

# What is Renormalization?

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## 1 Introduction

As everyone knows the quantized theory of electrodynamics was created in the late 1920's and early 1930's. The theory was analyzed in perturbation theory, and was quite successful to leading order in the fine-structure constant  $\alpha$ . However all sorts of infinities started to appear in calculations beyond that order, and it was almost twenty years before the technique of renormalization was developed to deal with these infinities. What resulted is quantum electrodynamics (QED), one of the most accurate physical theories ever created: the  $g$ -factor of the electron is predicted (correctly) by the theory to at least 12 significant digits! At first sight, renormalization appears to be a rather dubious procedure for hiding embarrassing infinities, and the success of QED seems nothing less than miraculous. Nevertheless, persuaded by success, most physicists decided that renormalizability was an essential ingredient in any physically relevant field theory. Such thinking played a crucial role first in the development of a fundamental theory of weak interactions and then in the discovery of the underlying theory of strong interactions. In this lecture I argue that renormalizability is *not* an essential characteristic of useful field theories. Indeed it is possible, some would say likely, that none of known interactions is described completely by a renormalizable field theory. Modern developments in renormalization theory have given meaning to nonrenormalizable field theories, thereby generalizing and greatly clarifying our understanding of quantum field theories. As a result nonrenormalizable interactions are possible and seem likely in most theories constructed to deal with the real world.

In the first part of this lecture we will examine the technique of renormalization, first illustrating the conventional ideas and then extending these to deal with nonrenormalizable interactions. Central to this discussion is the notion of a cut-off field theory as a low-energy approximation to some more general (and possibly unknown) theory. Much of this material warrants a more detailed discussion than we have time for, and so a number of exercises have been included to suggest topics for further thought.

In the second part of the lecture we will examine the implications of our new perspective on renormalization for theories of electromagnetic, strong, and weak

interactions. Here we will address such issues as the origins and significance of renormalizability, the importance of naturalness in physical theories, and the experimental limits on nonrenormalizable interactions in electromagnetic and weak interactions. We will also show how to use renormalization ideas to create rigorous nonrelativistic field theories that greatly simplify the analysis of such nonrelativistic systems as positronium or the  $\Upsilon$  meson.

Most of the ideas presented in this lecture are well known to many people. However, since little of the modern attitude towards renormalization has made it into standard texts yet, it seems appropriate to devote a lecture to the subject at this Summer School. The discussion presented here is largely self-contained; the annotated Bibliography at the end lists a few references that lead into the large literature on this diverse subject.

## 2 Renormalization Theory

### 2.1 The Problem with Quantum Fields

The infinities in quantum electrodynamics, for example, originate in the fact that the electric field  $E(x, t)$  becomes a quantum-mechanical operator in the quantum theory. Thus measurements of  $E(x, t)$  in identically prepared systems tend to differ: the electric field has quantum fluctuations. By causality, adjacent measurements of the field, say at points  $x$  and  $x + a$ , are independent and thus fluctuate relative to one another. As a result the quantum electric field is rough at all length scales, becoming infinitely rough at vanishingly small length scales.

**Exercise:** Define  $\bar{E}(x, t)$  to be the electric field averaged over a spherical region of radius  $a$  centered on point  $x$ . This might be roughly the field measured by a probe of size  $a$ . Show that

$$\langle 0 | (\bar{E}(x + a, t) - \bar{E}(x, t))^2 | 0 \rangle \rightarrow \frac{1}{a^4} \quad (1)$$

as  $a \rightarrow 0$ —i.e., the fluctuations in the field from point to point diverge as the probe size goes to zero, even for the vacuum state! Physically, the fluctuations arise because it is impossible to probe the electric field at a point without creating photons.

This structure at all length scales is characteristic of quantum fields, and is quite different from the behavior of classical fields which typically become smooth at some scale. The roughness of the field is at the root of the problem with defining the quantum field theory. For example, how does one define derivatives of a field  $E(x)$  when the difference  $E(x + a) - E(x)$  diverges as the separation  $a$  vanishes? In perturbation theory the roughness at short distances results in divergent integrations over loop momenta, divergences associated with intermediate states carrying arbitrarily large momenta (and having arbitrarily short wavelengths). The infinities that result demonstrate rather dramatically that the short-distance structure of the quantum fields plays an important role in determining the long-distance (low-momentum) behavior of the theory; the short-distance structure cannot be ignored.

