Effective Field Theories

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Literature

G.P. Lepage, “How to renormalize the Schrödinger equation”, arXiv:nucl-th/9706029

D.B. Kaplan, “Effective Field Theories”, arXiv:nucl-th/9506035

E. Braaten, HWH, Phys. Rep. **428** (2006) 259 [arXiv:cond-mat/0410417]

E. Braaten, HWH, G.P. Lepage, Phys. Rev. A **95** (2017) 012708 [arXiv:1607.08084]

Physics Near the Unitary Limit

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- Unitary limit: $a \to \infty$, $\ell \to 0 \quad \Longrightarrow \quad \mathcal{T}_2(k) \propto i/k$
- Use as starting point for effective field theory description
  - Large scattering length: $|a| \gg \ell \sim r_e, \ell_{vdW}, \ldots$
  - Natural expansion parameter: $\ell/|a|$, $k\ell$,...
  - Universal dimer with energy $B_2 = 1/(ma^2)$ $\quad (a > 0)$

    size $\langle r^2 \rangle^{1/2} \sim a \quad \Rightarrow \quad$ halo state
  - Reproduce tail of the wave function: $\psi(r) = \dfrac{e^{-r/a}}{r}$

- Nonperturb. resummation in EFT (van Kolck, Kaplan, Savage, Wise, 1998)

$$\mathcal{T}_2(k) \propto \frac{-a}{1+ika}\left[1 + \frac{r_e a k^2/2}{1+ika} + \ldots\right], \qquad Q \sim 1/a \sim k \ll 1/\ell$$

$\Longrightarrow$ universal properties (and perturbative corrections)

Physical Systems with Large a

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- Natural expansion parameter: $\ell/|a|$, $k\ell$,... $(\ell \sim r_e, \ell_{vdW}, ...)$
- Nuclear Physics: S-wave NN scattering, halo nuclei,...
  - $^1S_0$, $^3S_1$: $|a| \gg r_e \sim 1/m_\pi \longrightarrow B_d \approx 2.2$ MeV
  - neutron matter
  - halo nuclei (e.g. ^{11}Be = ^{11}Be + n, ^{11}Li = ^{9}Li + n + n, ...)

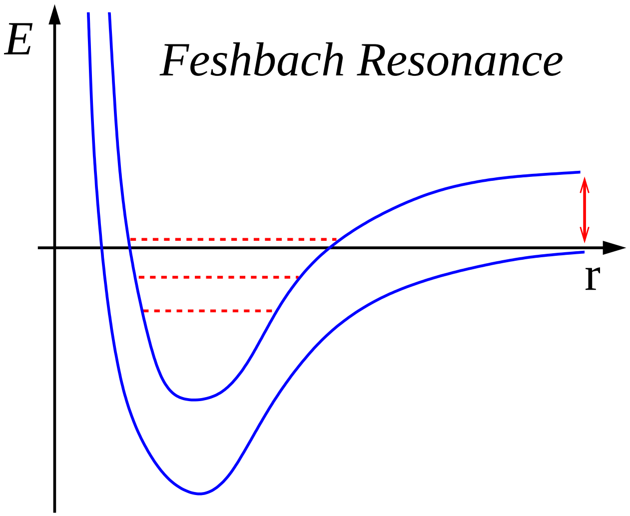
  $\Longrightarrow$ lecture by Daniel Phillips (week 3)
- Atomic Physics:
  - ^{4}He atoms: $a \approx 104$ Å $\gg r_e \approx 7$ Å $\sim \ell_{vdW}$, $\longrightarrow B_d \approx 100$ neV
  - ultracold atoms near Feshbach resonance $\Rightarrow$ variable a
- Particle Physics
  - Is the $X(3872)$ a $|D^0\bar{D}^{0*} + \bar{D}^0 D^{0*}\rangle$ molecule? $(J^{PC} = 1^{++})$

$$E_X = m_D + m_{D^*} - m_X = (0.07 \pm 0.12)\ \text{MeV}$$

Variable Scattering Length

- Feshbach Resonance:

  energy of molecular state in closed channel close to energy of scattering state



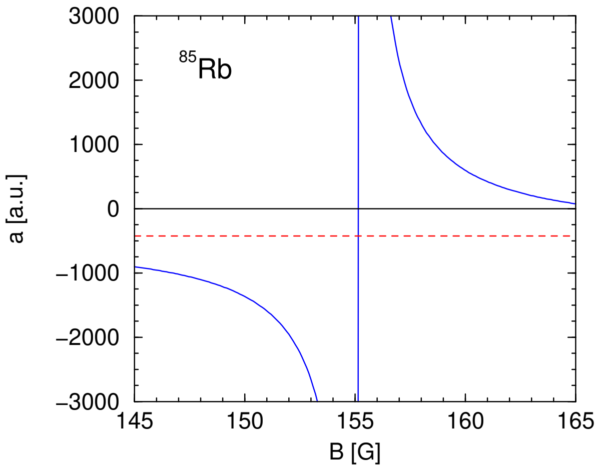
- Tune scattering length via external magnetic field

  (Tiesinga, Verhaar, Stoof, 1993)

- Example:

