Chiral Effective Field Theory and Nuclear Forces: hands-on - representation of nuclear forces and some first calculations

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TALENT school @MITP:
Effective field theories in light nuclei

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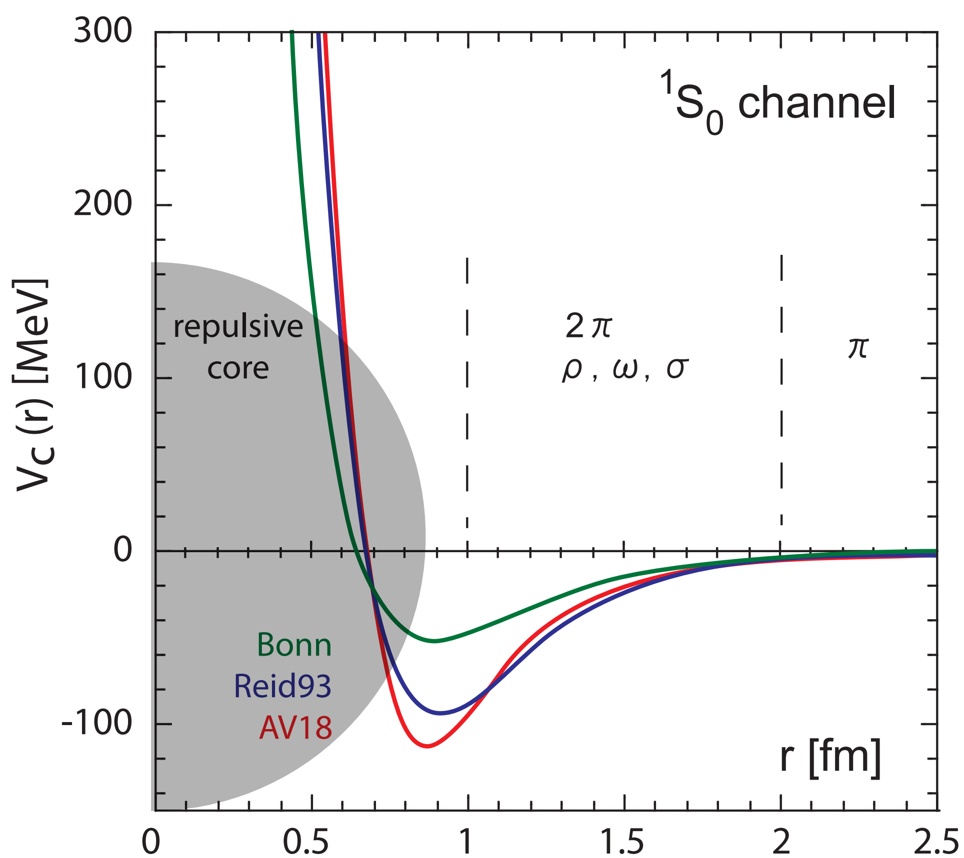
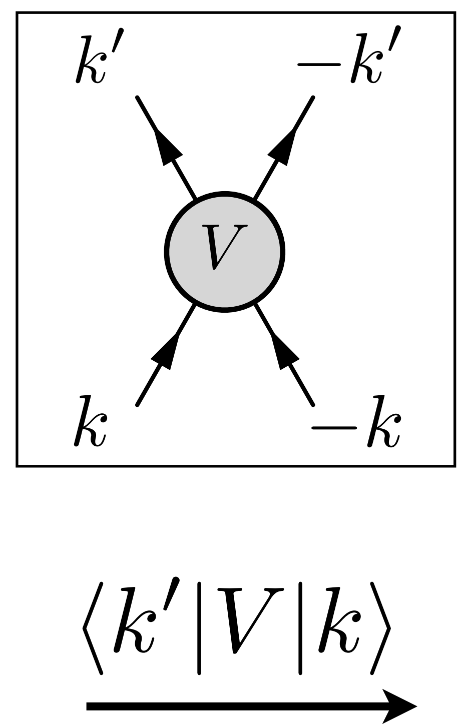
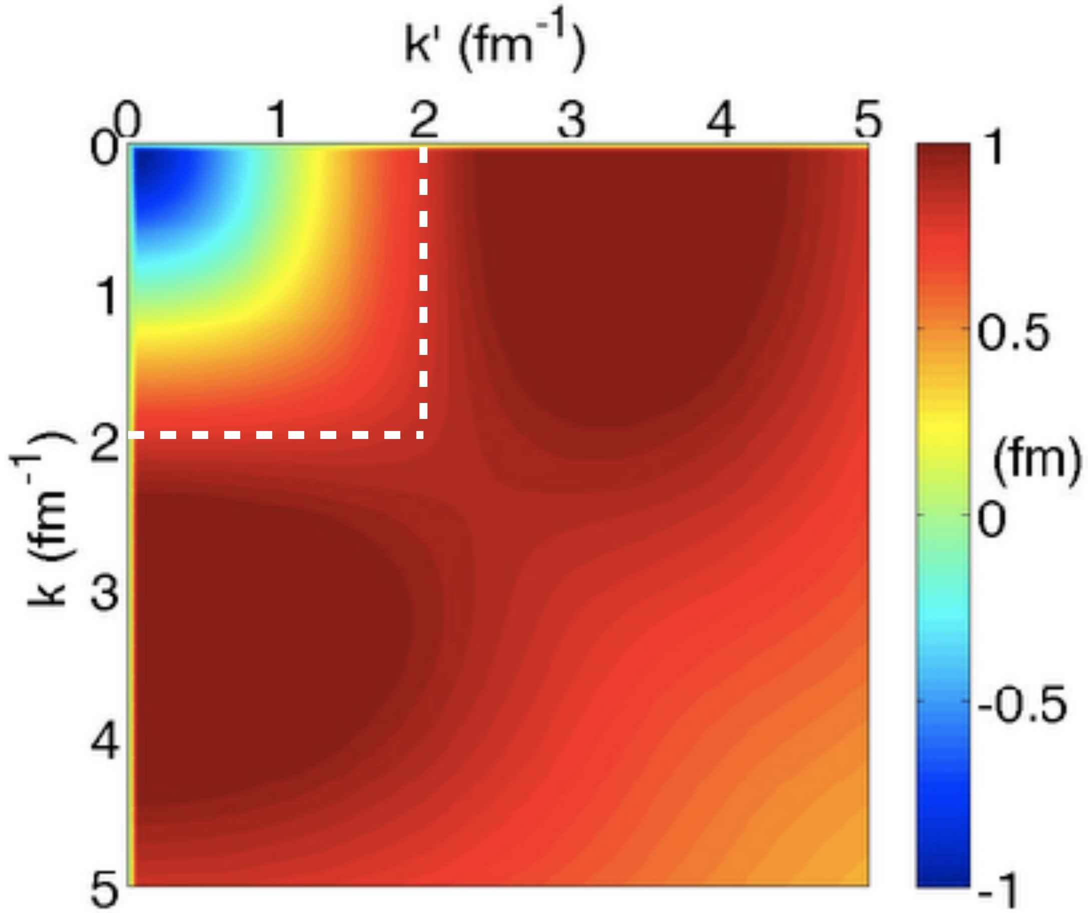
NN interactions