To give any meaning at all to a quantum field theory one must first regulate it, by in effect removing from the theory all states having energies much larger than some cutoff  $\Lambda$ . With a cutoff in place one is no longer plagued by infinities in calculations of the scattering amplitudes and other properties of the theory. For example, integrals over loop momenta in perturbation theory are cut off around  $\Lambda$  and thus are well defined. However the cutoff seems very artificial. The use of a cutoff apparently contradicts the notion, developed above, that the short-distance structure of the theory is important to the long-distance behavior; with the cutoff one is throwing away the short-distance structure. Furthermore  $\Lambda$  is a new and artificial parameter in the theory. Thus it is traditional to remove the cutoff by taking  $\Lambda$  to infinity at the end of any calculation. This last step is the source of much of the mystery in the renormalization procedure, and it now appears likely that this last step is also a wrong step in the nonperturbative analysis of many theories, including QED. Rather than follow this route we now will examine what it means to keep the cutoff finite.

## 2.2 Cut-off Field Theories

The basic idea behind renormalization is that all effects of the very high-energy states in the Hilbert space on the low-energy behavior of the theory can be simulated by a set of new local interactions. So we can discard the states with energy greater than some cutoff provided we modify the theory's Lagrangian to account for the effects that result from the discarded states. In this section we will see how this idea provides the basis for the conventional renormalization procedure, using QED as an example of a renormalizable field theory. This will lay the groundwork for our discussion, in the next section, of the origins and significance of nonrenormalizable interactions.

According to conventional renormalization theory QED is defined by a Lagrangian,

$$\mathcal{L}_0 = \bar{\psi}(i\partial \cdot \gamma - e_0 A \cdot \gamma - m_0)\psi - \frac{1}{2}(F^{\mu\nu})^2, \quad (2)$$

together with a regulator that truncates the theory's state space at some very large  $\Lambda_0$ .<sup>(a)</sup> The cut-off theory is correct up to errors of  $\mathcal{O}(1/\Lambda_0^2)$ . It is worth emphasizing that  $e_0$  and  $m_0$  are well-defined numbers so long as  $\Lambda_0$  is kept finite; in QED each can be specified to several digits (for any particular value of  $\Lambda_0$ ). Given these "bare" parameters one need know nothing else about renormalization in order to do calculations. One simply computes scattering amplitudes, cutting all loop momenta off at  $\Lambda_0$  and using the bare parameters in propagators and vertices. The renormalization takes care of itself automatically. To compute  $e_0$  and  $m_0$  for a particular  $\Lambda_0$  one chooses two convenient processes or quantities, computes them in terms of the bare parameters using  $\mathcal{L}_0$ , and

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<sup>(a)</sup>The simplest way to regulate perturbation theory is to simply cut off the integrals over loop momenta at  $\Lambda_0$ . Although such a regulator can (and has been) used, it complicates practical calculations because it is inconsistent with Lorentz invariance and gauge invariance. However the details of the regulator are largely irrelevant to our discussion, and so for simplicity we will speak of the regulator as though it is a simple cutoff. One of the more conventional regulators, such as Pauli-Villars or lattice regulators, is recommended for real calculations.

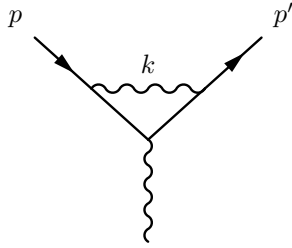


Figure 1: The one-loop vertex correction to the amplitude for an electron scattering off an external field.

adjusts the bare parameters until theory and experiment agree. Then all other predictions of the theory will be correct, up to errors of  $\mathcal{O}(1/\Lambda_0^2)$ .<sup>(b)</sup>

To understand the role of the cutoff in defining the theory, we start with QED as defined by  $\mathcal{L}_0$  and cutoff  $\Lambda_0$ , and we remove from this theory all states having energies or momenta larger than some new cutoff  $\Lambda$  ( $\ll \Lambda_0$ ). Then we examine how  $\mathcal{L}_0$  must be changed to compensate for this further truncation of the state space. Of course the new theory that results can only be useful for processes at energies much less than  $\Lambda$ , and so we restrict our attention to such processes. Furthermore we will analyze the effects of the new cutoff using perturbation theory, although our results are valid nonperturbatively as well.

