TALENT School Mainz

Few-Body Reactions

Summer 2022

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Problem 1.1: Continuity equation

Consider the charge $\rho(\mathbf{x})$ and current $\mathbf{J}(\mathbf{x})$ operators, which can be expanded in terms of a one-body operator plus a two-body operator, etc. The continuity equation states that charge and current operators are related as

$$\nabla \cdot \mathbf{J}(\mathbf{x}) = -i\big[\mathrm{H}, \rho(\mathbf{x})\big] \,,$$

where H is the Hamiltonian of the system made by the sum of the kinetic energy and the optential energy, $H = T + V$. Let the one-body operators be

$$\begin{aligned} \rho_{(1)}(\mathbf{x}) &= e\sum_i^A \Big(\frac{1+\tau_i^z}{2}\Big)\delta(\mathbf{x}-\mathbf{r}_i) \\ \mathbf{J}^c_{(1)}(\mathbf{x}) &= \frac{e}{2m}\sum_i^A \Big(\frac{1+\tau_i^z}{2}\Big)\big\{\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)\big\} \,. \end{aligned}$$

Show that

$$\nabla \cdot \mathbf{J}^c_{(1)}(\mathbf{x}) = -i\big[\mathrm{T}, \rho_{(1)}(\mathbf{x})\big] \,,$$

where

$$T = \sum_i^A \frac{p_i^2}{2m} \,. \tag{1}$$

NB.:We do not consider the spin current, because that is purely transverse and has divergence zero.
Hint: Work out the commutator and make use of this useful formula
$\Big[\mathbf{p}, f(\mathbf{r})\Big] = -i\nabla f(\mathbf{r})$

Solution: We start from

$$
\begin{aligned}
\big[\mathrm{T}, \rho_{(1)}(\mathbf{x})\big] &= \sum_{i,j}^{A} \Big[\frac{p_i^2}{2m}, e\Big(\frac{1+\tau_j^z}{2}\Big)\delta(\mathbf{x}-\mathbf{r}_j)\Big] \\
&= \sum_{i,j}^{A} \frac{e}{2m}\Big(\frac{1+\tau_j^z}{2}\Big) \Big[p_i^2, \delta(\mathbf{x}-\mathbf{r}_j)\Big] \qquad (2) \\
&= \sum_{i,j}^{A} \frac{e}{2m}\Big(\frac{1+\tau_j^z}{2}\Big) \Big\{\mathbf{p}_i \cdot \Big[\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_j)\Big] + \Big[\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_j)\Big] \cdot \mathbf{p}_i\Big\}
\end{aligned}
$$

The commutators are zero for $i \neq j$ since momentum and coordinate operators of different particles commute. We then have

$$
\big[\mathrm{T}, \rho_{(1)}(\mathbf{x})\big] = \sum_{i}^{A} \frac{e}{2m}\Big(\frac{1+\tau_i^z}{2}\Big) \Big\{\mathbf{p}_i \cdot \Big[\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)\Big] + \Big[\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)\Big] \cdot \mathbf{p}_i\Big\} \qquad (3)
$$

and by using the useful formula $\Big[\mathbf{p}, f(\mathbf{r})\Big] = -i\nabla f(\mathbf{r})$, we get to

$$
\begin{aligned}
\big[\mathrm{T}, \rho_{(1)}(\mathbf{x})\big] &= -\frac{ie}{2m}\sum_{i}^{A}\Big(\frac{1+\tau_i^z}{2}\Big)\nabla_{r_i} \cdot \Big\{\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)\Big\} \qquad (4) \\
&= \frac{ie}{2m}\nabla_x \cdot \sum_{i}^{A}\Big(\frac{1+\tau_i^z}{2}\Big)\Big\{\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)\Big\} = i\nabla_x \cdot \mathbf{J}^c_{(1)}(\mathbf{x})\,. \qquad (5)
\end{aligned}
$$