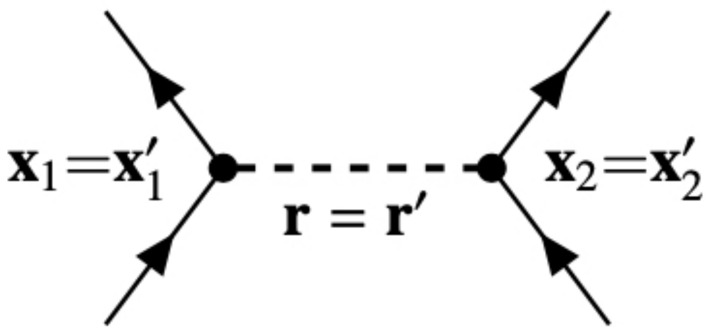


- constructed to fit low-energy scattering data
- "hard" NN interactions contain repulsive core at small relative distance
- strong coupling between low and high-momentum components

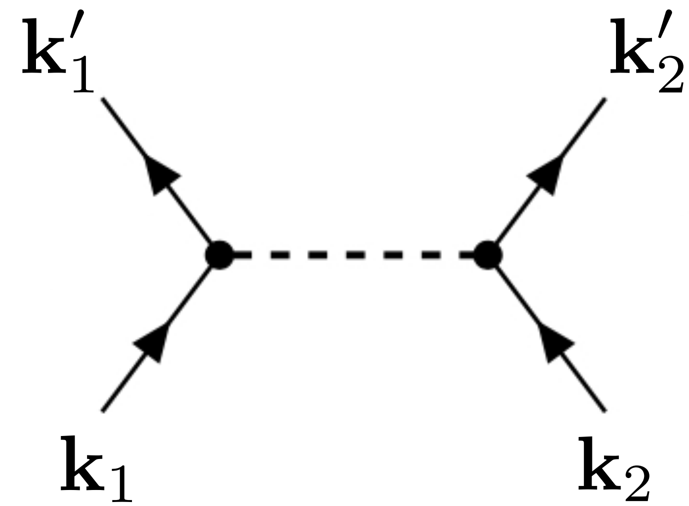
  $\Rightarrow$ nuclear many-body problem non-perturbative, hard to solve!

Nucleon-nucleon interactions

coordinate space:



momentum space:



In the following we will work in momentum space.

In general the interaction can be expressed in the following form (suppressing isospin):

$$\left\langle \mathbf{k}'_1 m_{s'_1} \mathbf{k}'_2 m_{s'_2} \middle| V_{\mathrm{NN}} \middle| \mathbf{k}_1 m_{s_1} \mathbf{k}_2 m_{s_2} \right\rangle$$

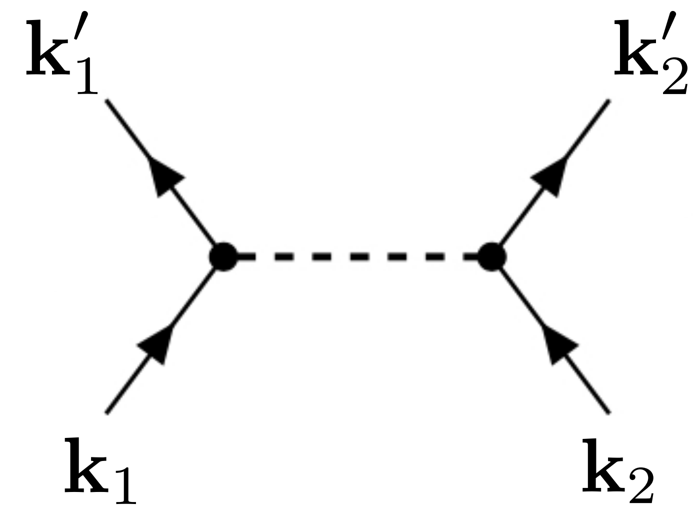
Single-particle spin projections

Local nucleon-nucleon interactions

Introduce relative momenta:

$$\mathbf{k}_1 = \frac{\mathbf{P}_{\rm cm}}{2} + \mathbf{p} \qquad \mathbf{k}_1' = \frac{\mathbf{P}_{\rm cm}'}{2} + \mathbf{p}'$$

$$\mathbf{k}_2 = \frac{\mathbf{P}_{\rm cm}}{2} - \mathbf{p} \qquad \mathbf{k}_2' = \frac{\mathbf{P}_{\rm cm}'}{2} - \mathbf{p}'$$



Use symmetries of interaction:

- conservation of center of mass momentum
- independence of P (Galilean invariance)
- rotational invariance

$$\left\langle \mathbf{k}_1' m_{s_1'} \mathbf{k}_2' m_{s_2'} \right| V_{\rm NN} \left| \mathbf{k}_1 m_{s_1} \mathbf{k}_2 m_{s_2} \right\rangle = \langle \mathbf{p}' S' m_{S'} | V_{\rm NN} | \mathbf{p} S m_S \rangle \, \delta(\mathbf{P}_{\rm cm} - \mathbf{P}_{\rm cm}')$$

