Additional Notes on Electron Scattering

by Nir Barnea

REFERENCES

- Quantum Field Theory
  C. Itzykson, J.-B. Zuber, Zuber.

- Electron scattering for Nuclear and Nucleon Scattering
  J. D. Walecka, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology **16**, (2001).

I. THE FEYNMAN RULES

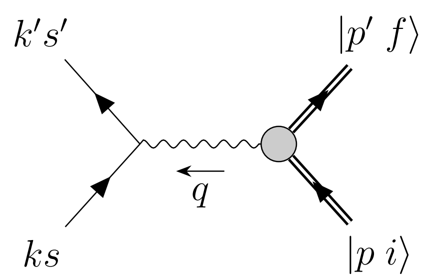

FIG. 1. Electron-nucleus scattering

The feynman rule for QED is given by (I&Z notations)

- The photon-electron vertex is given by

$$-ie\gamma_\mu(2\pi)^4\delta^4(\Sigma p) \tag{1}$$

- The interaction of the nucleus (nucleon) with the photon field is given by

$$-e\langle f|J^\nu(0)|i\rangle \tag{2}$$

  where $|i\rangle, |f\rangle$ are the nuclear initial and final states, and J_μ is the nuclear current operator.

- The photon propagator is given by

$$-i\frac{g_{\mu\nu}}{k^2+i\eta} \tag{3}$$

- Integrate over each internal momentum $\frac{d^4q}{(2\pi)^4}$

- A diagram is equal to $i\mathcal{A}(2\pi)^4\delta^4(\Sigma p)$ where $\mathcal{A}$ is the scattering matrix.

- External lines: $u(p,s), \bar{v}(p,s)$ for absorbed electron or positron, $\bar{u}(p,s), v(p,s)$ for emitted electron or positron. e^α_μ for emitted or absorbed photon.

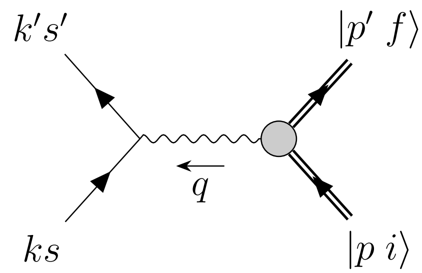

FIG. 2. Electron-nucleus scattering

II. THE CROSS-SECTION

Scattering of unpolarized electron from nucleus in the Lab system. The scattering amplitude is given by

$$\mathcal{A} = [(-i\ e)\bar{u}(k', s')\gamma^\mu u(k, s)]\,(-i)\frac{g_{\mu\nu}}{q^2 - i\eta}\,[e\langle f|J^\nu(0)|i\rangle] \tag{4}$$

and the cross section by

$$d\sigma = \frac{m_e}{|\boldsymbol{k}|}\frac{d\boldsymbol{k'}}{(2\pi)^3}\frac{m_e}{k'_0}\frac{1}{2}\sum_{s's}\bar{\sum_i}\sum_f |\mathcal{A}|^2(2\pi)^4\delta(P_f - P_i) \tag{5}$$

The flux is given by the velocity $v = \frac{|\boldsymbol{k}|}{\epsilon}$ times corrected with the normalization $\frac{m_e}{\epsilon}$. Due to our choice of Dirac spinors We need an extra $(2m_e)^2$ with respect to Peskin and Schroeder

$$(2m_e)^2 \times \frac{1}{2\epsilon|v|}\frac{d\boldsymbol{k'}}{(2\pi)^3}\frac{1}{2\epsilon'} = \frac{m_e}{|\boldsymbol{k}|}\frac{d\boldsymbol{k'}}{(2\pi)^3}\frac{m_e}{\epsilon'} \tag{6}$$

It can be written in covariant form

$$\frac{|\boldsymbol{k}|}{m_e} = \frac{M\sqrt{\epsilon^2 - m_e^2}}{Mm_e} = \frac{\sqrt{M^2\epsilon^2 - M^2m_e^2}}{Mm_e} = \frac{\sqrt{(p\cdot k)^2 - M^2m_e^2}}{Mm_e} \approx \frac{\sqrt{(p\cdot k)^2}}{Mm_e} \tag{7}$$

$$d\sigma = \frac{Mm_e}{\sqrt{(p\cdot k)^2}}\frac{d\boldsymbol{k'}}{(2\pi)^3}\frac{m_e}{k'_0}\frac{1}{2}\sum_{s's}\bar{\sum_i}\sum_f |\mathcal{A}|^2(2\pi)^4\delta(P_f - P_i) \tag{8}$$