Problem 1.2: Two-body currents

Looking at the previous problem, whenever the Hamiltonian has an interaction term V that does not commute with the charge operator $\left[V, \rho_{(1)}(\mathbf{x})\right] \neq 0$, then there has to be a two-body current $\mathbf{J}_{(2)}(\mathbf{x})$, so that

$$\nabla \cdot \mathbf{J}_{(2)}(\mathbf{x}) = -i\left[V, \rho_{(1)}(\mathbf{x})\right] .$$

If V has an exchange term in the isospin space that goes like $\sum_{ij} \vec{\tau}_i \cdot \vec{\tau}_j$, where $\vec{\tau}_{i,j}$ are the isospin vectors of particle i and j, respectively, then $\left[V, \rho_{(1)}(\mathbf{x})\right] \neq 0$ and there has to be a two-body current. Show that in this situation, the isospin part of the two-body current will go like $\sum_{ij}(\vec{\tau}_i \times \vec{\tau}_j)^z$.

Hint: Work out the commutator and remember that, if the two isospin operators act on the same particle, then $\left[\tau^k, \tau^m\right] = 2i\epsilon_{kml}\tau^l$ with (k, m, l) indicating the components (x, y, z)

Solution:

We start by working out the commutator:

$$\sum_{i,j,k}\left[\vec{\tau}_i \cdot \vec{\tau}_j, \tau_k^z\right]\delta(\mathbf{x}-\mathbf{r}_k) = \sum_{i,j,k}\left[\tau_i^x\tau_j^x + \tau_i^y\tau_y^y + \tau_i^z\tau_j^z, \tau_k^z\right]\delta(\mathbf{x}-\mathbf{r}_k) \quad (6)$$

$$= \sum_{i,j,k}\left[\tau_i^x\tau_j^x + \tau_i^y\tau_y^y, \tau_k^z\right]\delta(\mathbf{x}-\mathbf{r}_k) = \sum_{i,j,k}\left[\tau_i^x\tau_j^x, \tau_k^z\right]\delta(\mathbf{x}-\mathbf{r}_k) + \sum_{i,j,k}\left[\tau_i^y\tau_j^y, \tau_k^z\right]\delta(\mathbf{x}-\mathbf{r}_k)$$

$$= \sum_{i,j,k}\left(\tau_i^x\left[\tau_j^x, \tau_k^z\right] + \left[\tau_i^x, \tau_k^z\right]\tau_j^x\right)\delta(\mathbf{x}-\mathbf{r}_k) + \sum_{i,j,k}\left(\tau_i^y\left[\tau_j^y, \tau_k^z\right] + \left[\tau_i^y, \tau_k^z\right]\tau_j^y\right)\delta(\mathbf{x}-\mathbf{r}_k)$$

$$= 2i\sum_{i,j,k}\left(-\tau_i^x\tau_j^y\delta_{jk} - \tau_i^y\tau_j^x\delta_{ik} + \tau_i^y\tau_j^x\delta_{jk} + \tau_i^x\tau_j^y\delta_{ik}\right)\delta(\mathbf{x}-\mathbf{r}_k)$$

$$= 2i\sum_{i,j,k}\left(\tau_i^y\tau_j^x - \tau_i^x\tau_j^y\right)\delta_{jk}\delta(\mathbf{x}-\mathbf{r}_k) + \left(\tau_i^x\tau_j^y - \tau_i^y\tau_j^x\right)\delta_{ik}\delta(\mathbf{x}-\mathbf{r}_k)$$

$$= 2i\sum_{i,j}\left(\tau_i^y\tau_j^x - \tau_i^x\tau_j^y\right)\delta(\mathbf{x}-\mathbf{r}_j) + \left(\tau_i^x\tau_j^y - \tau_i^y\tau_j^x\right)\delta(\mathbf{x}-\mathbf{r}_i)$$

$$= 2i\sum_{i,j} -\left(\tau_i^x\tau_j^y - \tau_i^y\tau_j^x\right)\delta(\mathbf{x}-\mathbf{r}_j) + \left(\tau_i^x\tau_j^y - \tau_i^y\tau_j^x\right)\delta(\mathbf{x}-\mathbf{r}_i)$$