  ^{85}Rb atoms $\quad \dfrac{a(B)}{a_0} = 1 + \dfrac{\Delta}{B_0 - B}$

  $a_0 = -422$ a.u., $B_0 = 155.2$ G, $\Delta = 11.6$ G

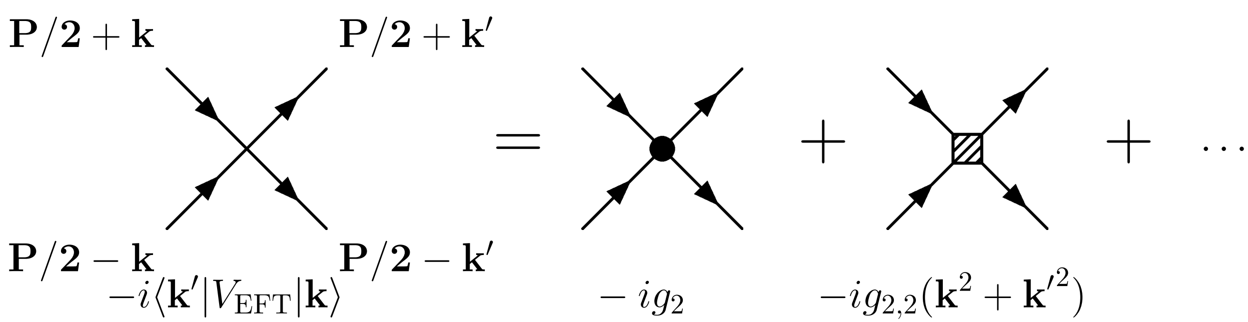


Two-Body System

- Large scattering length: $Q \sim k \sim 1/a \ll 1/\ell$
- Effective Lagrangian

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{1}{2m}\vec{\nabla}^2 \right) \psi - \frac{g_2}{4}(\psi^\dagger\psi)^2 - \frac{g_{2,2}}{4}\left[\vec{\nabla}(\psi^\dagger\psi)\right]^2 + \ldots$$

- Vertices:



- Scaling: $g_2 \sim 1/(mQ)$, $g_{2,2} \sim \ell/(mQ^2)$, $g_{2,2j} \sim \ell^j/(mQ^{j+1})$
- Scaling of diagrams with Q: Q^ν

$$\nu = 5L - \underbrace{2I}_{2(L+V-1)} + \sum_j (2j-(j+1))V_{2j} = 3L+2+\sum_j (j-3)V_{2j} \geq -1$$

Two-Body System

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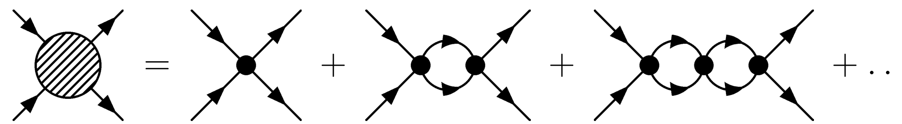
- All diagrams with only g_2 vertices scale as $1/Q$
- Scattering Amplitude:



$$\mathcal{T}_2(E) = -ig_2 - \frac{ig_2^2}{2}\int_0^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{1}{m\tilde{E}-q^2} + .. = -ig_2 + \frac{ig_2^2}{4\pi^2}\left(\Lambda - \frac{\pi}{2}\sqrt{-m\tilde{E}}\right) + ..$$

$$\equiv \frac{8\pi}{m}\frac{1}{-1/a+\sqrt{-m\tilde{E}}}$$

matching!

- Definitions: $m\tilde{E} \equiv mE + i\epsilon = k^2 + i\epsilon$
- Geometric series $\Rightarrow$ can be summed
- Loop divergent $\Rightarrow$ regulate with momentum cutoff Λ
- Running coupling constant: $g_2(\Lambda) = \frac{8\pi}{m}\left(\frac{1}{a} - \frac{2}{\pi}\Lambda\right)^{-1}$

Two-Body System

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- How does g_2 change with Λ (resolution scale)
- Renormalization group equation

$$\tilde{g}_2 \equiv \frac{m\Lambda}{4\pi^2} g_2 \implies \Lambda \frac{d}{d\Lambda} \tilde{g}_2 = \tilde{g}_2 (1 + \tilde{g}_2)$$

- Two fixed points:

$- \ \tilde{g}_2 = 0 \ \leftrightarrow \ a = 0 \quad \implies$ no interaction

$- \ \tilde{g}_2 = -1 \ \leftrightarrow \ 1/a = 0 \quad \implies$ unitary limit

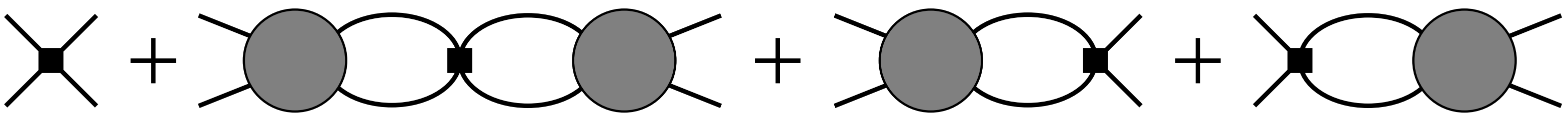
scale and conformal invariance

(Mehen, Stewart, Wise, 2000; Nishida, Son, 2007; ...)

- Interesting many-body physics: BEC/BCS crossover, universal viscosity bound $\Rightarrow$ perfect liquid, ...
(Kovtun, Son, Starinets, 2005; ...)