We now want to discard all contributions to the theory coming from loop momenta greater than the new cutoff  $\Lambda$ . Consider first, the one-loop radiative corrections to the amplitude for an electron to scatter off an external electromagnetic field.

Working in the  $\mathcal{L}_0$  theory with the original cutoff, the part of the vertex correction (Fig. 1) that is being discarded is

$$T^{(a)}(k > \Lambda) = -e_0^3 \int_{\Lambda}^{\Lambda_0} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \times \\ \times \bar{u}(p') \gamma^\mu \frac{1}{(p' - k) \cdot \gamma - m_0} A_{\text{ext}}(p' - p) \cdot \gamma \frac{1}{(p - k) \cdot \gamma - m_0} \gamma_\mu u(p). \quad (3)$$

Since the masses and external momenta are assumed to be much less than  $\Lambda$ , we can neglect  $m_0$ ,  $p$  and  $p'$  in the integrand as a first approximation, thereby

<sup>(b)</sup>Newcomers to field theory sometimes find it hard to believe that this is all there is to conventional renormalization, given the length and complexity of the treatment generally accorded the subject in texts. What happened to counterterms, subtraction points, and normalization conditions? These are all related to the detailed implementation of the general concepts we are discussing. Such implementations tend to be highly optimized for particular sorts of calculations—e.g., for high-order calculations in perturbation theory, or for lattice simulations—and as such can be fairly complex. Such details are important in actual calculations. Here however our focus is on conceptual issues and so we can dispense with much of the detail. Anyone planning to do real calculations is advised to consult the standard texts.

greatly simplifying the integral:

$$\begin{aligned}
T^{(a)}(k > \Lambda) &\approx -e_0^3 \int_{\Lambda}^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \bar{u}(p') \gamma^\mu \frac{k \cdot \gamma}{k^2} A_{\text{ext}}(p' - p) \cdot \gamma \frac{k \cdot \gamma}{k^2} \gamma_\mu u(p) \\
&\approx -e_0^3 \bar{u}(p') A_{\text{ext}}(p' - p) \cdot \gamma u(p) \int_{\Lambda}^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}. \tag{4}
\end{aligned}$$

Applying a similar analysis to the other one-loop corrections, we find that the part of the electron's scattering amplitude that is omitted as a result of the new cutoff has the form

$$T(k > \Lambda) \approx -ie_0 c_0 (\Lambda/\Lambda_0) \bar{u}(p') A_{\text{ext}}(p' - p) \cdot \gamma u(p) \tag{5}$$

where  $c_0$  is dimensionless and thus can depend only upon the ratio  $\Lambda/\Lambda_0$ , these being the only scales left in the loop integration.

**Exercise:** Show that

$$c_0(\Lambda/\Lambda_0) = -\frac{\alpha_0}{6\pi} \log(\Lambda/\Lambda_0). \tag{6}$$

Clearly  $T(k > \Lambda)$  is an important contribution to the electron's scattering amplitude; it cannot be dropped. However such a contribution can be reincorporated into the theory by adding the following new interaction to  $\mathcal{L}_0$ :

$$\delta\mathcal{L}_0 = -e_0 c_0 (\Lambda/\Lambda_0) \bar{\psi} A \cdot \gamma \psi. \tag{7}$$

Thus we can modify the Lagrangian to compensate for the removal of the states above the cutoff, at least for the purpose of computing the electron's scattering amplitude.