$$= 2i\sum_{i,j} -\left(\vec{\tau}_i \times \vec{\tau}_j\right)^z\delta(\mathbf{x}-\mathbf{r}_j) + \left(\vec{\tau}_i \times \vec{\tau}_j\right)^z\delta(\mathbf{x}-\mathbf{r}_i)$$

$$= 2i\sum_{i,j}\left(\vec{\tau}_i \times \vec{\tau}_j\right)\left(\delta(\mathbf{x}-\mathbf{r}_i) - \delta(\mathbf{x}-\mathbf{r}_j)\right) ,$$

so that we see that in the isospin part we do have a term that goes like

$$\sum_{ij}(\vec{\tau}_i \times \vec{\tau}_j)^z . \quad (7)$$

Problem 1.3: Electron scattering on the Deuteron

Let us consider the inelastic electron scattering of the deuteron, which (see lecture) is given by the longitudinal response function

$$R_L(\omega, \boldsymbol{q}) = \bar{\sum_i} \sum_f |\langle f|\rho(\boldsymbol{q})|i\rangle|^2 \delta\left(E_f + \frac{\boldsymbol{q}^2}{2M} - E_i - \omega\right) ,$$

where we consider also the recoil term $\frac{\mathbf{q}^2}{2M}$ and the average on the initial state projection of angular momentum $\bar{\sum}_i$. At low energy we can approximate the deutron initial wave function with its S-wave component as

$$|i\rangle \to \Psi(r) = N\frac{e^{-r/a}}{r} .$$

Assuming that the final state wave functions are plane waves $|f\rangle \to \frac{1}{\sqrt{(2\pi)^3}} \exp(i\boldsymbol{k}\cdot\boldsymbol{r})$, calculate the longitudinal response function.
Hint 1: Use the one-body charge operator

$$\rho(\mathbf{x}) = \rho_{(1)}(\mathbf{x}) = e\sum_i^A \Big(\frac{1+\hat{\tau}_i^z}{2}\Big)\delta(\mathbf{x}-\mathbf{r}_i) , \tag{8}$$

and perform the Fourier transform to get $\rho(\boldsymbol{q})$.
Hint 2: When calculating the matrix element, use the fact that the following two functions are connected by a Fourier transform

$$\frac{e^{-mr}}{r} \to_{\mathrm{FT}}\to \frac{4\pi}{m^2+k^2} \tag{9}$$

Solution: We first perform the Fourier transform of the charge density operator

$$\rho(\boldsymbol{q}) \quad = \quad \int d\boldsymbol{x} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\rho(\boldsymbol{x}) = e\sum_i \frac{1+\tau_i^z}{2} e^{i\boldsymbol{q}\cdot\mathbf{r}_i} .$$

For the duteron we have that $\boldsymbol{R} = \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, thus $\mathbf{r}_1 = \boldsymbol{R} - \frac{\mathbf{r}}{2}$ and $\mathbf{r}_2 = \boldsymbol{R} + \frac{\mathbf{r}}{2}$. The operator becomes then

$$\rho(\boldsymbol{q}) = \frac{1+\tau_1^z}{2} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}-\mathbf{r}/2)} + \frac{1+\tau_2^z}{2} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}+\mathbf{r}/2)} .$$

If we sit in the center of mass by putting $\boldsymbol{R}$=0, we get

$$\rho(\boldsymbol{q}) = \frac{1+\tau_1^z}{2} e^{i\boldsymbol{q}\cdot(-\mathbf{r}/2)} + \frac{1+\tau_2^z}{2} e^{i\boldsymbol{q}\cdot(\mathbf{r}/2)} .$$