Two-Body System: Corrections

- Higher-order derivative terms are perturbative
- Diagrams with $g_{2,2}$ scale like Q^0: first correction



$$\mathcal{T}_2^0(E) = \frac{1}{\left(-1/a + \sqrt{-m\tilde{E}}\right)^2} \left(-2 g_{2,2} m E \left(\frac{8\pi}{m g_2}\right)^2 + \text{const.} \right)$$

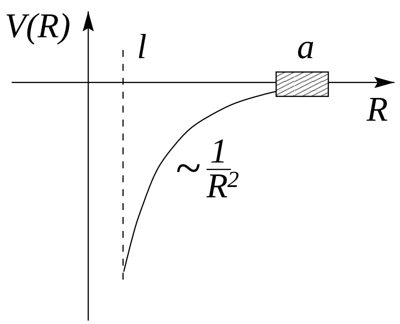
$$\equiv -\frac{8\pi}{m} \frac{m E r_e/2}{\left(-1/a + \sqrt{-m\tilde{E}}\right)^2} \qquad \text{matching!}$$

- Constant subtracted by counterterm $g_2^{(2)}$

Broken Scale Invariance

- Three-boson system near the unitary limit (Efimov, 1970)
- Hyperspherical coordinates: $R^2 = (r_{12}^2 + r_{13}^2 + r_{23}^2)/3$
- Schrödinger equation simplifies for $|a| \gg \ell \gg l$:

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial R^2} + \frac{s_0^2 + 1/4}{R^2}\right] f(R) = \underbrace{-\frac{\hbar^2\kappa^2}{m}}_{E} f(R)$$



- Singular Potential: renormalization required
- Boundary condition at small R: breaks scale invariance

  $\Longrightarrow$ scale invariance is anomalous
  $\Longrightarrow$ observables depend on boundary condition and a

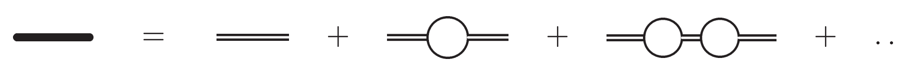
- Universality concept must be extended $\Rightarrow$ 3-body parameter
- Spin-1/2 fermions: Pauli repulsion $\Rightarrow$ higher order

Three-Body System

- Use auxilliary field $d = \psi^2 + \ldots$ (cf. Hubbard-Stratonovich, Kaplan, ...)
- Effective Lagrangian

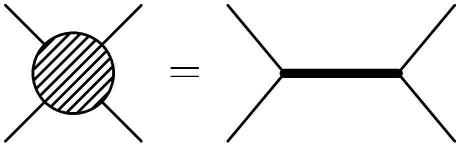
$$\mathcal{L}_d = \psi^\dagger \left(i\partial_t + \frac{1}{2m}\vec{\nabla}^2 \right) \psi + \frac{g_2}{4} d^\dagger d - \frac{g_2}{4}(d^\dagger \psi^2 + (\psi^\dagger)^2 d) - \frac{g_3}{36} d^\dagger d \psi^\dagger \psi + ..$$

- Equation of motion $\Longrightarrow$ eliminate $d \longrightarrow \mathcal{L}$
- Full dimeron propagator:



$$iD_d(P_0, P) = \frac{32\pi i}{g_2^2} \left[1/a - \sqrt{-mP_0 + P^2/4 - i\epsilon} \right]^{-1}$$

- Two-body amplitude:



Three-Body System in EFT

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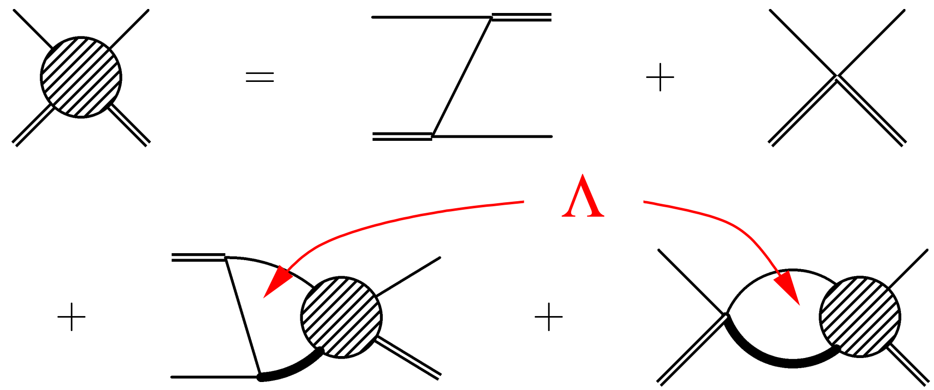
- Three-body equation:
  (S-waves)



$$\mathcal{T}_3(k,p) = M(k,p) + \frac{4}{\pi}\int_0^{\Lambda} dq\, \frac{q^2\, M(q,p)}{-\frac{1}{a} + \sqrt{\frac{3}{4}q^2 - mE - i\epsilon}}\, \mathcal{T}_3(k,q)$$

with $$M(k,p) = \underbrace{F(k,p)}_{\text{1-atom exchange}} \underbrace{-\frac{g_3}{9 g_2^2}}_{H(\Lambda)/\Lambda^2}$$

($g_3 = 0$, $\Lambda \to \infty \longrightarrow$ Skorniakov, Ter-Martirosian ’57)