It is important that the new interaction  $\delta\mathcal{L}_0$  is completely specified by a single number, the coupling constant  $c_0$ . In analyzing the vertex correction to electron scattering (Eq. (3)), we can neglect the external momenta relative to the internal momentum  $k$  with the result that the coupling  $c_0$  is independent of the external momenta. Thus the interaction is characterized by a number, rather than by some complicated function of the external momenta. Momentum independence, or more generally polynomial dependence on external momenta, indicates that these effective interactions are local in coordinate space; that is, they are polynomial in fields or derivatives of the fields all evaluated at the same point  $x$ . This important result actually follows from the uncertainty principle and is quite general: interactions involving intermediate states with momenta greater than the cutoff are local as far as the (low-energy) external particles are concerned. Since by assumption the external particles have momenta far smaller than  $\Lambda$ , intermediate states above the cutoff must be highly virtual. In quantum mechanics a state can be highly virtual provided it is short-lived, and so these highly-virtual intermediate states can exist for times and propagate over distances of only  $\mathcal{O}(1/\Lambda)$ . Such distances are tiny compared with the wavelengths of the external particles,  $\lambda \sim 1/p \gg 1/\Lambda$ , and thus the interactions are effectively local.

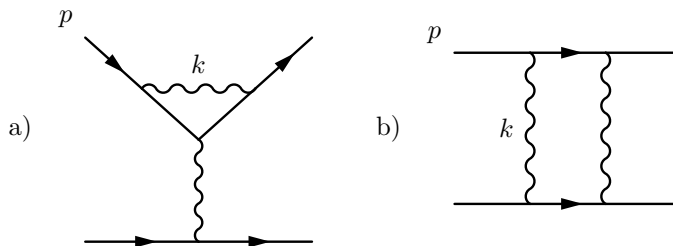


Figure 2: Examples of one-loop radiative corrections to electron-electron scattering.

Although electron scattering on external fields has been fixed, we must worry about the effect of the new cutoff on other processes. Consider for example the one-loop corrections to electron-electron scattering (Fig. 2). The  $k > \Lambda$  contributions due to vertex and self-energy corrections to the one-photon exchange process (e.g., Fig. 2a) are correctly simulated by  $\delta\mathcal{L}_0$ , just as they are in the case of an electron scattering on an external field. That leaves only the  $k > \Lambda$  contribution from the two-photon exchange diagrams (e.g., Fig. 2b). Again one can neglect the external momenta and masses in the internal propagators. The resulting amplitude must involve spinors for each of the external electrons, in combinations like  $\bar{u}\gamma_\mu u \bar{u}\gamma^\mu u$  or  $\bar{u}u \bar{u}u$ . These all have the dimension of  $[\text{energy}]^2$ , while in general a four-particle amplitude must be dimensionless. Thus the part of the amplitude involving loop momenta larger than  $\Lambda$  contributes something like

$$d(\Lambda/\Lambda_0) \frac{\bar{u}\gamma_\mu u \bar{u}\gamma^\mu u}{\Lambda^2}, \quad (8)$$

where  $d$  is dimensionless and where the factor in the denominator is  $\Lambda^2$ , since  $\Lambda$  is the only important scale left in the loop integral. Clearly such a contribution is suppressed by  $(p/\Lambda)^2$  and can be ignored (for the moment) since we are assuming  $p \ll \Lambda$ .

A similar analysis for, say, electron-electron scattering into four electrons and two positrons (e.g., Fig. 3) shows that intermediate states above the cutoff contribute something of order

$$\frac{(\bar{u}\gamma u)^2 (\bar{u}\gamma v)^2}{\Lambda^8}, \quad (9)$$

which is even less important. Evidently the more external particles involved in a loop, the larger the number of hard internal propagators, and the smaller the effect of the cutoff.

