A4 **Naturalness for square well potential**

$$a/r_0 = \left(1 - \frac{\tan x}{x}\right) \Rightarrow \frac{\tan x}{x} = 1 - a/r_0$$

$$P(a)\,da = \left|\frac{dx}{da}\right| da = \frac{1}{da/dx}\,da \qquad (\tan x)' = 1 + \tan^2 x$$

$$\frac{da}{dx} = r_0 \frac{d}{dx}\left(1 - \frac{\tan x}{x}\right) = r_0 \left(\underbrace{\cancel{\frac{\tan x}{x^2}}}_{\text{neglect}} - \frac{1+\tan^2 x}{x} \right)$$

$$= r_0 \left(-\frac{1}{x} - x\left(1 - a/r_0\right)^2 \right)$$

$$= -r_0 x \left(+\frac{1}{x^2} + \left(1 - a/r_0\right)^2 \right)$$

$x = k_0 r_0$

$$P(a)\,da = \frac{1}{r_0^2 k_0} \frac{N}{\frac{1}{k_0^2 r_0^2} + \left(1 - a/r_0\right)^2} = \frac{1}{k_0} \frac{N}{1/k_0^2 + (a - r_0)^2}\,da$$

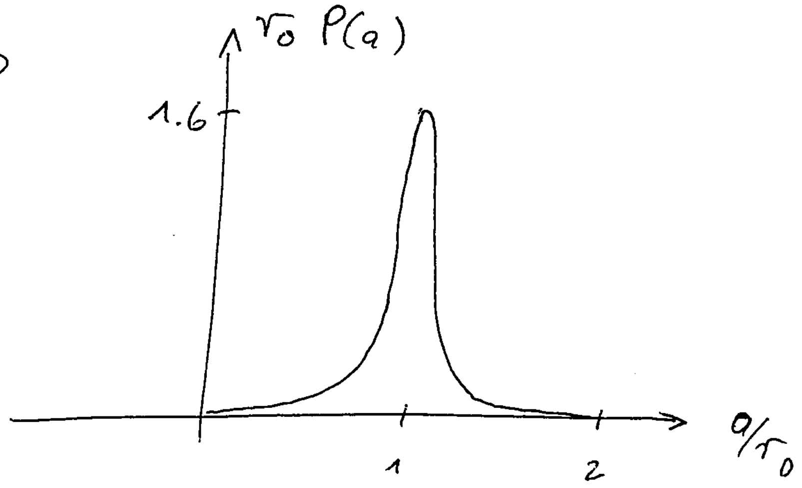
<u>Normalization:</u>

$$\int_{-\infty}^{\infty} P(a)\,da = 1 \Rightarrow N = 1/\pi$$

$$\Rightarrow \boxed{P(a)\,da = \frac{1}{1/k_0^2 + (a - r_0)^2}\,\frac{da}{\pi k_0}}$$

<u>Example:</u> $k_0 r_0 = 10$

$\Leftrightarrow$ most likely to have $a \approx r_0$ if k_0 uniformly distributed



A5 EFT for natural scattering length

(a)
$$\mathcal{L} = \psi^\dagger\left(i\partial_t + \frac{\nabla^2}{2m}\right)\psi - \frac{g_2}{4}(\psi^\dagger\psi)^2 - \frac{g_{2,2}}{4}\underbrace{\left(\vec\nabla(\psi^\dagger\psi)\right)}_{(\vec\nabla\psi^\dagger)\psi + \psi^\dagger(\vec\nabla\psi)}{}^2$$

Note:
$$\psi^\dagger\vec\nabla^2\psi = \psi^\dagger\vec\nabla(\vec\nabla\psi) = \cancel{\psi^\dagger(\vec\nabla\psi)}\Big|_{\text{surface}} - (\nabla\psi^\dagger)(\nabla\psi)$$

$$\Rightarrow \quad \mathcal{L} = \psi^\dagger(i\partial_t)\psi - \frac{(\vec\nabla\psi^\dagger)(\vec\nabla\psi)}{2m} - \frac{g_2}{4}(\psi^\dagger\psi)^2 - \frac{g_{2,2}}{4}\underbrace{\left(\nabla(\psi^\dagger\psi)\right)}_{(\nabla\psi^\dagger)\psi + \psi^\dagger(\nabla\psi)}{}^2$$

$$\frac{\delta\mathcal{L}}{\delta(\partial_t\psi)} = i\psi^\dagger, \qquad \frac{\delta\mathcal{L}}{\delta(\vec\nabla\psi)} = -\frac{(\vec\nabla\psi^\dagger)}{2m} - \frac{g_{2,2}}{4}\cdot 2\left((\nabla\psi^\dagger)\psi + \psi^\dagger(\nabla\psi)\right)\psi^\dagger$$

$$\frac{\delta\mathcal{L}}{\delta\psi} = -\frac{g_2}{4}\,2\,(\psi^\dagger\psi)\psi^\dagger - \frac{g_{2,2}}{4}\,2\,(\vec\nabla\psi^\dagger\psi)\vec\nabla\psi^\dagger$$

$$\partial_t\frac{\delta\mathcal{L}}{\delta(\partial_t\psi)} + \nabla\frac{\delta\mathcal{L}}{\delta(\nabla\psi)} - \frac{\delta\mathcal{L}}{\delta\psi} = 0 \quad \Longrightarrow$$