Renormalization

- Require invariance under $\Lambda \to \Lambda' = \Lambda + \delta\Lambda$

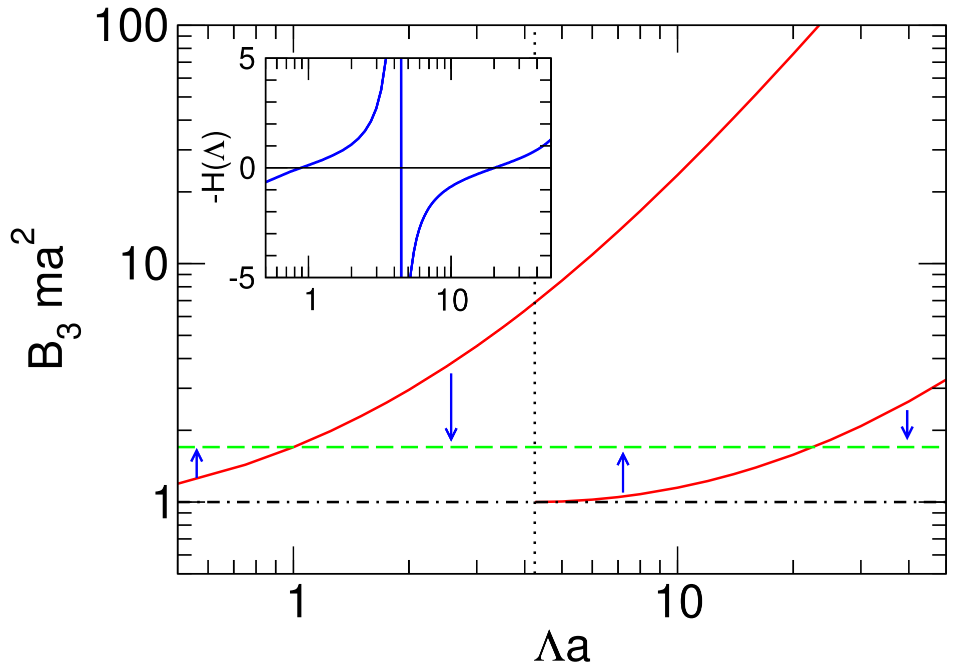
- $H(\Lambda)$ periodic: limit cycle

$$\Lambda \to \Lambda\, e^{n\pi/s_0} \approx \Lambda(22.7)^n$$

(Bedaque, HWH, van Kolck, 1999)

(Wilson, 1971)

- Anomaly: scale invariance broken to discrete subgroup



$$H(\Lambda) \approx \frac{\cos(s_0 \ln(\Lambda/\Lambda_*) + \arctan(s_0))}{\cos(s_0 \ln(\Lambda/\Lambda_*) - \arctan(s_0))}\,, \qquad s_0 \approx 1.00624$$

- Matching: $\Lambda_* \longleftarrow$ three-body observable
- Limit cycle $\iff$ Discrete scale invariance $\iff$ Efimov physics

Discrete Scale Invariance

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- Similarity to Matrjoschka doll

  $\longrightarrow$ "Russian Doll Renormalization"

- Other examples
  - $1/r^2$ potential in QM
    (Beane et al., ...)
  - Field theory models
    (Wilson, Glazek, LeClair et al.,...)
  - Turbulence, earthquakes, finance,...
    (cf. Sornette, Phys. Rep. **297** (1998) 239)

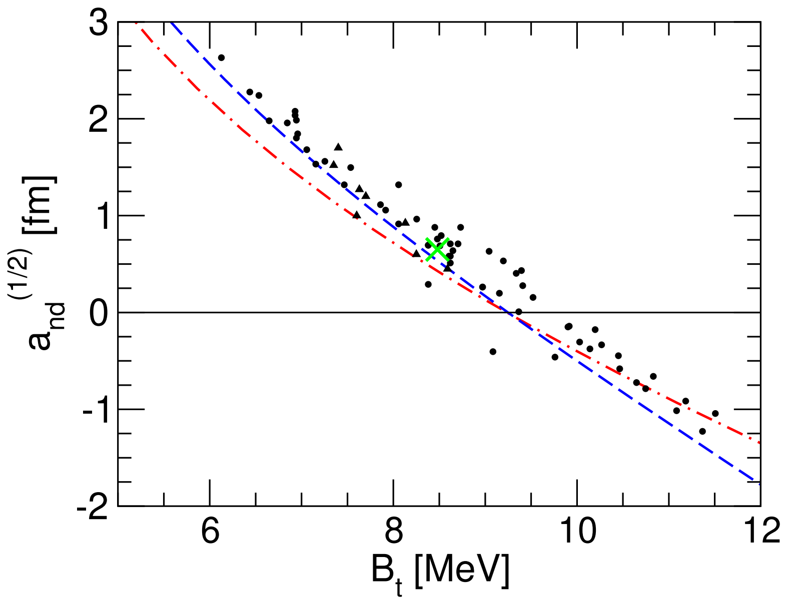
- Observable consequences?

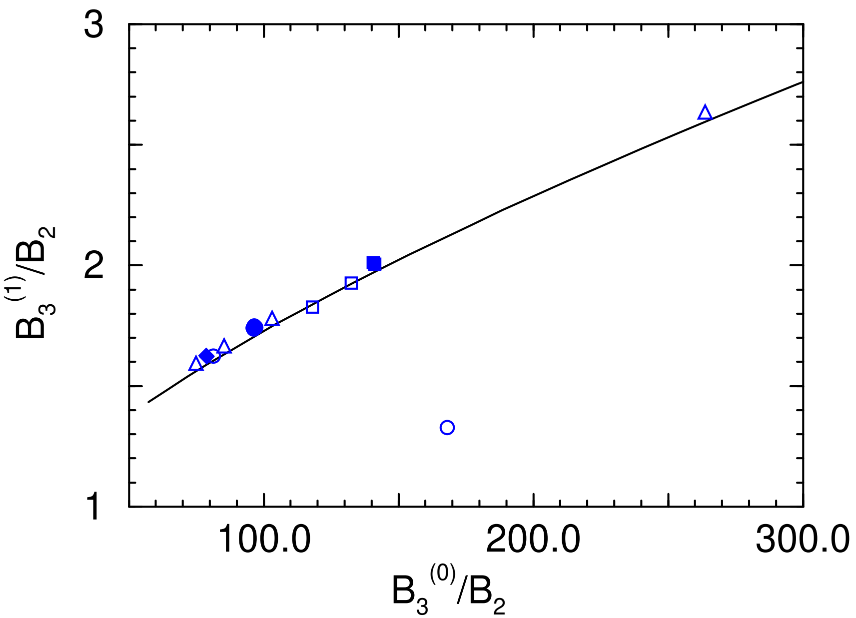
  $\longrightarrow$ Universal correlations (2 parameters at LO), Efimov effect, log-periodic dependence on scattering length,...