Two-body spin quantum numbers

Partial wave expansion

Expand the angular dependence of momentum vector states in a partial wave basis:

$$|\mathbf{p} S M_S\rangle = A_{\rm NN} \sum_{L,M_L,J,M_J} \mathcal{C}^{JM_J}_{LM_L SM_S} Y^*_{LM_L}(\hat{\mathbf{p}}) \, |p(LS)JM_J\rangle$$

Clebsch-Gordan coefficients

Spherical harmonics

here we use the convention: $A_{\rm NN} = 4\pi$

Rotational invariance implies $J = J', M_J = M_{J'}$ and independence of M_J

Free-space interactions are spin conserving, i.e. $S = S'$, but in general $M_S \neq M_{S'}$ (for example due to tensor interactions)

As a consequence we also have in general $L \neq L' (\text{for } S = S' = 1)$

$$\longrightarrow \quad \langle \mathbf{p}' S M_{S'} | V_{\rm NN} | \mathbf{p} S M_S\rangle = (4\pi)^2 \sum_{L,L',M_L,M'_L J,M_J} \mathcal{C}^{JM_J}_{LMSM_S} \mathcal{C}^{JM_J}_{L'M_{L'},SM_{S'}}$$
$$\times \, Y_{L'M_{L'}}(\hat{\mathbf{p}}') \langle p' (L'S) J | V_{\rm NN} | p(LS) J\rangle Y^*_{LM_L}(\hat{\mathbf{p}})$$

Partial wave interaction matrix elements

Partial wave matrix elements: nomenclature

$$\langle p' \, (L'S) \, J \, | V_{\rm NN} | \, p(LS) J \rangle$$

Nomenclature:

- uncoupled channel: $L = L' = J$
- coupled channel: $L, L' = J \pm 1$
- Spectroscopic notation: $^{2S+1}X_J$

while $X = {'S'}, {'P'}, {'D'}, {'F'}, {'G'}, \ldots$ for $L = 0, 1, 2, 3, \ldots$

e.g. $^1S_0 : S = L = L' = J = 0$

$^3S_1 : S = 1, L = L' = 0, J = 1$

$^3SD_1 : S = 1, L = 0, L' = 2, J = 1$

$^1P_1 : S = 0, L = L' = J = 1$

https://nn-online.org/

Operators in partial wave representation

Normalisation of vector-states:

$$\sum_{S,M_S} \int \frac{d\mathbf{p}}{(2\pi)^3} \left|\mathbf{p}SM_S\right\rangle \left\langle \mathbf{p}SM_S\right| = 1$$

$$\langle \mathbf{p}'S'M_{S'}|\mathbf{p}SM_S\rangle = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\delta_{SS'}\delta_{M_SM_{S'}}$$

Normalization of partial wave states: (exercise!)

