Busch formula (exercise)

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Two trapped

$\rightarrow$ two noninteracting particles of equal mass $m_1 = m_2 = m$ inside the Harmonic oscillator (HO) trap

$$H_0 = \frac{\boldsymbol{p}_1{}^2}{2m} + \frac{\boldsymbol{p}_2{}^2}{2m} + \frac{1}{2} m\omega^2 \boldsymbol{r}_1^2 + \frac{1}{2} m\omega^2 \boldsymbol{r}_2^2$$

with oscillator frequency ω

$\rightarrow$ introducing **RELATIVE** and **CM** coordinates

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad \boldsymbol{R}_{CM} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$$

$$\boldsymbol{p} = \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \qquad \boldsymbol{P}_{CM} = \boldsymbol{p}_1 + \boldsymbol{p}_2$$

we can separate H_0 into the **RELATIVE** and the **CM** part

$$H_0 = \left[\frac{\boldsymbol{p}^2}{2\mu} + \frac{1}{2}\mu\omega^2 r^2\right] + \left[\frac{\boldsymbol{P}_{\mathrm{CM}}^2}{2M_{\mathrm{CM}}} + \frac{1}{2} M_{\mathrm{CM}}\omega^2 \boldsymbol{R}_{\mathrm{CM}}^2\right]$$

where $M_{CM} = 2m$ (total mass) and $\mu = m/2$ (reduced mass)

Two trapped particles

Lets consider relative motion of two trapped particles.

$$E\Psi(r,\theta,\varphi) = \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}r^2 + \frac{\boldsymbol{L}^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2\right]\Psi(r,\theta,\varphi)$$

$\rightarrow$ Separating the wave function $\Psi(r,\theta,\varphi) = \frac{u_l(r)}{r}Y_{lm}(\theta,\varphi)$ (radial and angular part) and taking into account $\boldsymbol{L}^2 Y_{lm}(\theta,\varphi) = l(l+1)\hbar Y_{lm}(\theta,\varphi)$

yields **RADIAL EQUATION**

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u_l(r) = -\frac{2\mu}{\hbar^2}\left[E - \frac{\hbar^2 l(l+1)}{2\mu r^2} - \frac{1}{2}\mu\omega^2 r^2\right]u_l(r)$$

$\rightarrow$ **from now on I will assume $\hbar = c = 1$**

Two trapped particles

For simplicity we introduce **dimensionless variable** $x = \frac{r}{b_{\mathrm{HO}}}$, $b_{\mathrm{HO}} = \frac{1}{\sqrt{\mu\omega}}$

b_{HO} ... is quite often called Harmonic oscillator trap length

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u_l(x) + \frac{l(l+1)}{x^2} u_l(x) + x^2 u_l(x) = \eta u_l(x), \quad \eta = \frac{2E}{\omega}$$

$\rightarrow$ for this specific case the solution can be obtained in a form

$$u_l(x) = e^{-x^2/2} \left[c_1 x^{l+1} M\left(\frac{2l+3-\eta}{4}, l+\frac{3}{2}; x^2 \right) + c_2 x^{-l} M\left(\frac{-2l+1-\eta}{4}, -l+\frac{1}{2}; x^2 \right) \right]$$

where $M(\alpha, \beta; z)$ is the confluent hypergeometric function

Two trapped particles

$$u_l(x) = e^{-x^2/2}\left[c_1 x^{l+1} M\left(\frac{2l+3-\eta}{4}, l+\frac{3}{2}; x^2\right) + c_2 x^{-l} M\left(\frac{-2l+1-\eta}{4}, -l+\frac{1}{2}; x^2\right)\right]$$

Lets study $x \to \infty$

$\to$ in this asymptotic region $M(\alpha, \gamma, z \to \infty) \sim \dfrac{\Gamma(\gamma)}{\Gamma(\alpha)} z^{\alpha-\gamma} e^z$

$$u_l(x \to \infty) = c_1 \left\{ \frac{\Gamma\left[3/2 + l\right]}{\Gamma\left[(2l+3-\eta)/4\right]} + \frac{c_2}{c_1} \frac{\Gamma\left[1/2 - l\right]}{\Gamma\left[(-2l+1-\eta)/4\right]} \right\} x^{(1-\eta)/2} e^{-x^2/2}$$

$\to$ for HO trap there are only bound states, as a result $u_l(x)$ has to disappear as $x \to \infty$

$$\frac{c_2}{c_1} = -\frac{\Gamma\left[3/2 + l\right]}{\Gamma\left[(2l+3-\eta)/4\right]} \frac{\Gamma\left[(-2l+1-\eta)/4\right]}{\Gamma\left[1/2 - l\right]}$$

Two trapped particles

→ expression for c_2/c_1 can be further simplified

$$\Gamma\left[1/2 - l\right] = \frac{(-1)^l 2^l}{(2l-1)!!}\sqrt{\pi} \qquad \Gamma\left[3/2 + l\right] = \frac{(1+2l)!!}{2^{l+1}}\sqrt{\pi}$$

Semi-final result :

$$u_l(x) = c_1 x^{l+1} \left[M\left(\frac{2l+3-\eta}{4}, l+\frac{3}{2}; x^2\right) + \frac{c_2}{c_1} x^{-2l-1} M\left(\frac{-2l+1-\eta}{4}, -l+\frac{1}{2}; x^2\right)\right] e^{-x^2/2}$$

$$\frac{c_2}{c_1} = -(-1)^l 2^{-2l}(l+1/2)\left[(2l-1)!!\right]^2 \frac{\Gamma\left[(1-2l-\eta)/4\right]}{\Gamma\left[(2l+3-\eta)/4\right]}$$