$$|\mathcal{A}|^2 = \frac{e^4}{q^4}\,[\bar{u}(k', s')\gamma^\mu u(k, s)\bar{u}(k, s)\gamma^\nu u(k', s')]\,\langle f|J_\mu(0)|i\rangle\langle f|J_\nu(0)|i\rangle^* \tag{9}$$

defining

$$\eta^{\mu\nu} = 4m_e^2\frac{1}{2}\sum_{s's}\bar{u}(k', s')\gamma^\mu u(k, s)\bar{u}(k, s)\gamma^\nu u(k', s') \tag{10}$$

and

$$W_{\mu\nu} = (2\pi)^3\bar{\sum_i}\sum_f\langle i|J_\mu(0)|f\rangle\langle f|J_\nu(0)|i\rangle\delta(P_f - P_i) \tag{11}$$

we get

$$d\sigma = \frac{e^4}{q^4}\frac{M}{\sqrt{(p\cdot k)^2}}\frac{d\boldsymbol{k'}}{k'_0}\frac{1}{4(2\pi)^2}\eta^{\mu\nu}W_{\mu\nu} \tag{12}$$

we use now the relation $\alpha = \frac{e^2}{4\pi}$ and ignoring the electron mass $d\boldsymbol{k}' = d\Omega'\epsilon'^2 d\epsilon'$, $\epsilon' = k'_0$ we can write

$$\frac{d\sigma}{d\Omega' d\epsilon'} = \frac{\alpha^2}{q^4}\frac{\epsilon'}{\epsilon}\eta^{\mu\nu}W_{\mu\nu} \tag{13}$$

We can rewrite

$$\begin{aligned}\eta^{\mu\nu} &= 4m_e^2\frac{1}{2}\sum_{s's}\bar{u}(k',s')\gamma^\mu u(k,s)\bar{u}(k,s)\gamma^\nu u(k',s') \\ &= 4m_e^2\frac{1}{2}Tr\left[\sum_{s's}\gamma^\mu u(k,s)\bar{u}(k,s)\gamma^\nu u(k',s')\bar{u}(k',s')\right]\end{aligned} \tag{14}$$

Now

$$\sum_s u(k,s)\bar{u}(k,s) = \frac{\not{k}+m}{2m} \tag{15}$$

therefore

$$\eta^{\mu\nu} = \frac{1}{2}Tr\left[\gamma^\mu(\not{k}+m_e)\gamma^\nu(\not{k}'+m_e)\right] \tag{16}$$

The trace of an odd product of γ^μ matrices vanishes. For even products

$$\begin{aligned}Tr[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\ Tr[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma] &= 4(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu})\end{aligned} \tag{17}$$

substituting in η

$$\eta^{\mu\nu} = 2\left(g^{\mu\nu}(m_e^2 - k\cdot k') + k^\mu k'^\nu + k^\nu k'^\mu\right) \tag{18}$$

A. $W_{\mu\nu}$

Current conservation implies that

$$\partial^\mu\langle f|J_\mu(x)|i\rangle = \partial^\mu e^{i(p-p')\cdot x}\langle f|J_\mu(0)|i\rangle = e^{i(p-p')\cdot x}i(p-p')^\mu\langle f|J_\mu(0)|i\rangle = 0 \tag{19}$$

or

$$q^\mu\langle f|J_\mu(0)|i\rangle = 0 \tag{20}$$

Hence

$$q^\mu W_{\mu\nu} = W_{\mu\nu}q^\nu = 0 \tag{21}$$

The Dirac equation implies that

$$\bar{u}(k',s')\gamma^\mu q_\mu u(k,s) = \bar{u}(k',s')(\not{k}'-\not{k})u(k,s) = \bar{u}(k',s')(m_e - m_e)u(k,s) = 0 \tag{22}$$

It follows that lepton response tensor obeys the same relations

$$q^\mu\eta_{\mu\nu} = \eta_{\mu\nu}q^\nu = 0 \tag{23}$$

The nuclear response tensor may be a general function of q, p. We note that q^2 and $q\cdot p$ are the only independent scalars as $p^2 = M^2$. In the laboratory frame $p = (M, \mathbf{0})$ and we may write

$$\begin{aligned}q^2 &= (|\boldsymbol{k}'| - |\boldsymbol{k}|)^2 - (\boldsymbol{k}' - \boldsymbol{k})^2 = -2\epsilon\epsilon'(1-\cos\theta) = -4\epsilon\epsilon'\sin^2\frac{\theta}{2} \\ q\cdot p &= M(\epsilon' - \epsilon)\end{aligned} \tag{24}$$