Then we plug operator into the the longitudinal response

$$R_L(\omega, \boldsymbol{q}) = \bar{\sum_i} \sum_f |\langle f|\rho(\boldsymbol{q})|i\rangle|^2 \delta\left(E_f + \frac{\boldsymbol{q}^2}{2M} - E_i - \omega\right) ,$$

so that we have to calculate the matrix element $\langle f|\rho(\boldsymbol{q})|i\rangle$.
This matrix element will also have an isospin part, which we consider first. The deuteron ground state has isospin $T = 0$ and isospin projection $T_z = 0$. The operator $\rho(\boldsymbol{q})$ does not change the isospin projection, hence the final state must have $T_z = 0$ but can have both $T = 1, 0$. The isospin matrix elements give

$$\begin{aligned}
\langle T = 0\ T_z = 0|\frac{1+\tau_1^z}{2}|T = 0\ T_z = 0\rangle &= \frac{1}{2} \\
\langle T = 1\ T_z = 0|\frac{1+\tau_1^z}{2}|T = 0\ T_z = 0\rangle &= \frac{1}{2} \\
\langle T = 0\ T_z = 0|\frac{1+\tau_2^z}{2}|T = 0\ T_z = 0\rangle &= \frac{1}{2} \\
\langle T = 1\ T_z = 0|\frac{1+\tau_2^z}{2}|T = 0\ T_z = 0\rangle &= -\frac{1}{2}
\end{aligned} \tag{10}$$

so that we get

$$\begin{aligned}
\langle \boldsymbol{k}T|\rho(\boldsymbol{q})|i\rangle &= \int d\mathbf{r} \frac{1}{\sqrt{(2\pi)^3}} e^{-i\boldsymbol{k}\cdot\mathbf{r}} \langle T| \left[\frac{1+\tau_1^z}{2} e^{-i\boldsymbol{q}\cdot\mathbf{r}/2} + \frac{1+\tau_2^z}{2} e^{i\boldsymbol{q}\cdot\mathbf{r}/2}\right] |T = 0\rangle N \frac{e^{-r/a}}{r} \\
&= \frac{N}{2} \int d\mathbf{r} \frac{1}{\sqrt{(2\pi)^3}} e^{-i\boldsymbol{k}\cdot\mathbf{r}} \left[e^{-i\boldsymbol{q}\cdot\mathbf{r}/2} + (-)^T e^{i\boldsymbol{q}\cdot\mathbf{r}/2}\right] \frac{e^{-r/a}}{r}
\end{aligned}$$

Defining $\kappa = 1/a$ and using the second hint we get

$$\begin{aligned}
\langle \boldsymbol{k}T|\rho(\boldsymbol{q})|i\rangle &= \frac{N}{2} \int d\mathbf{r} \frac{1}{\sqrt{(2\pi)^3}} e^{-i\boldsymbol{k}\cdot\mathbf{r}} \left[e^{-i\boldsymbol{q}\cdot\mathbf{r}/2} + (-)^T e^{i\boldsymbol{q}\cdot\mathbf{r}/2}\right] \frac{e^{-\kappa r}}{r} \\
&= \frac{N}{2} \frac{1}{\sqrt{(2\pi)^3}} \left[\frac{4\pi}{(\boldsymbol{k}+\boldsymbol{q}/2)^2 + \kappa^2} + (-)^T \frac{4\pi}{(\boldsymbol{k}-\boldsymbol{q}/2)^2 + \kappa^2}\right] .
\end{aligned}$$

Plugging this into the response function we get

$$\begin{aligned}
R_L(\omega, \boldsymbol{q}) &= \int d\boldsymbol{k} \sum_{T=0,1} \frac{N^2}{2\pi} \left[\frac{1}{(\boldsymbol{k}+\boldsymbol{q}/2)^2 + \kappa^2} + (-)^T \frac{1}{(\boldsymbol{k}-\boldsymbol{q}/2)^2 + \kappa^2}\right]^2 \\
&\quad \delta(k^2/M + \frac{\boldsymbol{q}^2}{4M} - E_i - \omega) = \\
&= \int d\boldsymbol{k} \frac{2N^2}{\pi} \left(\frac{1}{(\boldsymbol{k}+\boldsymbol{q}/2)^2 + \kappa^2}\right)^2 \delta(k^2/M + \frac{\boldsymbol{q}^2}{4M} - E_i - \omega) .
\end{aligned} \tag{11}$$