TALENT summer school

Lippmann-Schwinger equation

1. Using the Schrödinger equation for the scattering of two particles with mass m,

$$(H_0 + V)|\psi_E\rangle = E|\psi_E\rangle \ ,$$

where H_0 is the free Hamiltonian, show that the Lippmann-Schwinger equation for the wave function,

$$|\psi_E^{\pm}\rangle = |\phi_E\rangle + \frac{1}{E - H_0 \pm i\epsilon} V|\psi_E^{\pm}\rangle \ ,$$

is satisfied. Here the plane wave state satisfies $H_0|\phi_E\rangle = E|\phi_E\rangle$. Why do you need the $\pm i\epsilon$?

2. We can define the T-matrix on-shell as the transition matrix that acting on the plane wave state yields the same result as the potential acting on the full scattering state. That is,

$$T^{(\pm)}(E)|\phi_E\rangle = V|\psi_E^{\pm}\rangle \tag{1}$$

What does it mean that the T-matrix is "on-shell"?

3. Show that matrix elements of the right-half on-shell T-matrix defined by

$$\left\langle \mathbf{p}'|T^{(\pm)}(E)|\mathbf{p}\right\rangle = \left\langle \mathbf{p}'|T^{(\pm)}(E = p^2/m)|\mathbf{p}\right\rangle \tag{2}$$

satisfy the Lippmann-Schwinger equation

$$\langle \mathbf{p}'|T^{(\pm)}(E)|\mathbf{p}\rangle = \langle \mathbf{p}'|V|\mathbf{p}\rangle + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\langle \mathbf{p}'|V|\mathbf{q}\rangle\langle \mathbf{q}|T^{(\pm)}(E)|\mathbf{p}\rangle}{E - \frac{q^2}{m} \pm i\epsilon} \ .$$

4. Start from the momentum-space partial-wave expansion and show that the Lippmann-Schwinger equation in a partial wave representation is given by:

$$\langle p'(L'S)J|T(E)|p(LS)J\rangle = \langle p'(L'S)J|V|p(LS)J\rangle \tag{3}$$

$$+ \frac{2}{\pi}\sum_{L''}\int_0^\infty dq\, q^2 \frac{\langle p'(L'S)J|V|q(L''S)J\rangle\langle q(L''S)J|T(E)|p(LS)J\rangle}{E - q^2/m + i\epsilon} \ . \tag{4}$$

The right half-on right shell T matrix is then given by

$$\langle p'(L'S)J|T(E = p^2/m)|p(LS)J\rangle = \langle p'(L'S)J|V|p(LS)J\rangle \tag{5}$$

$$+ \frac{2}{\pi}\sum_{L''}\int_0^\infty dq\, q^2 \frac{\langle p'(L'S)J|V|q(L''S)J\rangle\langle q(L''S)J|T(E = p^2/m)|p(LS)J\rangle}{p^2/m - q^2/m + i\epsilon} \ . \tag{6}$$

5. For the numerical solution of this equation we will focus for simplicity on the uncoupled case and write compactly (using $m = 1$)

$$\langle p'|K(p^2)|p\rangle = \langle p'|V|p\rangle + \frac{2}{\pi}\mathcal{P}\int_0^\infty dq\, q^2 \frac{\langle p'|V|q\rangle\langle q|K(p^2)|p\rangle}{p^2 - q^2} \ . \tag{7}$$

Here we introduced the K matrix which results from the real part of the integral kernel. Discuss the practical challenges for solving this equation, can it be solved by straightforward inversion?

6. Show that the integral on the right-hand side can be rewritten in the following form:

$$\mathcal{P}\int_0^\Lambda dq\, q^2 \frac{\langle p'|V|q\rangle\langle q|K(p^2)|p\rangle}{p^2-q^2} = \int_0^\Lambda dq \left[\frac{q^2\langle p'|V|q\rangle\langle q|K(p^2)|p\rangle - p^2\langle p'|V|p\rangle\langle p|K(p^2)|p\rangle}{p^2-q^2}\right] + p^2\langle p'|V|p\rangle\langle p|K(p^2)|p\rangle f_{ct}(p,\Lambda) \tag{8}$$

with the counter term

$$f_{ct}(p,\Lambda) = \frac{\log\left|\frac{p+\Lambda}{p-\Lambda}\right|}{2p} = \frac{\text{arctanh}\left(\frac{p}{\Lambda}\right)}{p}. \tag{9}$$

What is the advantage of this representation compared to the representation (7)?