Universality in the 3-Body System

- Three-body observables for different potentials are correlated

$\Longrightarrow$ Phillips line (Phillips, 1968)



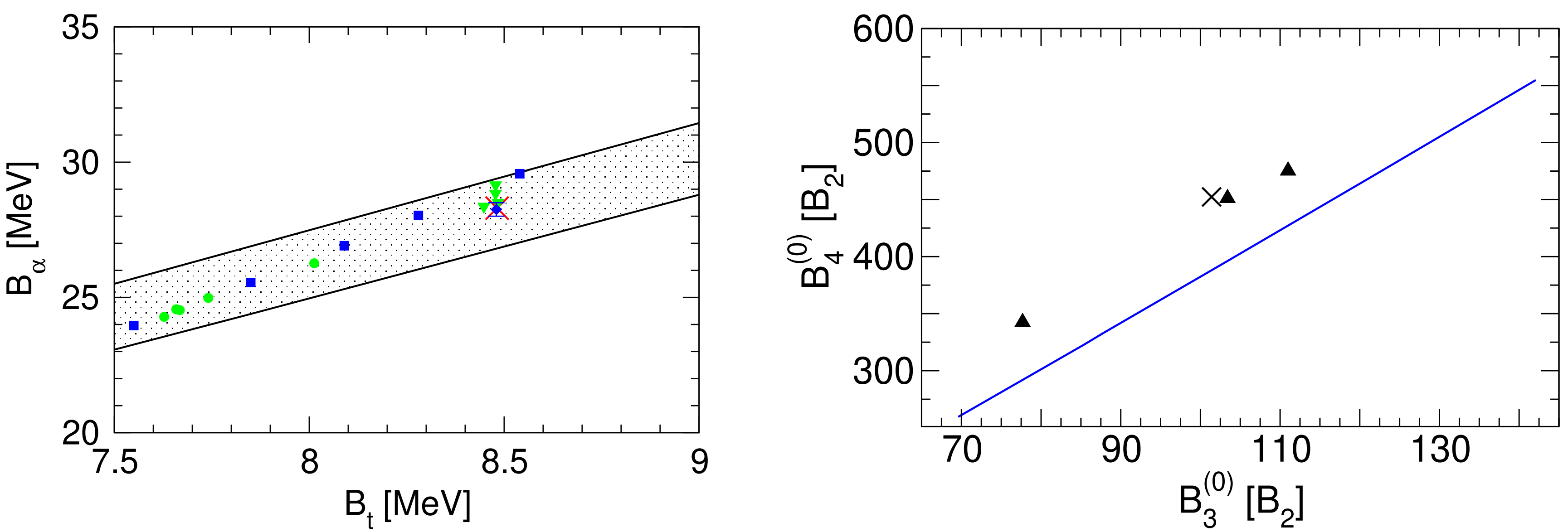


- Different values for Λ_* $\rightarrow$ physics at small distances
- Correlation universal: Nucleons, ^{4}He atoms,...

Universality in the 4-Body System

- No Four-body force required in LO (Platter, HWH, Meißner, 2004)
- Universal correlations persist in 4-body system

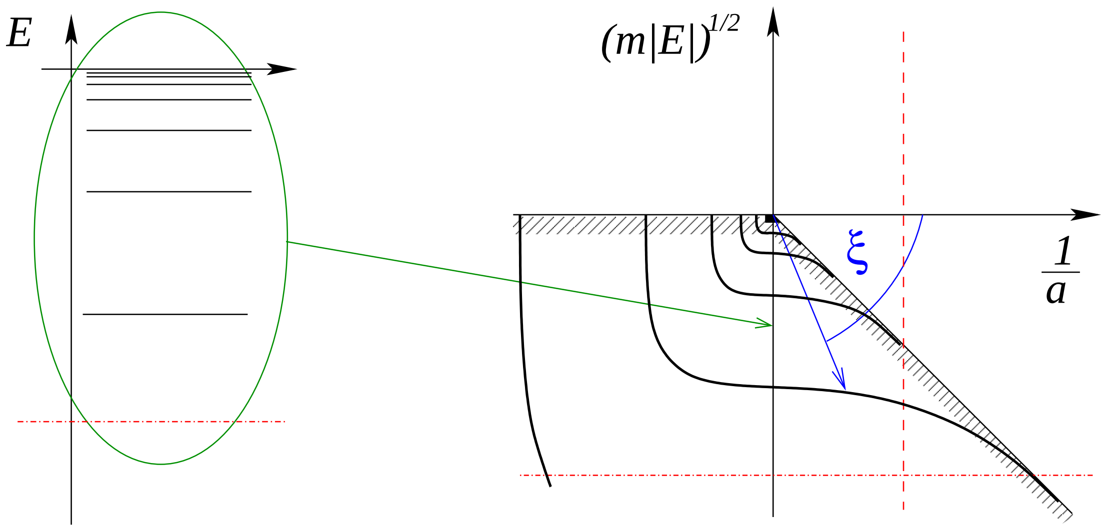
$\Longrightarrow$ Tjon line (Tjon, 1975)



- Variation of Λ_* parametrizes Tjon line
- Correlation universal: Nucleons, ^{4}He atoms,...