Current conservation implies that

$$W_{\mu\nu} = -W_1(q^2, q\cdot p)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$

$$+W_2(q^2, q\cdot p)\frac{1}{M^2}\left(p_\mu - \frac{q\cdot p}{q^2}q_\mu\right)\left(p_\nu - \frac{q\cdot p}{q^2}q_\nu\right) \tag{25}$$

This can be proved by writing the most general tensor

$$W_{\mu\nu} = -W_1 g_{\mu\nu} + W_2\frac{p_\mu p_\nu}{M^2} + A\frac{q_\mu q_\nu}{M^2} + B\frac{p_\mu q_\nu + p_\nu q_\mu}{M^2} + C\frac{p_\mu q_\nu - p_\nu q_\mu}{M^2} \tag{26}$$

using current conservation and the independence of the q, p vectors.

Combining the two tensors we get

$$\begin{aligned}\eta^{\mu\nu}W_{\mu\nu} &= 2\left(g^{\mu\nu}(-k\cdot k') + k^\mu k'^\nu + k^\nu k'^\mu\right)\left(-W_1 g_{\mu\nu} + W_2\frac{p_\mu p_\nu}{M^2}\right)\\ &= 2\left(-W_1(-2k\cdot k') + W_2\frac{1}{M^2}(2p\cdot kp\cdot k' - p^2 k\cdot k')\right)\end{aligned} \tag{27}$$

Since $k^2 = k'^2 = 0$

$$q^2 = -2k\cdot k' \tag{28}$$

$$\begin{aligned}\eta^{\mu\nu}W_{\mu\nu} &= 2\left(-W_1 q^2 + W_2\frac{1}{M^2}(2M^2\epsilon\epsilon' + M^2\frac{1}{2}q^2)\right)\\ &= 2\left(-W_1 q^2 + W_2(2\epsilon\epsilon' + \frac{1}{2}q^2)\right)\\ &= 4\epsilon\epsilon'\left(W_1 2\sin^2\frac{\theta}{2} + W_2(1-\sin^2\frac{\theta}{2})\right)\\ &= 4\epsilon\epsilon'\left(W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right)\end{aligned} \tag{29}$$

The cross-section

$$\begin{aligned}\frac{d\sigma}{d\Omega' d\epsilon'} &= \frac{\alpha^2}{4^2\epsilon^2\epsilon'^2\sin^4\frac{\theta}{2}}\frac{\epsilon'}{\epsilon}4\epsilon\epsilon'\left(W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right)\\ &= \frac{\alpha^2\cos^2(\theta/2)}{4\epsilon^2\sin^4(\theta/2)}\left(W_2(q^2, q\cdot p) + 2W_1(q^2, q\cdot p)\tan^2\frac{\theta}{2}\right)\end{aligned} \tag{30}$$

The left term on the right can be recongenized as the Mott cross-section in the lab frame for scattering a relativistic (massless) Dirac electron from a point charge

$$\sigma_M \equiv \frac{\alpha^2\cos^2(\theta/2)}{4\epsilon^2\sin^4(\theta/2)} \tag{31}$$

We finally get

$$\frac{d\sigma}{d\Omega' d\epsilon'} = \sigma_M\left(W_2(q^2, q\cdot p) + 2W_1(q^2, q\cdot p)\tan^2\frac{\theta}{2}\right) \tag{32}$$

B. Photoabsorption

The scattering amplitude for photoabsorption is given by

$$\mathcal{A} = e_\mu^{(\lambda)}\left[e\langle f|J^\mu(0)|i\rangle\right] \tag{33}$$

and the cross section by

$$\sigma_\gamma = \frac{1}{2\sqrt{(p\cdot k)^2}}\frac{1}{2}\sum_\lambda\bar{\sum_i}\sum_f|\mathcal{A}|^2(2\pi)^4\delta(P_f - P_i) \tag{34}$$

we recall the hadronic tensor

$$W_{\mu\nu} = (2\pi)^3\bar{\sum_i}\sum_f\langle i|J_\mu(0)|f\rangle\langle f|J_\nu(0)|i\rangle\delta(P_f - P_i) \tag{35}$$

substituting the scattering amplitude

$$\sigma_\gamma = \frac{e^2}{2\sqrt{(p\cdot k)^2}}\frac{1}{2}\sum_\lambda e_\mu^{(\lambda)\dagger} e_\nu^{(\lambda)}(2\pi)W^{\mu\nu}$$
$$= \frac{2\pi^2\alpha}{\sqrt{(p\cdot k)^2}}\sum_\lambda e_\mu^{(\lambda)\dagger} e_\nu^{(\lambda)} W^{\mu\nu} \tag{36}$$