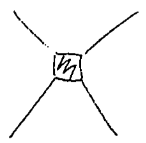
$$0 = i\partial_t\psi^\dagger + \nabla\left[-\frac{\vec\nabla\psi^\dagger}{2m} - \frac{g_{2,2}}{2}\psi^\dagger\vec\nabla(\psi^\dagger\psi)\right] - \left[-\frac{g_2}{2}\psi^\dagger(\psi^\dagger\psi)\;\overbrace{}^{-\frac{g_{2,2}}{2}(\vec\nabla\psi^\dagger)\vec\nabla(\psi^\dagger\psi)}\right]$$

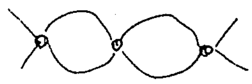
$$\Rightarrow \quad \boxed{-i\partial_t\psi^\dagger = -\frac{\nabla^2\psi^\dagger}{2m} + \frac{g_2}{2}\psi^\dagger(\psi^\dagger\psi) - \frac{g_{2,2}}{2}\psi^\dagger\left(\vec\nabla^2(\psi^\dagger\psi)\right)}$$

$$\Rightarrow \quad \boxed{i\partial_t\psi = -\frac{\nabla^2\psi}{2m} + \frac{g_2}{2}(\psi^\dagger\psi)\psi - \frac{g_{2,2}}{2}\left(\vec\nabla^2(\psi^\dagger\psi)\right)\psi}$$

(6) $\nu = 2$:

conventions as in lecture!

 $= -i\, g_{2,2} \cdot 2k^2 \quad , \quad |\vec{k}| = |\vec{k}'|$

 $= -i g_2 \left(- \frac{m\, g_2}{4\pi^2} \left(\Lambda + \frac{\pi}{2} i k \right) \right)^2$

$$\Longrightarrow \quad T = -g_2 + \frac{m (g_2)^2}{4\pi^2} \left(\Lambda + \frac{\pi}{2} i k \right) - g_{2,2} \cdot 2k^2 - g_2 \cdot \frac{m^2 (g_2)^2}{16\pi^4} \left(\Lambda + \frac{\pi}{2} i k \right)^2$$

Define: $$\frac{8\pi a}{m} = g_2 \left(1 - \frac{g_2 m \Lambda}{4\pi^2} + \frac{g_2^2 m^2 \Lambda^2}{16\pi^4} + \dots \right)$$

$$\Longrightarrow \quad g_2 = \frac{8\pi a}{m} \left(1 + \frac{2a\Lambda}{\pi} + \frac{4a^2\Lambda^2}{\pi^2} + \dots \right)$$

$$\Longrightarrow \quad T = + \frac{8\pi a}{m} \Bigg[\left(1 + \frac{2a\Lambda}{\pi} + \frac{4a^2\Lambda^2}{\pi^2} + \dots \right)(-1) + \frac{m}{4\pi^2} \left(\frac{8\pi a}{m} \right) \left(1 + \frac{4a\Lambda}{\pi} \dots \right) \left(\Lambda + \frac{\pi}{2} i k \right) - \frac{m^2}{16\pi^4} \left(\frac{8\pi a}{m} \right)^2 \left(\Lambda^2 + \pi i k \Lambda - k^2 \frac{\pi^2}{4} \right) \Bigg] - g_{2,2} \cdot 2k^2$$

$$= + \frac{8\pi a}{m} \left[-1 + i k a + k^2 a^2 \right] - g_{2,2} \cdot 2k^2$$

$$\overset{!}{=} - \frac{8\pi a}{m} \left[1 - i k a - a^2 k^2 + \frac{a r_e}{2} k^2 + \dots \right]$$

$$\Longrightarrow \quad \boxed{g_{2,2} = \frac{2\pi}{m} a^2 r_e}$$

A6 Power counting

~~A4~~

perturbative N-body counting

(i) $L = I - V + 1$

(ii) $\sum_n 2n\, V^n = E + 2I$ — # of n-body vertices

$V^n = \sum_i V^n_{2i}$ ← # of n-body vertices w/ $2i$ derivatives

$\int \frac{d^4k}{(2\pi)^4} \sim k^5$

$V = \sum_n V^n$ ← # vertices

propagator $\sim 1/k^2$

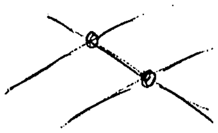
$g_{n,2i}\, p^{2i} \sim k^{2i}$ $\quad \boxed{\text{Diagram} \sim k^\nu}$

$\Longrightarrow \quad \nu = 5L - 2I + \sum_n \sum_i 2i\, V^n_{2i}$

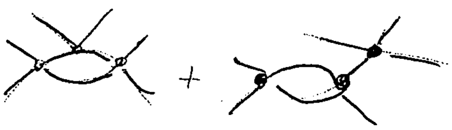
use $\quad I = L + V - 1 = L - 1 + \sum_n \sum_i V^n_{2i}$

$\Longrightarrow \quad \boxed{\nu = 3L + 2 + \sum_n \sum_i (2i - 2)\, V^n_{2i}} \geq \underbrace{2 - 2(N-1)}_{2(2-N)}$

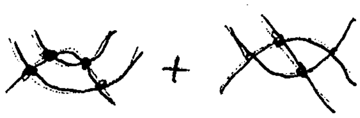
3-body



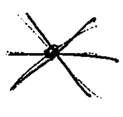
$\nu = 2 - 2 \cdot 2 = -2$



$\nu = 3 + 2 - 2 \cdot 3 = -1$



$\nu = 3 \cdot 2 + 2 - 2 \cdot 4 = 0$



$\nu = 2 - 2 \cdot 1 = 0$

ν_{min}	N
2	1
0	2
−2	3
−4	4
−6	5