**Exercise:** Show that an amplitude with  $n$  external particles has dimension  $[\text{energy}]^{4-n}$  when relativistic normalization (i.e.,  $\langle p \dots | p' \dots \rangle \propto \sqrt{p^2 + m^2} \delta^3(p - p')$ ) is used for all states. Use this fact to show that the addition of an extra pair of external fermions to a  $k > \Lambda$  loop results in an extra factor of  $1/\Lambda^3$ , while adding an extra external photon leads to an extra factor of  $1/\Lambda$ . For some processes additional factors of external momenta or of the electron mass may also be required

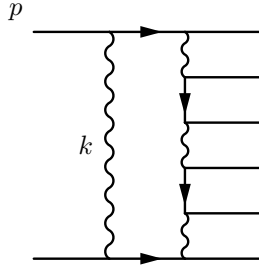


Figure 3: A one-loop radiative correction to electron-electron scattering i into four electrons and two positrons.

respectively by gauge invariance or chiral symmetry (i.e., electron-helicity conservation when  $m = 0$ ). Such factors result in additional factors of  $1/\Lambda$ . (Note that the relativistic normalization condition for Dirac spinors is  $\bar{u}u = 2m$ , and that photon polarization vectors are normalized by  $\varepsilon \cdot \varepsilon = 1$ .)

Using simple dimensional and power-counting arguments of this sort, one can show that the only scattering amplitudes that are strongly affected by the cutoff are those involving the electron-photon vertex, and for these the loss of the  $k > \Lambda$  states is correctly compensated by adding the single correction  $\delta\mathcal{L}_0$  to the Lagrangian. The only other physical quantity that is strongly affected by the cutoff is the mass of the electron: the physical mass of the electron is  $m_0$  plus a self-energy correction that involves the  $k > \Lambda$  states. The effect of these states on the mass is easily simulated by adding a term of the form

$$- m_0 \tilde{c}_0(\Lambda/\Lambda_0) \bar{\psi}\psi \quad (10)$$

to the Lagrangian.<sup>(c)</sup> Thus the theory with Lagrangian

$$\mathcal{L}_\Lambda = \bar{\psi}(i\partial \cdot \gamma - e_\Lambda A \cdot \gamma - m_\Lambda)\psi - \frac{1}{2}(F^{\mu\nu})^2, \quad (11)$$

cutoff  $\Lambda$ , and coupling parameters

$$e_\Lambda = e_0(1 + c_0(\Lambda/\Lambda_0)) \quad (12)$$

$$m_\Lambda = m_0(1 + \tilde{c}_0(\Lambda/\Lambda_0)) \quad (13)$$

gives the same results as the original theory with cutoff  $\Lambda_0$  (up to corrections of  $\mathcal{O}(1/\Lambda^2)$ ).

<sup>(c)</sup>It is not obvious at first glance that this mass correction is proportional to  $m_0$ . On strictly dimensional grounds one might expect a term proportional to  $\Lambda_0$ . However such a term is ruled out by the chiral symmetry of the massless theory. If  $m_0$  is set equal to zero, then the original theory is symmetric under chiral transformations of the form  $\psi \rightarrow \exp(i\omega\gamma_5)\psi$ . Changing the cutoff does not affect this symmetry and therefore any new interaction that violates chiral symmetry must vanish if  $m_0$  vanishes. Thus upon calculating the coefficient of the new  $\bar{\psi}\psi$  interaction one finds that it is proportional to  $m_0$  rather than  $\Lambda_0$ ; that is, the mass renormalization is logarithmically divergent rather than linearly divergent. Note by way of contrast that chiral symmetry is explicitly broken in Wilson's implementation of fermions on a lattice, and consequently the mass renormalization in such a theory is proportional to  $\Lambda_0$  rather than  $m_0$ .

With this result we see that a change in the cutoff can be compensated by changing the bare coupling and mass in the Lagrangian in such a way that the low-energy physics of the theory is unaffected. This is the classical result of renormalization theory.