To calculate the polarization sums we first define acoordinate system in which the z direction coincides with the direction of $\boldsymbol{k}$, namely $\boldsymbol{e}_{\boldsymbol{k}1}, \boldsymbol{e}_{\boldsymbol{k}2}, \boldsymbol{e}_{\boldsymbol{k}3} = \hat{\boldsymbol{k}}$. In polar basis we may write

$$\boldsymbol{e}_{\boldsymbol{k}\pm} = \mp\frac{1}{\sqrt{2}}(\boldsymbol{e}_{\boldsymbol{k}1} \pm i\boldsymbol{e}_{\boldsymbol{k}2})$$
$$\boldsymbol{e}_{\boldsymbol{k}0} = \hat{\boldsymbol{k}} \tag{37}$$

The polarization is given by $e^{(\lambda)} = (0, \boldsymbol{e}^{(\lambda)})$, and the sums are

$$\sum_\lambda e_i^{(\lambda)\dagger} e_j^{(\lambda)} = \frac{1}{2}[(\boldsymbol{e}_{\boldsymbol{k}1} - i\boldsymbol{e}_{\boldsymbol{k}2})_i(\boldsymbol{e}_{\boldsymbol{k}1} + i\boldsymbol{e}_{\boldsymbol{k}2})_j + (\boldsymbol{e}_{\boldsymbol{k}1} + i\boldsymbol{e}_{\boldsymbol{k}2})_i(\boldsymbol{e}_{\boldsymbol{k}1} - i\boldsymbol{e}_{\boldsymbol{k}2})_j]$$
$$= (\boldsymbol{e}_{\boldsymbol{k}1})_i(\boldsymbol{e}_{\boldsymbol{k}1})_j + (\boldsymbol{e}_{\boldsymbol{k}2})_i(\boldsymbol{e}_{\boldsymbol{k}2})_j$$
$$= \delta_{ij} - \frac{\boldsymbol{k}_i\boldsymbol{k}_j}{\boldsymbol{k}^2} \tag{38}$$

we have seen that

$$k_\mu W^{\mu\nu} = 0$$
$$k_i W_{ij} = |\boldsymbol{k}| W_{0j} \tag{39}$$

therefore

$$\sum_\lambda e_\mu^{(\lambda)\dagger} e_\nu^{(\lambda)} W^{\mu\nu} = \sum_{ij}\left(\delta_{ij} - \frac{\boldsymbol{k}_i\boldsymbol{k}_j}{\boldsymbol{k}^2}\right)W^{ij}$$
$$= \sum_i W^{ii} - W^{00}$$
$$= -g_{\mu\nu}W^{\mu\nu} \tag{40}$$

using the construction

$$W_{\mu\nu} = -W_1(q^2, q\cdot p)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$
$$+W_2(q^2, q\cdot p)\frac{1}{M^2}\left(p_\mu - \frac{q\cdot p}{q^2}q_\mu\right)\left(p_\nu - \frac{q\cdot p}{q^2}q_\nu\right) \tag{41}$$

we get

$$-g_{\mu\nu}W^{\mu\nu} = 3W_1 - W_2\frac{1}{M^2}\left(p^2 - \frac{(p\cdot q)^2}{q^2}\right) \tag{42}$$

For the photon we need to take the limit $q^2 \longrightarrow 0$. The hadronic tensor must stay finite in this limit, thus

$$W_2 \to O(q^2)$$
$$W_1 + W_2\frac{1}{M^2}\frac{(p\cdot q)^2}{q^2} \to O(q^2) \tag{43}$$

and consequently

$$-g_{\mu\nu}W^{\mu\nu} = 3W_1 - \frac{W_2}{M^2}\left(p^2 - \frac{(p\cdot q)^2}{q^2}\right)$$
$$= 2W_1 + (W_1 + \frac{W_2}{M^2}\frac{(p\cdot q)^2}{q^2}) + \frac{W_2 p^2}{M^2}$$
$$\to 2W_1 + O(q^2) \tag{44}$$

