2.3 Two-body system with large scattering length

- $|a| \gg \ell \sim r_e \sim \ldots$ [requires fine tuning]

$\Longrightarrow$ cannot expand in (ka), but expansion in $(k\ell)$, (kr_e), ... still works well

- Physics:

$$T(k) = \frac{8\pi}{m} \frac{1}{-1/a + \frac{r_e}{2}k^2 + \ldots - ik}$$

$$= -\frac{8\pi a}{m} \frac{1}{1+ika} \left[1 - \frac{r_e a k^2/2}{1+ika} \right]^{-1}$$

$$= -\frac{8\pi a}{m} \left[\frac{1}{1+ika} + \frac{r_e a k^2/2}{(1+ika)^2} + \ldots \right]$$

- count $k \sim 1/a \equiv Q$ $\longrightarrow$ expansion in $(Q\ell)$

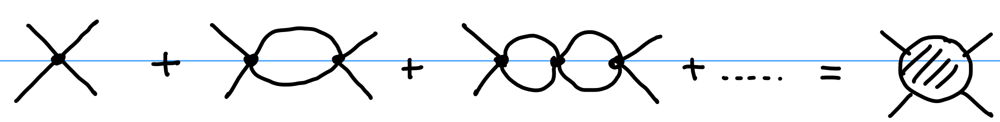
Conjecture: $g_2 \sim \frac{1}{mQ}$, $g_{2,2} \sim \frac{\ell}{mQ^2}$, ..., $g_{2,2i} \sim \frac{\ell^i}{mQ^{i+1}}$

$\hookrightarrow$ Q^ν where $\nu = 5L - 2I + \sum_i \underbrace{(2i - (i+1))}_{i-1}$

$\hookrightarrow$ $\boxed{\nu = 3L + 2 + \sum_i (i-3) V_{2i} \geq -1}$

- What does this mean for Feynman diagrams?

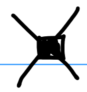
g_2:

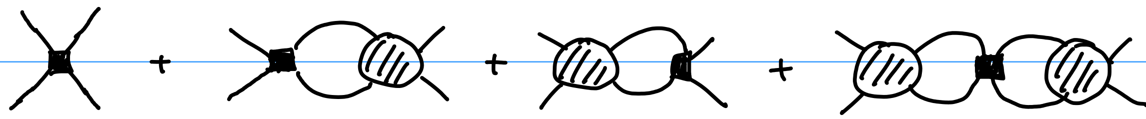


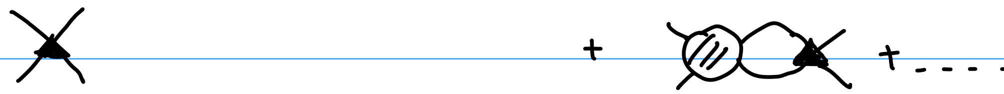
$\sim Q^{-1}$

$\nu = -1$

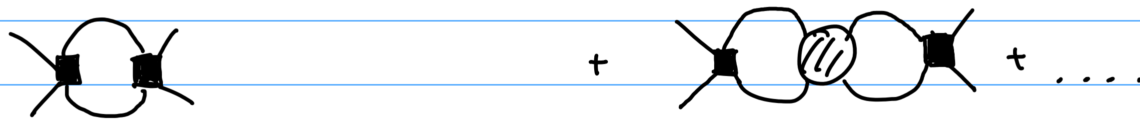
For each loop $\nu = \ldots \underbrace{+3}_{\Delta L} \ldots \underbrace{-(0-3)\cdot 1}_{\Delta V_{20}} = \nu$

$g_{2,2}$:  $\nu = 0 + 2 + (1-3)\cdot 1 = 0$

 $\nu = 0$

$g_{2,4}$: 

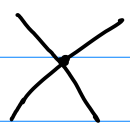
$\nu = 0 + 2 + (2-3)\cdot 1 = 1$

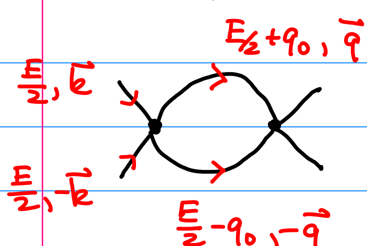


$\nu = 3 + 2 + (1-3)\cdot 2 = 1$

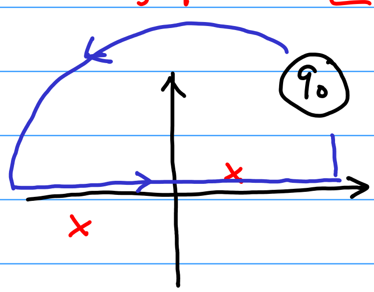
....

