mu\nu} \right| \Omega \right\rangle = \left\langle \Omega \left| [1 + \mathcal{O}(\alpha)] \frac{\alpha}{\pi} G^c_{\mu\nu} G^c_{\mu\nu} \right| \Omega \right\rangle$$

and the Wilson coefficient (trace anomaly: $\beta(\alpha) P \propto T_{\mu\mu}$):

$$C_{\rm G}(\alpha) = 1 + \sum_{k\geq 0} c_k \alpha^{k+1} = -\frac{\beta_0 \alpha^2}{2\pi\beta(\alpha)}$$

$$= 1 - \frac{\beta_1}{\beta_0} \frac{\alpha}{4\pi} + \frac{\beta_1^2 - \beta_0\beta_2}{\beta_0^2} \left(\frac{\alpha}{4\pi}\right)^2 - \frac{\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3}{\beta_0^3} \left(\frac{\alpha}{4\pi}\right)^3 + \mathcal{O}(\alpha^4)\,.$$

Gluon condensate II

Gluon condensate can in principle be obtained by subtracting perturbative series from non-perturbative Monte-Carlo data. Problems:

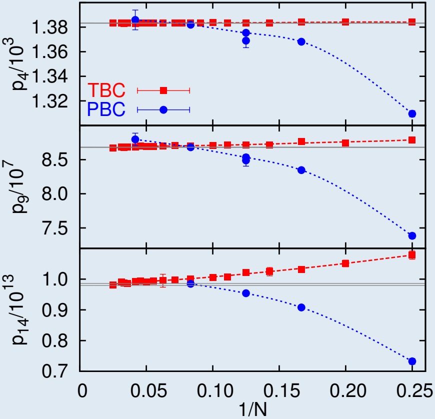
- Series has ambiguity that is not necessarily much smaller than $\langle GG \rangle$.
- $p_n/(np_{n-1}) \sim \beta_0/(2\pi d)$. If renormalon dominance sets in around $n \sim 7$ at $d = 1$, $n \gtrsim 25$ is expected for $d = 4$ in the lattice scheme.
- Di Renzo et al. hep-th/9502095, 8 loops,
- Di Renzo et al. hep-lat/0011067, 10 loops,
- Rakow hep-lat/0510046, 16 loops,
- Horsley et al. 1205.1659, 20 loops, $V \leq 12^4$.

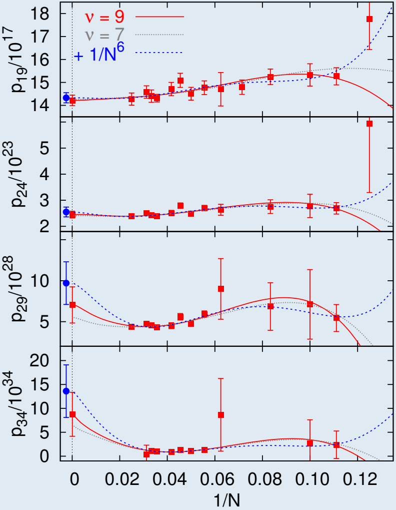
Horsley et al (Sec IV.A)

We do not observe such a factorial growth up to loop order $n = 20$. This is a fact which we have to accept and appreciate theoretically [Suslov, Zakharov].

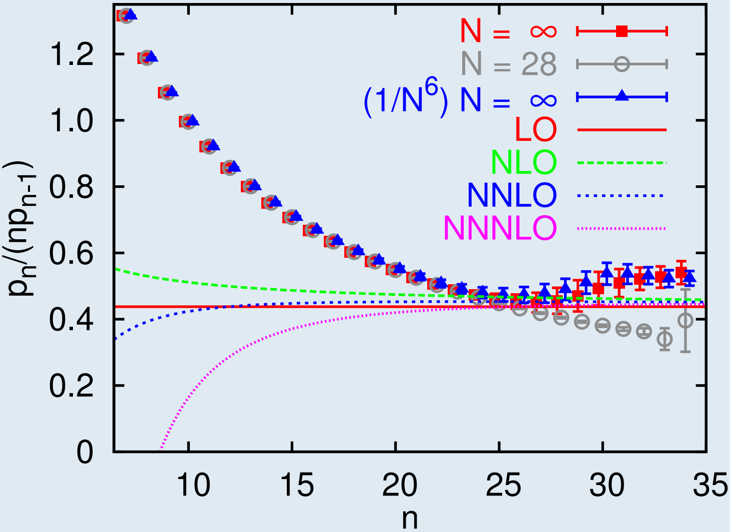
Comparison with other data, extrap. $V \leq 40^4 \to \infty^4$

Our TBC p_n vs. periodic (PBC) of hep-lat/0011067 and 1205.1659.