The code version ... contains routines that allow to compute the K matrix for coupled and uncpoupled channels? Try to understand the routine for the uncoupled channels and compute the matrix elements of the K matrix for different channels.

7. The Lippmann Schwinger equation can now be solved numerically by discretizing all the matrices, i.e.

$$\langle p_i|K(p^2)|p_j\rangle = \langle p_i|V|p_j\rangle + \frac{2}{\pi}\sum_k w_k \left[\frac{p_k^2\langle p_i|V|p_k\rangle\langle p_k|K(p^2)|p_j\rangle - p_j^2\langle p_i|V|p_j\rangle\langle p_j|K(p^2)|p_j\rangle}{p_j^2-p_k^2}\right] + \frac{2}{\pi}p_j^2\langle p_i|V|p_j\rangle\langle p_j|K(p^2)|p_j\rangle f_{ct}(p_j,\Lambda)\ . \tag{10}$$

The pole in the integrand is now formally properly subtracted, but the expression is still numerically ill-defined. In order to solve the equation we introduce an additional mesh point p_{N+1} that is slightly shifted elative to the pole position: $p_{N+1} = p_j + \epsilon$ (with ϵ small) and write

$$\langle p_i|K(p^2)|p_j\rangle = \langle p_i|V|p_j\rangle + \frac{2}{\pi}\sum_{k=1}^N w_k \left[\frac{p_k^2\langle p_i|V|p_k\rangle\langle p_k|K(p^2)|p_j\rangle - p_{N+1}^2\langle p_i|V|p_{N+1}\rangle\langle p_{N+1}|K(p^2)|p_j\rangle}{p_{N+1}^2-p_k^2}\right] + \frac{2}{\pi}p_{N+1}^2\langle p_i|V|p_{N+1}\rangle\langle p_{N+1}|K(p^2)|p_j\rangle f_{ct}(p_{N+1},\Lambda)\ . \tag{11}$$

Show that the solution for the K matrix for a particular energy $E = p_{N+1}^2$ can be written in form of the following $(N+1)\times(N+1)$ matrix problem:

$$A_{ik}K_{kj} = V_{ij} \tag{12}$$

with $V_{N+1,i} = \langle p_{N+1}|V|p_i\rangle$ etc. and $(i,k \in [1,N])$

$$\begin{aligned}
A_{i,k} &= \delta_{ik} - \frac{2}{\pi}w_k p_k^2 \frac{\langle p_i|V|p_k\rangle}{p_{N+1}^2-p_k^2}\\
A_{N+1,k} &= -\frac{2}{\pi}w_k p_k^2 \frac{\langle p_{N+1}|V|p_k\rangle}{p_{N+1}^2-p_k^2}\\
A_{i,N+1} &= +\frac{2}{\pi}\langle p_i|V|p_{N+1}\rangle p_{N+1}^2\left[\left(\sum_{k=1}^N \frac{w_k}{p_{N+1}^2-p_k^2}\right) - f_{ct}(p_{N+1},\Lambda)\right]\\
A_{N+1,N+1} &= 1 + \frac{2}{\pi}\langle p_{N+1}|V|p_{N+1}\rangle p_{N+1}^2\left[\left(\sum_{k=1}^N \frac{w_k}{p_{N+1}^2-p_k^2}\right) - f_{ct}(p_{N+1},\Lambda)\right]
\end{aligned} \tag{13}$$

8. For the uncoupled channels the phase shifts are related to the diagonal matrix elements of the right half-on shell K matrix via:

$$\delta_L(p) = \arctan\left[-p \langle p|K(p^2)|p\rangle\right] \tag{14}$$

Usually the phase shifts are given as a function of the laboratory energy $E_{\mathrm{lab}} = 2p^2$. Compute the phase shifts $\delta_L(E_{\mathrm{lab}})$ for different partial wave channels and compare to the experimental values. These experimental results can be obtained from the following website:

`https://nn-online.org/NN/`