Limit Cycle: Efimov Effect

- Universal spectrum of three-body states
  (Efimov, 1970)



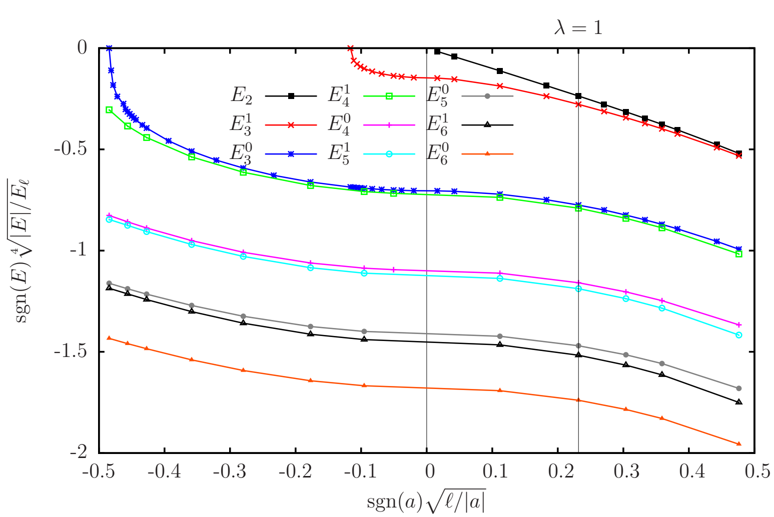
  - Discrete scale invariance for fixed angle ξ
  - Geometrical spectrum for $1/a \to 0$

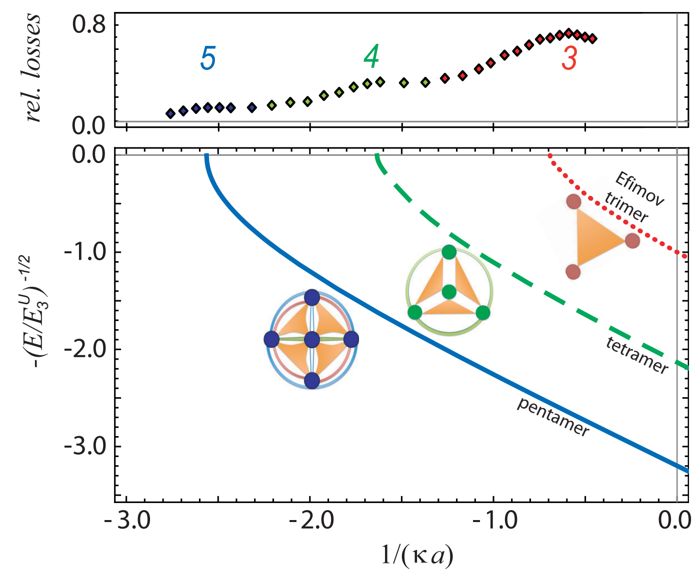
$$B_3^{(n)}/B_3^{(n+1)} \quad \xrightarrow{1/a \to 0} \quad \left(e^{\pi/s_0}\right)^2 = 515.035\ldots$$

- Ultracold atoms $\Longrightarrow$ variable scattering length

Universal Tetramers and Beyond

- No four-body force required at LO (Platter, HWH, Meißner, 2004)
- Universal tetramers: $B_4^{(0)} = 4.610(1)\ B_3\,, \quad B_4^{(1)} = 1.00227(1)\ B_3$
  (Platter, HWH, 2004, 2007; von Stecher et al., 2009; Deltuva 2010-2013)
- Two tetramers attached to each trimer
- Universal states up to $N = 16$ calculated
  (von Stecher, 2010, 2011; Gattobigio, Kievsky, Viviani, 2011-2014)
- Observation up to $N = 5$ in Cs losses (Grimm et al. (Innsbruck), 2009, 2013)





Three-Body Recombination

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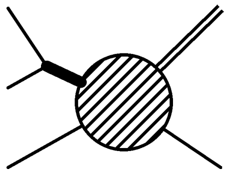

- Three-body recombination:

  3 atoms $\rightarrow$ dimer + atom $\Rightarrow$ loss of atoms

- Recombination constant: $\dot{n}_A = -K_3\, n_A^3$
- K_3 has log-periodic dependence on scattering length

  (Nielsen, Macek, 1999; Esry, Greene, Burke, 1999; Bedaque, Braaten, HWH, 2000)

- Deep dimers: Efimov trimers aquire width $\Rightarrow$ resonances
- Loss term in short distance b.c.: $\Lambda_* \longrightarrow \Lambda_* \exp^{i\eta_*/s_0}$

  $\Longrightarrow$ non-hermitian Hamiltonian (“optical potential”)

- Universal line shape of recombination resonance ($a < 0$)

$$K_3^{deep} = \frac{64\pi^2(4\pi - 3\sqrt{3})\ \coth(\pi s_0)\sinh(2\eta_*)}{\sin^2[s_0 \ln(a/a_-)] + \sinh^2 \eta_*} \frac{\hbar\, a^4}{m}\,, \qquad s_0 \approx 1.00624..$$

and other observables . . .

