2. Nonrelativistic Effective Field Theories

2.1 N-body Quantum Mechanics & Quantum Field Theory

- consider N-particle system w/ 2-body interaction

$$\hat{H}^{(N)} = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j} V(r_{ij}) \ , \quad r_{ij} = |\vec{r}_i - \vec{r}_j|$$

where $V(r_{ij})$ has "natural low-energy length scale" ℓ

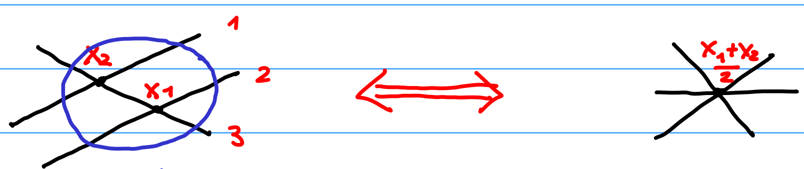
(V finite range $\longrightarrow$ $\ell \triangleq$ range of potential)

- describe physics w/ $|E| \ll \frac{1}{m\ell^2}$ $\Longrightarrow$ use effective potential

$$\hat{H}^{(N)}_{\text{eff}} = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j} V^{(2)}_{\text{eff}}(r_{ij}) + \sum_{i<j<k} V^{(3)}_{\text{eff}}(r_{ij}, r_{jk}) + \dots$$

- $V^{(N)}_{\text{eff}}$ has free parameters to be tuned to reproduce low-energy observables of $\hat{H}^{(N)}$

- V_{eff} may have many-body forces even if V has not



resolution scale $\lambda \sim 1/p$

- which terms are required?

  - only k-body forces with $k \leq N$
  - power counting

- equivalent formulation: Quantum Field Theory through "second quantization"

$\hookrightarrow$ instead of operators $\vec{p}$ & $\vec{r}$, the theory is formulated in terms of quantum field operators $\hat{\psi}(\vec{r})$ [$\hat{\psi}^+(\vec{r})$] that annihilate [create] particles at $\vec{r}$. For bosons, we have

$$[\hat{\psi}(\vec{r},t), \hat{\psi}(\vec{r}\,',t)] = 0$$

$$[\hat{\psi}(\vec{r},t), \hat{\psi}^+(\vec{r}\,',t)] = \delta^3(\vec{r}-\vec{r}\,')$$

where $\hat{\psi}(\vec{r},t)$ can be written in Fourier decomposition

$$\hat{\psi}(\vec{r},t) = \sum_k \psi_k(\vec{r})\,\hat{c}_k\, e^{-i\omega_k t} \quad , \quad \psi_k(\vec{r}) = \frac{1}{\sqrt{V}}\, e^{i\vec{k}\cdot\vec{r}}$$

$$\hat{\psi}^+(\vec{r},t) = \sum_k \psi_k^*(\vec{r})\,\hat{c}_k^+\, e^{i\omega_k t}$$

free particles quantized in box

with the operators $\hat{c}_k, \hat{c}_k^+$

$$[\hat{c}_{\vec{k}}, \hat{c}_{\vec{k}'}] = 0 \qquad\qquad \hat{c}_{\vec{k}}|n_{\vec{k}}\rangle = (n_{\vec{k}})^{1/2}|n_{\vec{k}}-1\rangle$$

$$[\hat{c}_{\vec{k}}, \hat{c}_{\vec{k}'}^+] = \delta_{\vec{k},\vec{k}'} \quad \Longrightarrow \quad \hat{c}_{\vec{k}}^+|n_{\vec{k}}\rangle = (n_{\vec{k}}+1)^{1/2}|n_{\vec{k}}+1\rangle$$

cf. harmonic oscillator

- Hamiltonian becomes

$$\hat{H} = \int d^3\vec{r}\; \hat{\psi}^+(\vec{r})\left(\frac{-\vec{\nabla}^2}{2m}\right)\hat{\psi}(\vec{r}) + \frac{1}{2}\int d^3r \int d^3r'\; \underline{\hat{\psi}^+\hat{\psi}(\vec{r})}\;\underbrace{V(\vec{r}-\vec{r}\,')}_{\text{non-local}}\;\underline{\hat{\psi}^+\hat{\psi}(\vec{r}\,')}$$