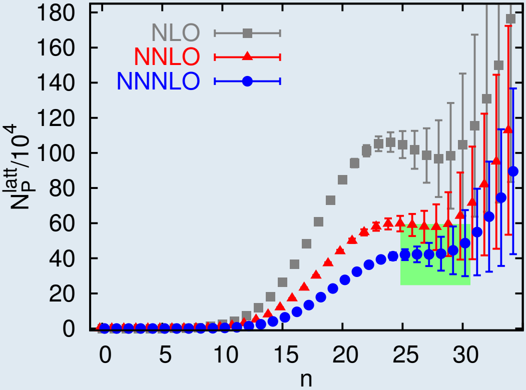


Ratios of subsequent coefficients



Normalization of the gluon condensate renormalon

$$p_n \stackrel{n\to\infty}{=} N_P^{\rm latt} \left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \left\{1 + \frac{20.08931\ldots}{n+db} + \frac{505\pm 33}{(n+db)(n+db-1)} + \cdots\right\}$$



$$N_P^{\rm latt} = 42(17)\cdot 10^4$$

$$N_{\rm G}^{\rm latt} := \frac{36}{\pi^2} N_P^{\rm latt} = 1.5(6)\cdot 10^6$$

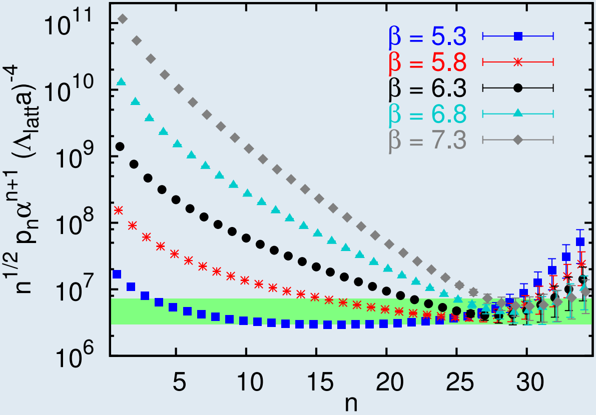
$$N_{\rm G}^{\overline{\rm MS}} = 2.2(9)$$

$$N_{\rm G,large-\beta_0}^{\overline{\rm MS}} = \frac{3e^{10/3}}{2\pi^3} \approx 1.36$$

Is ambiguity of size $\sim 2.2\,\Lambda_{\overline{\rm MS}}^4$? Much "worse"!

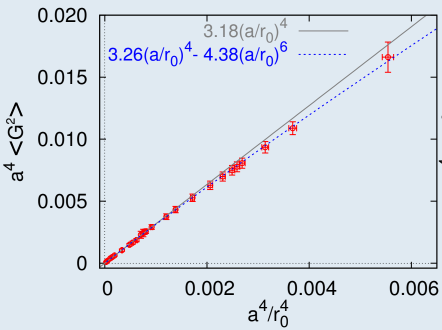
Asymptotic divergence: ambiguity and minimal term

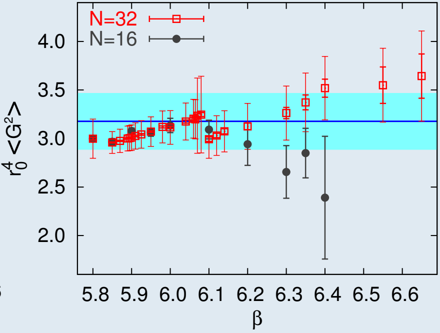
$$d = 4 : \quad \left.\frac{\sqrt{n} p_n \alpha^{n+1}}{(\Lambda a)^4}\right|_{n=n_0} \approx 12.06\, N_P \quad \Rightarrow \delta\langle GG\rangle \simeq 27(11)\Lambda^4_{\overline{\mathrm{MS}}} \sim 0.085\,\mathrm{GeV}^4$$