Efimov Physics in Ultracold Atoms

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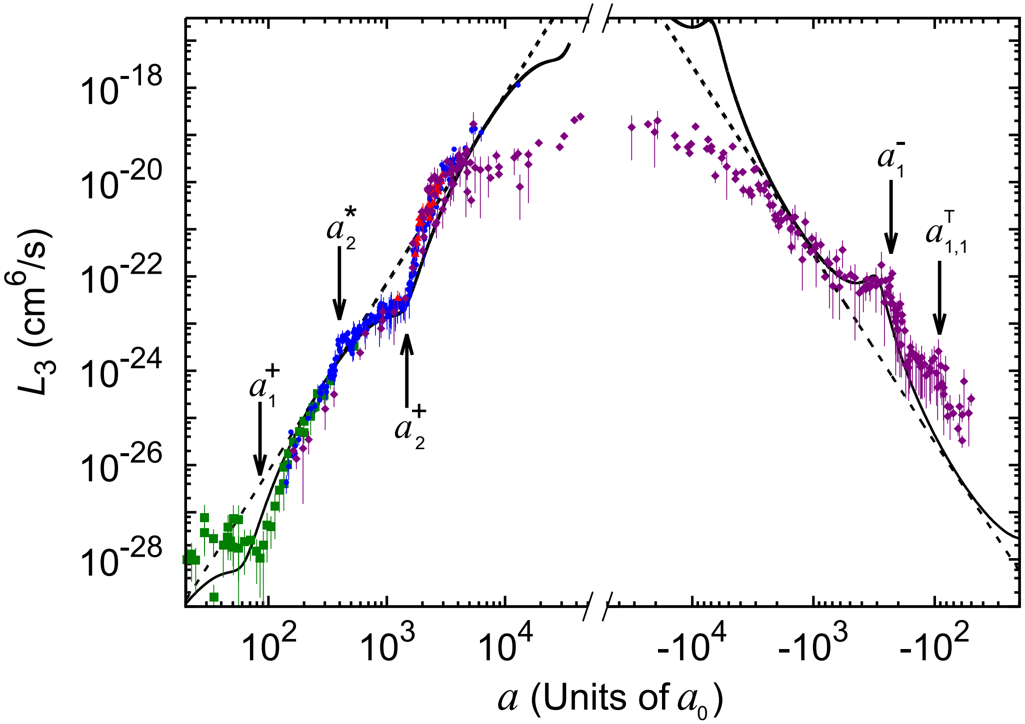
- First experimental evidence in ^{133}Cs (Kraemer et al. (Innsbruck), 2006)
  now also ^{6}Li, ^{7}Li, ^{39}K, ^{41}K/^{87}Rb, ^{6}Li/^{133}Cs
- Example: Efimov spectrum in ^{7}Li



Pollack et al. (Rice), Science **326** (2009) 1683; Phys. Rev. A **88** (2013) 023625

- vdW tail determines resonance position: $a_-/l_{vdW} \approx -10\ (\pm 15\%)$
  but not width (Wang et al., 2012; Naidon et al. 2012, 2014; ...)

Inelastic Processes

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- Loss coefficients used in few-body rate equations
- Complete many-body description requires density matrix
- Effective density matrix from tracing over high-energy states?
- Naive evolution equation for $H_{eff} = H - iK$

$$i\hbar\, \partial_t \rho = H_{\text{eff}}\,\rho - \rho H_{\text{eff}}^{\dagger} = [H, \rho] - i\{K, \rho\}$$

- Implies $\partial_t \mathrm{Tr}\,(\rho) = -\mathrm{Tr}(2K\rho)/\hbar$

$\Longrightarrow$ probability not conserved

- Need evolution equation for open system coupled to particle reservoir

$\Longrightarrow$ Lindblad equation (Lindblad; Gorini, Kossakowski, Sudarshan, 1976)

Inelastic Processes: Open EFT

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- Consider general Hamiltonian with a loss term ("optical potential")

$$H_{\text{eff}} = H - iK, \qquad K = \sum_i \gamma_i \int d^3r\, \Phi_i^\dagger \Phi_i$$

- Derive effective density matrix for low-energy particles by tracing over high-energy particles
- Evolution equation for effective density matrix $\Rightarrow$ Lindblad equation

$$i\hbar\partial_t\rho = [H,\rho] - i\sum_i \gamma_i \int d^3r \left(\Phi_i^\dagger\Phi_i\rho + \rho\Phi_i^\dagger\Phi_i - 2\Phi_i\rho\Phi_i^\dagger\right)$$

(Braaten, HWH, Lepage, 2017)

$\Longrightarrow$ Concept of Open EFT (Burgess et al., 2015)

Inelastic 2-Body Losses

- Application to inelastic 2-body losses
- Fermionic atoms with a loss channel $\Rightarrow$ a complex

$$K = (4\pi\hbar^2/m)\,\mathrm{Im}\big(1/a\big)\int d^3r\,\Phi^\dagger\Phi, \qquad \Phi = 4\pi a\,\psi_2\psi_1$$

- Particle losses: $\langle N\rangle = \mathrm{Tr}(\rho N)$

$$\frac{d}{dt}\langle N_1\rangle = \frac{d}{dt}\langle N_2\rangle = -\frac{\hbar}{2\pi m}\mathrm{Im}\big(1/a\big)\int d^3r\,\left\langle\Phi^\dagger\Phi\right\rangle$$

where $\mathcal{C} = \left\langle\Phi^\dagger\Phi\right\rangle$ contact operator evaluated in state $\langle\ldots\rangle$

$\Rightarrow$ “contact density”

$C = \int d^3r\,\mathcal{C}$ $\Rightarrow$ “contact”

2-Atom Losses and the Contact

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- Universal relations involving the contact: $C = \int d^3r\, \mathcal{C}(\boldsymbol{r})$
  measures number of pairs at short distances (Tan, 2005-2008)

  e.g. adiabatic relation $$\frac{d}{da^{-1}} E = -\frac{\hbar^2}{4\pi m}\, C$$

  also RF spectroscopy, photoassociation, ...
- Here: inelastic loss rate for mixture of atom species $\sigma = 1, 2$
- Inelastic short-distance processes parameterized by complex scattering length (Tan, 2008; Braaten, Platter, 2008)

$$\frac{d}{dt} N_\sigma = -\frac{\hbar}{2\pi m}\, \mathrm{Im}(1/a)\, C, \qquad \sigma = 1, 2$$