$V_{eff}(\vec{r}-\vec{r}\,')$

- use a "local" effective field theory

$$\hat{H}_{eff} = \int d^3r \, \hat{\mathcal{H}}_{eff}$$ Hamiltonian density

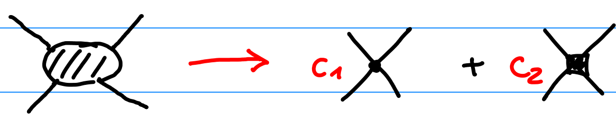
$$\mathcal{H}_{eff} = \psi^\dagger \left(-\frac{\vec{\nabla}^2}{2m}\right)\psi + \frac{g_2}{4}(\psi^\dagger\psi)^2 + \frac{g_{2,2}}{4}\vec{\nabla}(\psi^\dagger\psi)\cdot\vec{\nabla}(\psi^\dagger\psi) + \dots$$

$$+ \frac{g_3}{6}(\psi^\dagger\psi)^3 + \dots$$

$$+ \frac{g_4}{4!}(\psi^\dagger\psi)^4 + \dots$$

all ψ's are evaluated at same point $\vec{r}$

- $V(\vec{r}-\vec{r}\,') \longrightarrow c_1\,\delta(\vec{r}-\vec{r}\,') + c_2\,"\vec{\nabla}^2"\delta(\vec{r}-\vec{r}\,') + \dots$



$\mathcal{H}_{eff}$ in accordance with symmetries of $\mathcal{H}$

- Practical calculations: Lagrangian formalism

$$\mathcal{L}_{eff} = \pi\dot{\psi} - \mathcal{H}_{eff}$$ Legendre transformation

$$\hookrightarrow \quad \mathcal{L}_{eff} = \psi^\dagger\left(+i\partial_t + \frac{\vec{\nabla}^2}{2m}\right)\psi - \frac{g_4}{4}(\psi^\dagger\psi) + \dots$$

Euler-Lagrange equations $\longrightarrow$ equations of motion

2.2 Two-body system with natural scattering length

- simplify: bosons, S-wave interactions
- generic two-boson system with short-range interactions

$\longrightarrow$ only one length scale ℓ

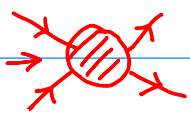
→ $a, r_e, \ldots \sim \ell$

- **Aside:** effective range parameters

  For a short-range potential: $k\cot\delta$ is analytic in k^2

  $$\hookrightarrow \; k\cot\delta = -\frac{1}{a} + \frac{r_e}{2}k^2 + \frac{P}{4}k^4 + \ldots$$

  scattering length — effective range — shape parameter

  $E = k^2/m$ 

  Physics:
  $$T(k) = \frac{8\pi}{m}\frac{1}{k\cot\delta(k) - ik} = \frac{8\pi}{m}\left(-\frac{1}{a} + \frac{r_e}{2}k^2 + \ldots - ik\right)^{-1}$$

  scattering amplitude

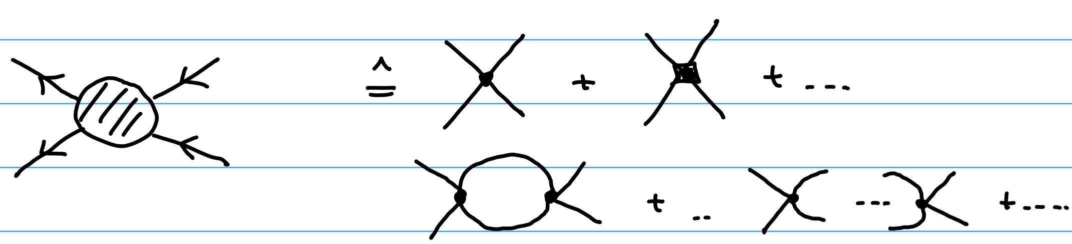
  $$\overset{k \ll 1/\ell}{=} -\frac{8\pi a}{m}\Big(1 - \underbrace{iak}_{k\ell} + \underbrace{\left(\frac{r_e a}{2} - a^2\right)k^2}_{(k\ell)^2} + \ldots\Big)$$