The non-perturbative gluon condensate

$$\langle GG \rangle = a^{-4}(\alpha)\frac{36}{\pi^2 C_G(\alpha)}\left[\langle P \rangle_{\mathrm{MC}}(\alpha) - \sum_{n=0}^{n_0} p_n \alpha^{n+1}\right] + \mathcal{O}(a^2)$$

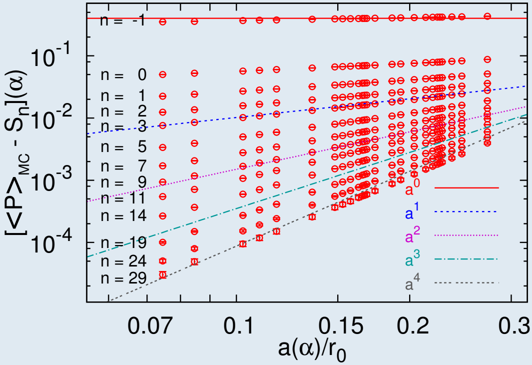




$$\langle GG \rangle = 3.18(29) r_0^{-4} = 24.2(8.0)\Lambda_{\overline{\mathrm{MS}}}^4 \simeq 0.077\,\mathrm{GeV}^4\,,$$

$$\delta\langle GG \rangle \simeq 27(11)\Lambda_{\overline{\mathrm{MS}}}^4 \sim 0.085\,\mathrm{GeV}^4 \qquad (r_0 \approx 0.5\,\mathrm{fm})$$

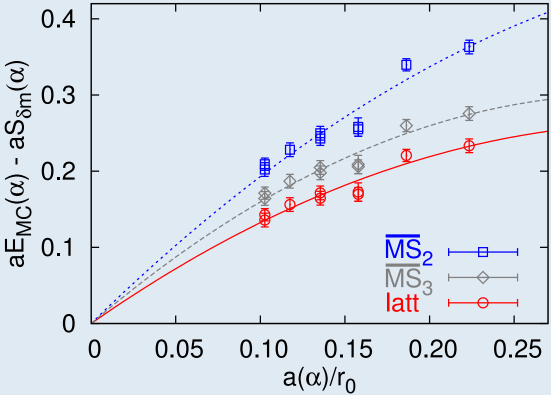
Fixed order truncations



Low order truncations can, within a certain range of lattice spacings (or momenta), mimic “condensates” of dimensions $d < 4$. However, these (non-universal) slopes effectively describe short-distance physics rather than non-perturbative features.

Is ambiguity really so large and scheme-independent?

We can test this for the example of the HQET binding energy, subtracting different truncated resummations from Monte-Carlo data:



$\delta\overline{\Lambda} = 0.450(44)r_0^{-1}$. Indeed: $\overline{\Lambda}r_0 = 1.55(8), 1.89(8), 2.17(8)$.

Summary

- Factorial growth in a lattice scheme from order α^9 to α^{20} of the coefficients of the static energy and from order α^{25} to α^{35} for the plaquette, in accordance with the OPE renormalon expectations.
- Normalizations of the leading heavy quark and heavy gluino pole mass renormalons in the $\overline{\mathrm{MS}}$ scheme:

$$N_m^{\overline{\mathrm{MS}}} = 0.620(35)\,, \quad C_F/C_A\, N_{m_{\tilde{g}}}^{\overline{\mathrm{MS}}} = 0.610(41)\,.$$

- The heavy quark pole mass can only be defined within an ambiguity of order $1.2\, N_m^{\overline{\mathrm{MS}}} \Lambda_{\overline{\mathrm{MS}}} \approx 180$ MeV.
- Similarly, for the plaquette $36/\pi^2 P$: $N_{\mathrm{G}}^{\overline{\mathrm{MS}}} = 2.2(9)$. This means that $\delta\langle GG\rangle \sim 0.085\,\mathrm{GeV}^4$, while $\langle GG\rangle \sim 0.077\,\mathrm{GeV}^4$.
  Impact on α_s determinations from τ decays??
- Outlook: in view of e.g. precision top quark physics we consider determining N_m with sea quarks. Improved actions with smaller $\Lambda_{\overline{\mathrm{MS}}}/\Lambda_{\mathrm{latt}}$-ratios may help detecting asymptotics at lower orders.