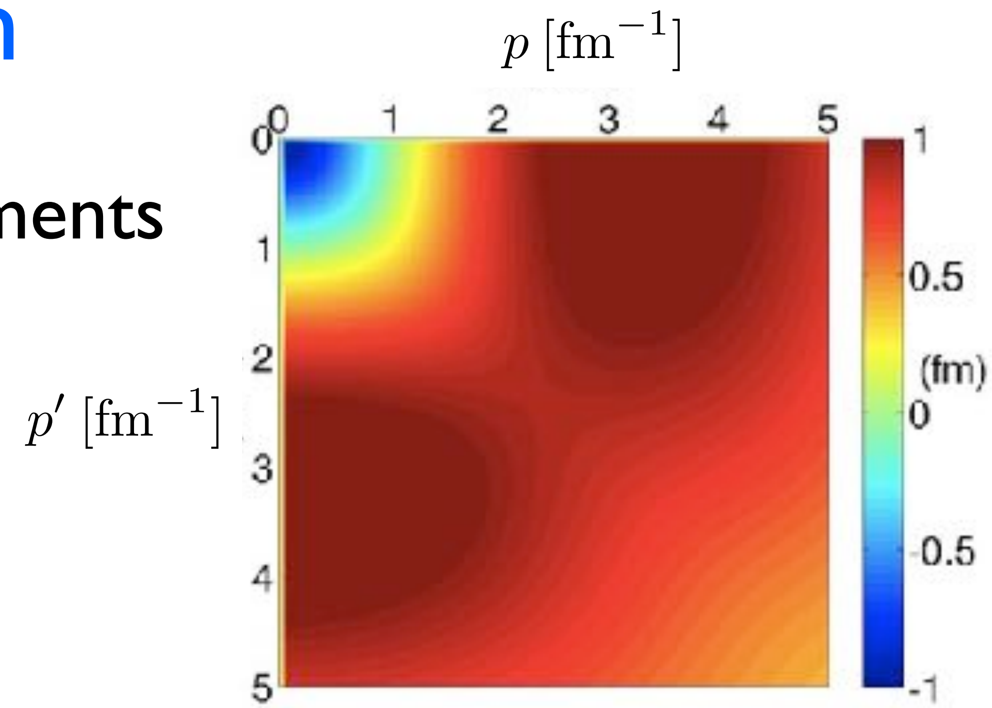
$$\frac{2}{\pi}\sum_{S,L,J,M_J} \int dpp^2 \left|p(LS)JM_J\right\rangle \left\langle p(LS)JM_J\right| = 1$$

$$\langle p'(L'S')J'M_{J'}|p(LS)JM_J\rangle = \frac{\pi}{2}\frac{\delta(p-p')}{pp'}\delta_{LL'}\delta_{SS'}\delta_{JJ'}\delta_{M_JM_{J'}}$$

$$\hat{V}_{\rm NN}\hat{V}_{\rm NN} \to \frac{2}{\pi}\int d\bar{p}\bar{p}^2 \sum_{\bar{L}} \left\langle p'(L'S)J|V_{\rm NN}|\bar{p}(\bar{L}S)J\right\rangle \left\langle \bar{p}(\bar{L}S)J|V_{\rm NN}|p(LS)J\right\rangle$$

Discretization

For practical calculations tabulate matrix elements on a discrete mesh system $\{p_i\}$



$$\langle p'_i\,(L'S)\,J\,|V_{\mathrm{NN}}|\,p_j(LS)J\rangle$$

By this all objects become discrete matrices and integrals turn into sums:

$$C\,(p,p') = \frac{2}{\pi}\int d\bar{p}\bar{p}^2 A(p,\bar{p})B\,(\bar{p},p') \rightarrow C_{ij} = \frac{2}{\pi}\sum_n w_n\bar{p}_n^2 A_{in}B_{nj}$$

mesh weights   mesh points

By defining $\bar{A}_{ij} = \frac{2}{\pi}\sqrt{w_i}p_i A_{ij}\sqrt{w_j}p_j$

we can absorb all extra factors in the objects and conveniently write matrix products in the form: $\bar{C}_{ij} = \bar{A}_{in}\bar{B}_{nj}$ (sum convention)

A first look at some real-world interactions

Python notebook version 0

Warm up problem: the deuteron bound state

Schroedinger equation:

$$(T_{\mathrm{rel}} + V_{\mathrm{NN}}) \left|\psi_E\right\rangle = E \left|\psi_E\right\rangle$$

Relative kinetic energy diagonal: $\langle \mathbf{p}' | T_{\mathrm{rel}} | \mathbf{p} \rangle = \frac{\mathbf{p}^2}{m_N} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')$

The interaction matrix elements are given in units of $[V_{\mathrm{NN}}] = \mathrm{fm}$
corresponding to a unit system with $m_N = 1$

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What is the suitable basis representation for the problem?

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$V_{\mathrm{NN}}^{00}(p_i, p_j)$	$V_{\mathrm{NN}}^{02}(p_i, p_j)$
$V_{\mathrm{NN}}^{20}(p_i, p_j)$	$V_{\mathrm{NN}}^{22}(p_i, p_j)$

Exercise: represent the problem in matrix form and determine binding energy via numerical diagonalization.

Calculate and visualize the wave function, including the separate S-wave and D-wave contributions