Two trapped particles interacting with a short-range potential $V(r)$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} u_l(r) = -2\mu \left[E - \frac{l(l+1)}{2\mu r^2} - V(r) - \frac{1}{2}\mu\omega^2 r^2 \right] u_l(r)$$

Two different scales :

→ potential range b

→ HO trap scale b_{HO}

assuming separation of scales $b << b_{\mathrm{HO}}$

Three different regions :

$r < b$: HO term can be neglected

$b < r < b_{HO}$: $V(r)$ and HO terms can be neglected

$b_{HO} < r$: $V(r)$ term can be neglected

$b < r < b_{HO}$ region

For $r \to b_+$ HO potential can be neglected but we are still in an external region with respect to $V(r)$ of range b.

EXTERNAL REGION :

$$u_l(r) = A\,(kr)\,[j_l(kr) + n_l(kr)\,\tan(\delta_l)]$$

$\to$ for $r > b$ but still r clos to zero

$$j_l(z \to 0) \to \frac{z^2}{(2l+1)!!} \qquad\qquad n_l(z \to 0) \to \frac{(2l-1)!!}{x^{l+1}}$$

$$u_l(r \to 0) \to B_l r^{l+1} \left\{ 1 + (2l+1)\,[(2l-1)!!]^2\,\frac{\tan(\delta_l)}{(kr)^{2l+1}} \right\}, \qquad B_l = \frac{Ak^{l+1}}{(2l+1)!!}$$

$b < r < b_{HO}$ region

In this specific region we have also a solution of radial equation with HO trap but no $V(r)$.

$$u_l(x) = c_1 x^{l+1} \left[M\left(\frac{2l+3-\eta}{4}, l+\frac{3}{2}; x^2\right) + \frac{c_2}{c_1} x^{-2l-1} M\left(\frac{-2l+1-\eta}{4}, -l+\frac{1}{2}; x^2\right)\right] e^{-x^2/2}$$

$$M(\alpha, \beta, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)}} \qquad\qquad e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$$

where $a^{(n)}$ is a rising factorial $a^{(0)} = 1$, $a^{(1)} = a$, $a^{(2)} = a(a+1)$, $a^{(3)} = a(a+1)(a+2)$, ...

$\rightarrow$ for $r > b$ but still r clos to zero $M(..., ..., x^2) \simeq 1$ and $e^{-x^2/2} \simeq 1$

$$u_l(x \to 0) \to c_1 x^{l+1} \left[1 + \frac{c_2}{c_1} \frac{1}{x^{2l+1}}\right]$$

Matching condition

$$u_l(r \to 0) \to B_l r^{l+1} \left\{ 1 + (2l+1)\left[(2l-1)!!\right]^2 \frac{\tan(\delta_l)}{(kr)^{2l+1}} \right\}, \qquad B_l = \frac{Ak^{l+1}}{(2l+1)!!}$$

$$u_l(r \to 0) \to c_1 r^{l+1} \left[1 + \frac{c_2}{c_1} \frac{b_{\mathrm{HO}}^{2l+1}}{r^{2l+1}} \right], \qquad (\text{we used } x = r/b_{HO})$$

Matching two solutions :

$$(2l+1)\left[(2l-1)!!\right]^2 \frac{\tan(\delta_l)}{(k)^{2l+1}} = \frac{c_2}{c_1} b_{\mathrm{HO}}^{2l+1}$$

$$\frac{c_2}{c_1} = -(-1)^l 2^{-2l}(l+1/2)\left[(2l-1)!!\right]^2 \frac{\Gamma\left[(1-2l-\eta)/4\right]}{\Gamma\left[(2l+3-\eta)/4\right]}$$

Busch formula for phase shifts

Busch formula

$$k^{2l+1}\mathrm{cotg}(\delta_l) = (-1)^{l+1}(\sqrt{4\mu\omega})^{2l+1}\frac{\Gamma\left[(2l+3-\eta)/4\right]}{\Gamma\left[(1-2l-\eta)/4\right]}$$

$$\eta = \frac{2E}{\omega}, \qquad k = \sqrt{2\mu E}$$

We can use Harmonic oscillator trap bound state energies calculated with the given short-ranged potential $V(r)$ to extract the corresponding scattering phaseshifts !

Few-body systems

Lets assume N-body system in a HO trap (equal masses $m_i = m$)

$$H_0 = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i=1}^{N} \frac{1}{2} m\omega \left(\boldsymbol{r}_i - \boldsymbol{x}_N\right)^2$$

using simple identity $\sum_{i=1}^{N} (\boldsymbol{r}_i - \boldsymbol{x}_N)^2 = \sum_{i<j}^{N} \frac{1}{N} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2$ we get

$$H_0 = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i=1}^{N} \frac{1}{2} \frac{m\omega}{N} \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)^2$$

Introducing Jacobi coordinates :

$$H_0^{\mathrm{intrinsic}} = \sum_{i=1}^{N-1} \left[\frac{\boldsymbol{\pi}_\rho^2}{2m} + \frac{1}{2} m\omega^2 \boldsymbol{x}_\rho \right]$$