H.-W. Hammer TU Darmstadt

# Exercises

TALENT School "Effective Field Theories in Light Nuclei"
Introduction to Effective Field Theories

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This is a collection of exercises which accompany the lecture. The exercises vary in difficulty and length. You may use mathematica for analytical/numerical calculations where appropriate. Discussions are strongly encouraged.

## 1 Dimensional analysis I: Planck units

Construct the Planck length $l_P$, the Planck mass $m_P$, and the Planck time $t_P$ from the fundamental constants $G$, $\hbar$ and $c$. Estimate their size in SI units and compare to typical time scales from particle physics. What is the significance of these units?

## 2 Dimensional analysis II: hydrogen atom

Use dimensional analysis to estimate the size and energy of the ground state of the hydrogen atom. Which dependencies are determined by dimensional analysis and which are not?

## 3 Dimensional analysis III: photon-photon scattering

Consider photon-photon scattering in quantum electrodynamics. Estimate the amplitude for photon-photon scattering at energies well below the electron mass, $E_\gamma \ll m_e$. How can you obtain the cross section?

## 4 Naturalness for a square well potential

Consider the attractive radially symmetric square well potential

$$\begin{aligned} V(r) &= -V_0\,, \qquad r < r_0\,, \\ &= 0\,, \qquad r \geq r_0\,, \end{aligned}$$

where $V_0 > 0$. The $S$-wave scattering length for scattering of a particle of mass $m$ off this potential is

$$a = r_0 \left(1 - \frac{\tan(\kappa_0 r_0)}{\kappa_0 r_0}\right)\,, \tag{1}$$

where $\kappa_0 = \sqrt{mV_0}$. Assume a constant probability distribution $P(\kappa_0)$ for $\kappa_0$ ($r_0$ fixed) and derive the resulting probability distribution $P(a)$ for $a$. Use the relation

$$P(a)da = P(\kappa_0)d\kappa_0 \,.$$

For $\kappa_0 r_0 \gg \pi$ the term $\tan(\kappa_0 r_0)$ in Eq. (1) changes for small variations of $\kappa_0 r_0$ much more strongly than $1/\kappa_0 r_0$. The variation of $1/\kappa_0 r_0$ can thus be neglected. Sketch the result and give an interpretation. What the most probable value of $a$?

# 5 EFT for natural scattering length

Consider the non-relativistic effective field theory with the Lagrangian

$$\mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2m} \right) \psi - \frac{g_2}{4}(\psi^\dagger\psi)^2 - \frac{g_{2,2}}{4}\nabla(\psi^\dagger\psi) \cdot \nabla(\psi^\dagger\psi) \,.$$

(a) Derive the classical equation of motion for the field $\psi$ by using the Euler-Lagrange equations:

$$\partial_t \frac{\delta\mathcal{L}}{\delta(\partial_t\psi)} + \nabla\frac{\delta\mathcal{L}}{\delta(\nabla\psi)} - \frac{\delta\mathcal{L}}{\delta\psi} = 0 \,.$$

(b) Consider now 2-body scattering in this EFT. In the lecture, the scattering amplitude in leading order ($\nu = 0$) and next-to-leading order ($\nu = 1$) was discussed. Calculate the contributions at order $\nu = 2$. Determine $g_2(\Lambda)$ and $g_{2,2}(\Lambda)$ at this order.

# 6 Power Counting

Consider an effective field theory with natural interaction terms. In the lecture it was shown that a Feyman diagram with arbitrary 2-particle exchanges for small momenta $k$ scales like $k^\nu$. The following expression was derived for the exponent $\nu$:

$$\nu = 3L + 2 + \sum_i (2i-2)V_{2i} \,,$$

where $L$ denotes the number of loops and $V_{2i}$ the number of 2-body interactions with $2i$ derivatives. Generalize this expression to include arbitrary natural $N$-particle interactions. Give the diagrams for $3 \to 3$ particle scattering in the lowest three orders in $\nu$.

# 7 Dimer formalism I

Show that the Lagrangian with an explicit dimer field

$$\mathcal{L}' = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2m} \right) \psi + \frac{g_2}{4}d^\dagger d - \frac{g_2}{4}(d^\dagger\psi^2 + (\psi^\dagger)^2 d) - \frac{g_3}{36}d^\dagger d\psi^\dagger\psi + \dots \,, \qquad (2)$$

is equivalent to the usual Lagrangian without dimer field

$$\mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2m} \right) \psi - \frac{g_2}{4}(\psi^\dagger\psi)^2 - \frac{g_3}{36}(\psi^\dagger\psi)^3 + \ldots\,,$$

up to $N$-body interactions with $N \geq 4$. Use the classical equation of motion for the dimer field. Do you obtain the same observables in 3- and 4-body systems with the two Lagrangians?

# 8 Dimer formalism II

Consider the two-particle problem in the theory with an explicit dimer field using the Lagrangian in Eq. (2).

1. Which diagrams contribute to the full dimeron propagator? How is the 2-body scattering amplitude obtained from the full dimer propagator?

2. Calculate the full dimeron propagator and the 2-body scattering amplitude.

# 9 Size of the dimer

In the lecture we discussed the universal properties of the weakly bound dimer for $a \gg l$. The wave function for large relative distances $r$ between the two particles is

$$\psi_D(r) = \frac{\mathcal{N}}{r} e^{-r/a}\,, \qquad r \gg l\,.$$

How large is the dimer?

# 10 Asymptotic normalization constants

Consider a system with the scattering amplitude

$$\langle \boldsymbol{k}' | t(E) | \boldsymbol{k} \rangle = \frac{2\pi}{\mu} \left[ \frac{1}{a} - \frac{r}{2}k^2 + ik \right]^{-1} \qquad \text{with } E = k^2/(2\mu)\,, \tag{3}$$

where $a$ and $r$ are the scattering length and effective range of the particles while $\mu$ is their reduced mass. Near the bound state pole, the scattering amplitude can be expanded about the pole at $E = -\gamma^2/(2\mu)$:

$$\langle \boldsymbol{k}' | t(E) | \boldsymbol{k} \rangle = \frac{Z}{E + \gamma^2/(2\mu)} + \text{regular}\,. \tag{4}$$

In a bound two-body system, the residue $Z$ is connected to the *asymptotic normalization coefficient* (ANC) $A$ of the bound-state wave function. Near the bound state pole, the full Green's function has the general form

$$\langle \boldsymbol{k}' | \frac{1}{E - H} | \boldsymbol{k} \rangle = \frac{\psi(\vec{k}')\psi^*(\vec{k})}{E + \gamma^2/(2\mu)} + \text{regular}\,, \tag{5}$$

where $\psi(\vec{k})$ is the asymptotic wave function for the $S$-wave bound state, whose co-ordinate space representation is

$$\psi(\boldsymbol{r}) = \frac{A}{\sqrt{4\pi}} \frac{\exp(-\gamma r)}{r}\,, \tag{6}$$

and $H$ is the full Hamiltonian. Relate $A$ to $Z$ in order to determine the ANC.

# 11 Efimov effect

In a system of three identical bosons with large scattering length, $|a| \gg l$, the Efimov effect occurs. It leads to a geometric spectrum of 3-body bound states in the energy interval $1/ma^2 \ll B_3 \ll 1/ml^2$. For two neighbouring states, the relation

$$\frac{B_3^{(n+1)}}{B_3^{(n)}} \approx \exp(-2\pi/s_0)\,, \qquad s_0 \approx 1.00624\ldots\,.$$

Give an estimate for the number of Efimov states. What happens in the limit $|a| \to \infty$?