The total photoabsorption cross-section is therefore given by

$$\sigma_\gamma = \frac{(2\pi)^2\alpha}{\sqrt{(p\cdot k)^2}}W_1(k^2, -p\cdot k) \quad ; \quad k^2 = 0 \tag{45}$$

C. The Nuclear response function

Let us inspect a bit the nuclear response tensor. We give up the relativistic covariance as anyway at the moment we don't have a full reltivistic nuclear theory. In the lab frame we define

$$\begin{aligned} S_L = W_{00} &= -W_1\left(1-\frac{q_0q_0}{q^2}\right)+W_2\frac{1}{M^2}\left(M-\frac{q_0M}{q^2}q_0\right)\left(M-\frac{q_0M}{q^2}q_0\right) \\ &= W_1\frac{\boldsymbol{q}^2}{q^2}+W_2\frac{\boldsymbol{q}^4}{q^4} \\ &= \frac{\boldsymbol{q}^2}{q^2}\left(W_1+W_2\frac{\boldsymbol{q}^2}{q^2}\right) \end{aligned} \tag{46}$$

$$\begin{aligned} S_T &= \left(\delta_{ij}-\frac{q_iq_j}{\boldsymbol{q}^2}\right)W_{ij} \\ &= \left(\delta_{ij}-\frac{q_iq_j}{\boldsymbol{q}^2}\right)\left(-W_1(-\delta_{ij}-\frac{q_iq_j}{q^2})+W_2\frac{1}{M^2}\left(-\frac{q_0M}{q^2}q_i\right)\left(-\frac{q_0M}{q^2}q_j\right)\right) \\ &= W_1\left(3+\frac{\boldsymbol{q}^2}{q^2}-\frac{\boldsymbol{q}^2}{\boldsymbol{q}^2}-\frac{\boldsymbol{q}^2}{q^2}\right) \\ &= 2W_1 \end{aligned} \tag{47}$$

Inversly

$$\begin{aligned} W_1 &= \frac{1}{2}S_T \\ W_2 &= \frac{q^2}{\boldsymbol{q}^2}\left(\frac{q^2}{\boldsymbol{q}^2}S_L-\frac{1}{2}S_T\right) \end{aligned} \tag{48}$$

Therefore

$$\frac{d\sigma}{d\Omega' d\epsilon'} = \sigma_M\left\{\left(\frac{q^2}{\boldsymbol{q}^2}\right)^2 S_L+\left(-\frac{q^2}{2\boldsymbol{q}^2}+\tan^2\frac{\theta}{2}\right)S_T\right\} \tag{49}$$

We have seen that

$$W_{\mu\nu} = (2\pi)^3\bar{\sum_i}\sum_f\langle i|J_\mu(0)|f\rangle\langle f|J_\nu(0)|i\rangle\delta(P_f-P_i) \tag{50}$$

The sum over the final states includes a sum over the internal states and an integration over the total nuclear momentum

$$\sum_f = \sum_n\int\frac{d\boldsymbol{p}'}{(2\pi)^3} \tag{51}$$

substituting into the nuclear response tensor

$$W_{\mu\nu} = \bar{\sum_i}\sum_n\langle i|J_\mu(0)|n\rangle\langle n|J_\nu(0)|i\rangle\delta(E_n+\frac{\boldsymbol{q}^2}{2M}-E_i-\omega) \tag{52}$$

where we have used the non relativistic kinetic energy, and defined the energy trasnfer $\omega=-q_0=\epsilon-\epsilon'$.

$$J_\mu = (\rho,\boldsymbol{J}) \tag{53}$$

Thus

$$S_L(\omega,\boldsymbol{q}) = \bar{\sum_i}\sum_n|\langle i|\rho|n\rangle|^2\delta(E_n+\frac{\boldsymbol{q}^2}{2M}-E_i-\omega) \tag{54}$$

