10 Asymptotic normalization constants

(i) $\langle \vec{k}' | t(E) | \vec{k} \rangle = \frac{2\pi}{\mu} \left[\frac{1}{a} - \frac{r}{2} k^2 + ik \right]^{-1}$

$E = k^2/2\mu$ $\hookrightarrow$ has pole(s) at $E = -\frac{\gamma_{1/2}^2}{2\mu}$

<u>Poles:</u> $k = i\gamma$, $\gamma > 0$

$\hookrightarrow$ $1/a + \frac{r}{2}\gamma^2 - \gamma = 0$

$\hookrightarrow$ $\gamma^2 - \frac{2}{r}\gamma + \frac{2}{ra} = 0$

$\gamma_{1/2} = \frac{1}{r}\left(1 \pm \sqrt{1 - 2r/a}\right)$ $\nearrow$ $\gamma_1 \sim 2/r + \ldots$ $\searrow$ $\boxed{\gamma_2 \sim 1/a + \ldots}$

physical solution

Consider

(i) near pole

$$\langle \vec{k}' | t(E) | \vec{k} \rangle = \frac{2\pi}{\mu} \frac{1}{r/2} \left[(-ik - \gamma_1)(-ik - \gamma_2) \right]^{-1} \frac{(ik-\gamma}{(ik-\gamma}$$

Set $-ik \equiv \gamma_2$ except in pole term

$$\simeq \frac{4\pi}{\mu r} \frac{-2\gamma_2}{(\gamma_2 - \gamma_1)(k^2 + \gamma_2^2)}$$

rename $\gamma_2 \to \gamma$

$$\cong \frac{4\pi}{\mu r} \frac{-2\gamma}{2\mu} \frac{1}{(\gamma - \gamma_1)} \frac{1}{\underbrace{\frac{k^2}{2\mu}}_{E} + \frac{\gamma^2}{2\mu}}$$

$$\gamma - \gamma_1 = -2\sqrt{1 - 2r/a} \approx -2(1 - \gamma r)$$

$$\hookrightarrow \quad \boxed{Z = \frac{2\pi}{\mu^2} \frac{\gamma}{1 - \gamma r}}$$

(ii) Full Green's fct.

$$(1) \quad \frac{1}{E - H} = \frac{1}{E - H_0} + \frac{1}{E - H_0} t(E) \frac{1}{E - H_0}$$

$$(2) \quad \langle \vec{k}' | \frac{1}{E - H} | \vec{k} \rangle \Big|_{E = -\gamma^2/2\mu} = \frac{\psi(\vec{k}')\, \psi^*(\vec{k})}{E + \frac{\gamma^2}{2\mu}} + \text{regular}$$

only 2nd term of (1) can contribute to pole
$\longrightarrow$ evaluate in momentum space

$$\int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \underbrace{\langle \vec{k}' | \frac{1}{E - H_0} | \vec{p} \rangle}_{\frac{(2\pi)^3 \delta^3(\vec{k}' - \vec{p})}{E - \frac{p^2}{2\mu}}} \langle \vec{p} | t(E) | \vec{q} \rangle \underbrace{\langle \vec{q} | \frac{1}{E - H_0} | \vec{k} \rangle}_{\frac{(2\pi)^3 \delta^3(\vec{q} - \vec{k})}{E - \frac{q^2}{2\mu}}}$$

$$\hookrightarrow \langle \vec{k}' | \frac{1}{E - H} | \vec{k} \rangle \Big|_{E = \gamma^2} = \frac{-1}{\frac{\gamma^2}{2\mu} + \frac{k'^2}{2\mu}} \; \frac{-1}{\frac{\gamma^2}{2\mu} + \frac{k^2}{2\mu}} \; \frac{Z}{E + \frac{\gamma^2}{2\mu}} + \ldots$$

Need $\psi(\vec{k}) = \int d^3\vec{r}\; e^{+i\vec{k}\cdot\vec{r}} \underbrace{\psi(\vec{r})}_{A/\sqrt{4\pi}\,\frac{e^{-\gamma r}}{r}}$

$$= \frac{2\pi A}{\sqrt{4\pi}} \int_{-1}^{1} dx \int_0^\infty r^2 dr\; e^{ikrx} \frac{e^{-\gamma r}}{r}$$

$$= \int_0^\infty \frac{2\pi}{\sqrt{4\pi}} A \int_0^\infty \cancel{r}\, dr\; \frac{1}{ik\cancel{r}} \underbrace{\left(e^{ikr} - e^{-ikr}\right)}_{2i\sin(kr)} e^{-\gamma r}$$

$$= \sqrt{4\pi}\; A/\cancel{k} \underbrace{\int_0^\infty dr \sin(kr)\, e^{-\gamma r}}_{\cancel{k}/\gamma^2+k^2}$$

Now match coefficients

$$\frac{(2\mu)^2}{(\gamma^2+k'^2)(\gamma^2+k^2)} \frac{Z}{E+\frac{\gamma^2}{2\mu}} \overset{!}{=} \frac{|A|^2}{E+\frac{\gamma^2}{2\mu}} \frac{4\pi}{(\gamma^2+k^2)(\gamma^2+k'^2)}$$

$\hookrightarrow$ A real $\quad A = \frac{\mu}{\sqrt{\pi}} Z^{1/2}$

$$\boxed{A = \frac{\cancel{\mu}}{\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{\cancel{\mu}} \left(\frac{2\gamma}{1-\gamma r}\right)^{1/2}}$$