- Now calculate the LO ($\nu = -1$) contribution:

 $= -i g_2$



$$= \underbrace{\tfrac{1}{2}}_{\text{symmetry factor}} (-\cancel{i} g_2)^2 \underbrace{\int \frac{d^4 q}{(2\pi)^4}}_{\int dq_0 \int d^3\vec{q}} \frac{\cancel{i}}{\frac{E}{2} + q_0 - \frac{\vec{q}^{\,2}}{2m} + i\epsilon} \; \frac{\cancel{i}}{\frac{E}{2} - q_0 - \frac{\vec{q}^{\,2}}{2m} + i\epsilon}$$



$$= \frac{g_2^2}{2} \frac{\cancel{2\pi} i}{\cancel{2\pi}} (-1) \int^{\Lambda} \frac{d^3 q}{(2\pi)^3} \frac{m}{mE - \vec{q}^{\,2} + i\epsilon}$$

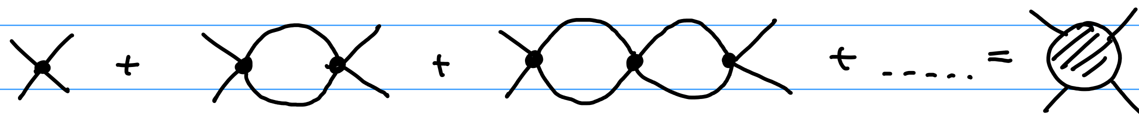
Regulator

$$\int d^3 q = \underbrace{\int d\Omega_q}_{4\pi} \int q^2 dq$$

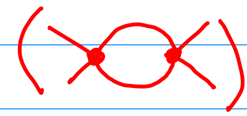
$$= \frac{i m g_2^2}{2} \frac{1}{2\pi^2} \int_0^{\Lambda} dq \, \frac{q^2}{q^2 - mE - i\epsilon}$$

$$\int_0^{\Lambda} dq\, \frac{q^2}{q^2 - mE - i\epsilon} = \Lambda - \frac{\pi}{2}\sqrt{-mE - i\epsilon} + \cancel{\mathcal{O}(1/\Lambda)}$$

neglect, can be made arbitrarily small

- Need: 

"Bubble sum" $\hat{=}$ geometric series

$\propto$  ^{2} iT

$$iT = -ig_2 \left(1 - \frac{mg_2}{4\pi^2}\left(\Lambda - \frac{\pi}{2}\sqrt{-mE-i\epsilon}\right) + \left[\frac{mg_2}{4\pi^2}\left(\Lambda - \frac{\pi}{2}\sqrt{-mE-i\epsilon}\right)\right]^2 + \dots\right)$$

$$= -i\left(1/g_2 + \frac{m}{4\pi^2}\left(\Lambda - \pi/2\sqrt{-mE-i\epsilon}\right)\right)^{-1}, \quad E = k^2/m$$

$$\overset{!}{=} i\,\frac{8\pi}{m}\left[-1/a - ik\right]^{-1}$$

matching / renormalization condition

$$\Rightarrow \quad \| \; g_2(\Lambda) = \frac{8\pi a}{m}\left[1 - \frac{2a\Lambda}{\pi}\right]^{-1} \; \|$$

Running coupling constant

- Expansion for $a \sim \ell$ reproduces natural case
- Change of $\Lambda \Rightarrow$ renormalization group transformation
- Dimensionless coupling: $\tilde{g}_2(\Lambda) = \frac{\Lambda m\, g_2(\Lambda)}{4\pi^2}$

↪ term in $\mathcal{L}$: $g_2(\Lambda)\,(\psi^\dagger\psi)^2$

↪ separate scaling of operators and couplings

- RG-equation: $\Lambda \frac{d}{d\Lambda} \tilde{g}_2(\Lambda) = \dots = \tilde{g}_2 (1 + \tilde{g}_2)$

$\hookrightarrow$ 2 Fixed points [theory is scale invariant at fixed points]

(i) $\tilde{g}_2 = 0 \rightarrow$ free fixed point, corresponds to $a \cong 0$ "natural case"

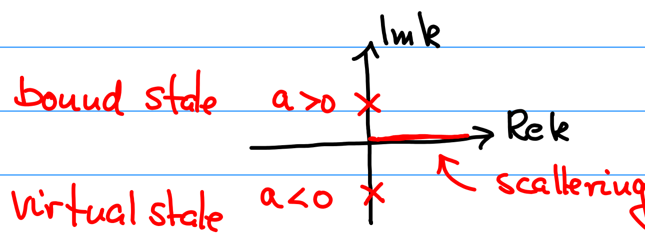
(ii) $\tilde{g}_2 = -1 \rightarrow$ interacting fixed point, corresponds to $1/a \cong 0$ "unnatural case"

$\hookrightarrow$ unitary limit, since $T(k) \sim 1/k$

- Bound/virtual states : poles in T-matrix

$$T(k) = \frac{8\pi}{m} \frac{1}{-1/a - ik} \Rightarrow k = i/a$$

$$E = \frac{k^2}{m} = -\frac{1}{ma^2}$$



$$T(E) = \frac{8\pi}{m} \frac{1}{-1/a + \sqrt{-mE - i\epsilon}} \quad , \quad E = k^2/m$$