Before we evaluate S_T in a similar fashion we need note that

$$\left(\delta_{ij}-\frac{q_iq_j}{\boldsymbol{q}^2}\right)J_iJ_j = \boldsymbol{J}\cdot\boldsymbol{J}-(\boldsymbol{J}\cdot\hat{\boldsymbol{q}})(\boldsymbol{J}\cdot\hat{\boldsymbol{q}})$$

$$= (\boldsymbol{J} - (\boldsymbol{J}\cdot\hat{\boldsymbol{q}})\hat{\boldsymbol{q}}) \cdot (\boldsymbol{J} - (\boldsymbol{J}\cdot\hat{\boldsymbol{q}})\hat{\boldsymbol{q}}) \tag{55}$$

defining the trasverse current

$$\boldsymbol{J}_T = \boldsymbol{J} - (\boldsymbol{J}\cdot\hat{\boldsymbol{q}})\hat{\boldsymbol{q}} \tag{56}$$

we may now write

$$S_T(\omega, \boldsymbol{q}) = \overline{\sum_i}\sum_n |\langle i|\boldsymbol{J}_T|n\rangle|^2 \delta(E_n + \frac{\boldsymbol{q}^2}{2M} - E_i - \omega) \tag{57}$$

The one-body current operator

$$\langle \boldsymbol{p}'\sigma'\tau'|J^\mu(0)|\boldsymbol{p}\sigma\tau\rangle = i\bar{u}(\boldsymbol{p}',\sigma')\eta^\dagger_{\tau'}[F_1(q^2)\gamma^\mu + F_2(q^2)\sigma^{\mu\nu}q_\nu]\eta_\tau u(\boldsymbol{p},\sigma) \tag{58}$$

The isospin structure of form factors must

$$F_i = \frac{1}{2}(F_i^S + \tau_3 F_i^V) \tag{59}$$

For $q^2 = 0$ the form factors coincide with the electric charge and the magnetic moments therefore

$$\begin{aligned} F_1^S(0) &= F_1^V(0) = 1 \\ 2mF_2^S(0) &= \kappa_p + \kappa_n - 1 = -0.120 \\ 2mF_2^V(0) &= \kappa_p - \kappa_n - 1 = +3.706 \end{aligned} \tag{60}$$

Here κ_i is the anomalus magnetic moment. Substituting the explicit form of the Dirac spinors

$$u(p,s) = \sqrt{\frac{E_p + M}{2M}} \begin{pmatrix} I \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+M} \end{pmatrix} \chi_s \tag{61}$$

($E_p = p^0 = \sqrt{\boldsymbol{p}^2 + M^2}$), and expanding to order M^2

$$\langle \boldsymbol{p}'\sigma'\tau'|J^\mu(0)|\boldsymbol{p}\sigma\tau\rangle = \eta^\dagger_{\tau'}\chi^\dagger_\sigma J^\mu_N \chi_\sigma \eta_\tau \tag{62}$$

$$\begin{aligned} \rho_N &= F_1 - (F_1 + 4MF_2)\left[\frac{\boldsymbol{q}^2}{8M^2} - \frac{i\boldsymbol{q}\cdot(\boldsymbol{\sigma}\times\boldsymbol{p})}{4M^2}\right] + O\left(\frac{1}{M^3}\right) \\ \boldsymbol{J}_N &= \frac{F_1}{2M}(\boldsymbol{p}+\boldsymbol{p}') + (F_1 + 2MF_2)\left[\frac{i(\boldsymbol{\sigma}\times\boldsymbol{q})}{2M}\right] + O\left(\frac{1}{M^3}\right) \end{aligned} \tag{63}$$

The 1-body charge density operator

$$\rho(q) = \sum_j^Z F_1^p(q^2)e^{i\boldsymbol{q}\cdot\mathbf{r}_j} + \sum_j^N F_1^n(q^2)e^{i\boldsymbol{q}\cdot\mathbf{r}_j} \tag{64}$$

and the 1-body current

$$\begin{aligned} \boldsymbol{J} =& \sum_j^Z F_1^p(q^2)\left\{\frac{\boldsymbol{p}}{2M}, e^{i\boldsymbol{q}\cdot\mathbf{r}_j}\right\} + \sum_j^N F_1^n(q^2)\left\{\frac{\boldsymbol{p}}{2M}, e^{i\boldsymbol{q}\cdot\mathbf{r}_j}\right\} \\ &+ \sum_j^Z (F_1^p(q^2) + \kappa_p F_2^p(q^2))\frac{i\sigma_j\times\boldsymbol{q}}{2M}e^{i\boldsymbol{q}\cdot\mathbf{r}_j} \\ &+ \sum_j^N (F_1^n(q^2) + \kappa_n F_2^n(q^2))\frac{i\sigma_j\times\boldsymbol{q}}{2M}e^{i\boldsymbol{q}\cdot\mathbf{r}_j} \end{aligned} \tag{65}$$

Deriving the currents form the effective Lagrangian leads to further corrections to these terms.