**Exercise:** Sketch out arguments for the validity of this result to two-loop order by examining some simple process like electron-electron scattering. Divide each loop in a diagram into high-energy and low-energy parts, with  $\Lambda$  as the dividing line. This means that each two-loop diagram will be divided into four contributions depending upon loop energies: high-high, high-low, low-high, and low-low. The low-low contribution is still present with the new cutoff. The low-high and high-low contributions are removed by the cutoff, but these are either local in character, down by  $1/\Lambda$ , or are automatically simulated in the new theory by diagrams in which the high-energy loop is replaced by one of the new interaction introduced above to correct one-loop results (Eqs. (7) and (10)). The part of the low-high and high-low contributions that is local can be treated with the high-high contribution. The high-high contribution must be completely local and can be simulated by a local interaction in the Lagrangian. Again by power-counting, only the electron-photon vertex and the electron mass are appreciably changed by the cutoff, and thus these new local interactions are taken care of by introducing corrections of relative order  $e_0^4$  into  $e_\Lambda$  and  $m_\Lambda$ .

**Exercise:** Parameters  $e_\Lambda$  and  $m_\Lambda$  vary as the cutoff  $\Lambda$  is varied. The couplings are said to “run” as more or less of the state space is included in the cut-off theory. Show that these couplings satisfy “evolution equations” of the form:

$$\Lambda \frac{de_\Lambda}{d\Lambda} = \beta(e_\Lambda) \tag{14}$$

$$\Lambda \frac{dm_\Lambda}{d\Lambda} = m_\Lambda \gamma_m(e_\Lambda). \tag{15}$$

(In these equations we assume that  $m_\Lambda$  is negligible compared with  $\Lambda$ ; more generally  $\beta$  and  $\gamma_m$  depend also on the ratio  $m_\Lambda/\Lambda$ .) Compute  $\beta$  and  $\gamma_m$  to leading order in  $e_\Lambda$  for QED.

The bare parameters  $e_\Lambda$  and  $m_\Lambda$  can be thought of as the effective charge and mass of an electron at energy-momentum scales of  $\mathcal{O}(\Lambda)$ . This is particularly relevant in analyzing a process that is characterized by only a single scale, say  $Q$ . To compute the amplitude for such a process in a cut-off theory one must take  $\Lambda$  much larger than  $Q$ . However the main effect of vertex and self-energy corrections to the amplitude is to replace  $e_\Lambda$  and  $m_\Lambda$  by  $e_Q$  and  $m_Q$  everywhere in the amplitude. Thus such amplitudes are most naturally expressed in terms of the bare parameters for the theory with cutoff  $\Lambda = Q$ . This result is not surprising insofar as physical amplitudes are independent of the actual value of the cutoff used. The natural way to express this independence is to calculate with  $\Lambda \gg Q$  but then to reexpress the result in terms of the running parameters at scale  $Q$ . In this way one removes all explicit reference to the actual cutoff, and in particular one removes large logarithms of  $\Lambda/Q$  that otherwise tend to spoil the convergence of perturbation theory. This procedure, while useful in QED, has proven essential in perturbative QCD where the convergence of perturbation theory is marginal at best.



### 2.3 Beyond Renormalizability

It is clear from our analysis in the last section that the cut-off theory is accurate only up to corrections of  $\mathcal{O}(p^2/\Lambda^2)$  where  $p$  is typical of the external momenta. In practice such errors may be negligible, but if one wishes to remove them there are two options. One is the traditional option of taking  $\Lambda$  to infinity. The other is to keep  $\Lambda$  finite but to add further corrections to the Lagrangian. This second choice is far more informative and useful.

To illustrate the procedure consider again the  $k > \Lambda$  part of the amplitude for an electron scattering on an external field as in Eq. (3). In our earlier analysis we neglected external momenta and masses relative to the loop momentum. We can correct this approximation by making a Taylor expansion of the amplitude in powers of  $p/\Lambda$ ,  $p'/\Lambda$ , and  $m_0/\Lambda$  to obtain terms of the form

$$\begin{aligned} T(k > \Lambda) = & -ie_0 c_0 \bar{u} A_{\text{ext}} \cdot \gamma u \\ & - \frac{ie_0 m_0 c_1}{\Lambda^2} \bar{u} A_{\text{ext}}^\mu \sigma_{\mu\nu} (p - p')^\nu u \\ & - \frac{ie_0 c_2}{\Lambda^2} (p - p')^2 \bar{u} A_{\text{ext}} \cdot \gamma u + \dots, \end{aligned} \quad (16)$$