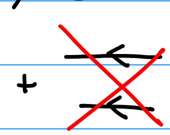
  $\hookrightarrow$ expansion in $k\ell$

- In practice it is convenient to work with Feynman diagrams → graphical representation

- Feynman rules → translate diagrams in mathematical expressions

- Typical diagrams:



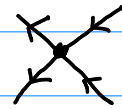


no contribution to T-matrix

- <u>Feynman rules</u>

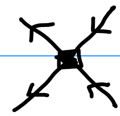
Propagator: $\xleftarrow{p_0, \vec{p}} \;\hat{=}\; \dfrac{i}{p_0 - \frac{\vec{p}^{\,2}}{2m} + i\epsilon}$ ← take $\epsilon \to 0$ in the end

Vertices:



$\hat{=}\; -ig_2$

$\frac{\vec{P}}{2} + \vec{k}'$ $\quad$ $\frac{\vec{P}}{2} + \vec{k}$



$\frac{\vec{P}}{2} - \vec{k}'$ $\quad$ $\frac{\vec{P}}{2} - \vec{k}$

$\hat{=}\; -ig_{2,2}\,(k^2 + k'^2)$

independent of $\vec{P}$ (Galilei invariance)

- <u>Size of coefficients:</u> dimensional analysis

$\hbar = c = 1 \;\longrightarrow\;$ two scales: m, e

$[m] = [E] = [p] = 1 \;\hat{=}\; (E^1)$

$[t] = [x] = -1$

Action $= \int d^4x\, \mathcal{L}_{eff} \;\longrightarrow\; [\mathcal{L}] = 4$

kinetic term: $\psi^\dagger \left(i\partial_t + \frac{\nabla^2}{m}\right)\psi \;\longrightarrow\; [\psi] = 3/2$

interactions: a) $g_2 (\psi^\dagger \psi)^2 \;\longrightarrow\; [g_2] + 6 = 4 \;\Rightarrow\; g_2 = [-2]$

b) $g_{2,2} \left(\vec{\nabla}(\psi^\dagger \psi)\right)^2 \;\longrightarrow\; \dots \;\Rightarrow\; g_{2,2} = [-4]$

- <u>Galilei invariance:</u> all coefficients contain factor $1/m$

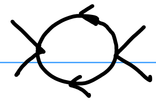
$\hookrightarrow$ $\quad g_2 \sim e/m$

$g_{2,2} \sim e^3/m$

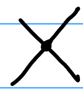
- **Power counting:** scaling of given diagram with k

in general: $L = I - V + 1$ "topological identity"

L: loops, I: internal lines, V: vertices



for our theory: $4V = 2I + E$ (E: external lines)



$V = \sum_i V_{2i}$ ← # vertices with $2i$ derivatives

now consider scaling of a given diagram with k:

$$\underbrace{\int \frac{d^4k}{(2\pi)^4}}_{\text{loops}} \sim \frac{k^5}{\cancel{m}}\,, \qquad \longrightarrow \;\sim \frac{\cancel{m}}{k^2}\,, \qquad g_{2,2i}\,k^{2i} \sim \frac{\cancel{\ell^{2i+1}}}{\cancel{m}}\,k^{2i}$$

need only consider powers of k! [ℓ, m will work out automatically]

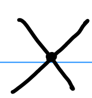
- For a given diagram $\mathcal{M} \sim k^{\nu}$ where

$$\nu = 5L - \underbrace{2I}_{\substack{\parallel \\ L + V - 1 \;=\; L + \sum_i V_{2i} - 1}} + \sum_i V_{2i}\cdot(2i)$$

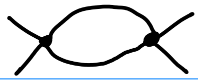
$$\Longrightarrow \quad \boxed{\nu = 3L + 2 + \sum_i (2i-2)\,V_{2i} \;\geq\; 0}$$

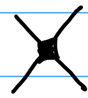
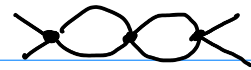
For 2-body interactions only

$\nu = 0$:



g_2

$\nu = 1$: 

$\nu = 2$:  $+$ 

$g_{2,2}$

$\vdots$

$\hookrightarrow$ Calculate diagrams, ... (cf. Sec. 8 of B&H)

<u>Next:</u> look at strong interactions: $|a| \gg \ell$