- Short-range correlations in nuclear physics
  $\Longrightarrow$ lecture by Nir Barnea (week 2)

Inelastic 3-Atom Losses

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- Inelastic three-atom loss rate

$$\frac{d}{dt}\langle N\rangle = -\frac{6\hbar}{m s_0}\sinh(2\eta_*)C_3$$

(linear term in η_*: Werner, Castin, 2012; Smith, Braaten, Kang, Platter, 2014)

- Three-body contact: $C_3 = f(\Lambda)\int d^3r\,\langle(\psi^3)^\dagger\psi^3\rangle$

where $f(\Lambda)$ is scheme-dependent

- Equivalent definition (Braaten, Kang, Platter, 2011)

$$\text{with} \qquad \Lambda_* \frac{\partial\langle H\rangle}{\partial\Lambda_*}\bigg|_a = -\frac{2\hbar^2}{m}C_3$$

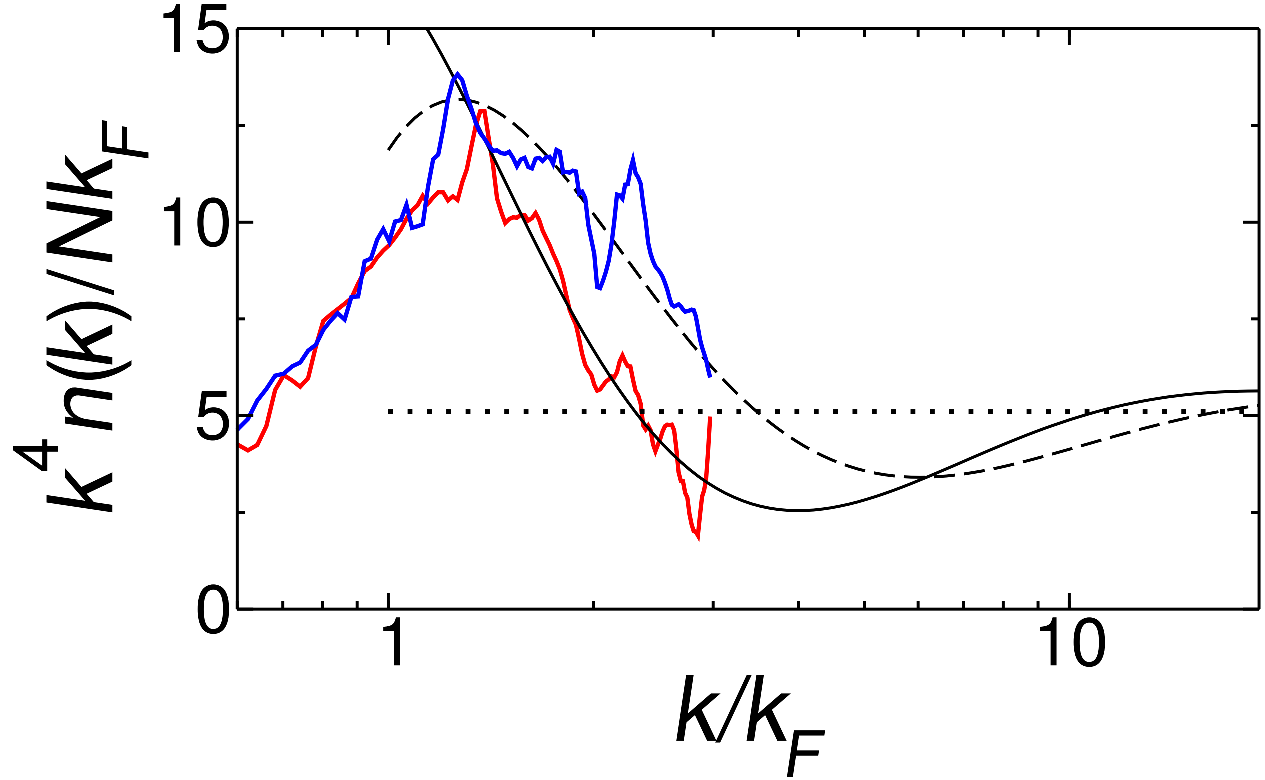
- Tail of momentum distribution (Braaten, Kang, Platter, 2011)

$$k^4 n(k) \longrightarrow C_2 + A\sin[2s_0\ln(k/\kappa*) + \phi]C_3/k$$

Three-body contact

- Consistent with experiment ($\langle n_1 \rangle < \langle n_2 \rangle$)

$$k^4 n(k) \longrightarrow C_2 + A \sin[2s_0 \ln(k/\kappa *) + \phi] C_3/k$$



Exp.: Makotyn, Klauss, Goldberger, Cornell, Jin, Nature Phys. **88**, 116 (2014)

Theo.: Braaten, Kang, Platter, Phys. Rev. Lett **112**, 110402 (2014)

Summary

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- Effective (field) theory

  $\Rightarrow$ systematic expansion to exploit separation of scales
- Universality

  $\Rightarrow$ long-distance observables independent of details of short-range interaction
- Physics of the unitary limit
  - scale invariance in 2-body system

    $\Rightarrow$ anomalously broken to discrete scale invariance for ≥ 3 particles (in certain cases)
  - universal correlations between few-body observables
  - Efimov physics $\Leftrightarrow$ RG limit cycle
  - short-range correlations encoded in Tan contact
  - unitary limit is universal: particles, nuclei, cold atoms,...

Additional Transparencies