where coefficients  $c_0, c_1 \dots$  are all dimensionless, and where the structure of the amplitude is constrained by the need for current conservation and for chiral invariance in the limit  $m_0 = 0$ . The effects of all of these terms can be simulated by adding new local interactions to the Lagrangian. The first term was handled in the previous section by simply replacing the bare charge  $e_0$  by  $e_\Lambda$ ; the only difference now is that contributions to  $c_0$  of  $\mathcal{O}(m_0^2/\Lambda^2)$  must be retained. The remaining terms in  $T(k > \Lambda)$  require the introduction of new types of interaction:

$$\delta\mathcal{L}_2^a = \frac{e_0 m_0 c_1}{\Lambda^2} \bar{\psi} F^{\mu\nu} \sigma_{\mu\nu} \psi + \frac{e_0 c_2}{\Lambda^2} \bar{\psi} i \partial_\mu F^{\mu\nu} \gamma_\nu \psi. \quad (17)$$

These interactions are designed by including a field for each external particle in  $T(k > \Lambda)$ , and a derivative for each power of an external momentum. By augmenting the Lagrangian with such terms we can systematically remove all  $\mathcal{O}(p^2/\Lambda^2)$  errors from the cut-off theory. Of course such errors arise in processes other than electron scattering off a field, and further terms must be added to the Lagrangian to compensate for these. For example, the  $p^2/\Lambda^2$  contribution coming from  $k > \Lambda$  in electron-electron scattering (Eq. (8)) is compensated by interactions like

$$\delta\mathcal{L}_2^b = \frac{d}{\Lambda^2} (\bar{\psi} \gamma_\mu \psi)^2. \quad (18)$$

Luckily power-counting and dimensional analysis tell us that only a few processes are affected by the new cutoff to this order, and therefore only a finite number of terms need be added to the Lagrangian to remove all errors of  $\mathcal{O}(p^2/\Lambda^2)$  in all processes.

It seems remarkable that the  $p^2/\Lambda^2$  errors for an infinity of processes can be removed from the cut-off theory by adding a finite (even small) number of new interactions to the Lagrangian. However, one can show, even without examining

particular processes, that there is only a finite number of possible new interactions that might have been relevant to this order. Interactions that simulate  $k > \Lambda$  physics must be local—i.e., polynomial in the fields, and derivatives of the fields—and they must have the same symmetries as the underlying theory. In QED, these symmetries include Lorentz invariance, gauge invariance, parity conservation, and so on.<sup>(d)</sup> In addition the interaction terms must have (energy) dimension four; thus an interaction operator of dimension  $n + 4$  must have a coefficient of  $\mathcal{O}(1/\Lambda^n)$  or smaller,  $\Lambda$  being the smallest scale in the  $k > \Lambda$  part of the theory. Dimension-six operators are obviously important in  $\mathcal{O}(1/\Lambda^2)$ , but operators of higher dimension are suppressed by additional powers of  $1/\Lambda$  and so are irrelevant to this order. There are very few operators of dimension six or less that are Lorentz and gauge invariant, and that can be constructed from polynomials of  $\psi$  (dimension  $3/2$ ),  $A_\mu$  (dimension 1), and  $\partial_\mu$  (dimension 1). And these few are the only ones needed to correct the Lagrangian of the cut-off theory through order  $p^2/\Lambda^2$ .

**Exercise:** It is critical to our discussion that an operator of dimension  $n + 4$  in the Lagrangian only affect results in order  $(p/\Lambda)^n$  (or less). So introducing, for example, a dimension-eight operator should not affect the predictions of the theory through order  $(p/\Lambda)^2$ . This is obviously the case at tree level in perturbation theory, since the coefficient of the dimension-eight operator has a factor  $1/\Lambda^4$  that must appear in the final result for any tree diagram involving the interaction. Such a contribution will then be suppressed by a factor  $(p/\Lambda)^4$ . However in one-loop order (and beyond) loop integrations can supply powers of  $\Lambda$  that cancel the powers of  $1/\Lambda$  explicit in the interaction, resulting in contributions from the new interaction that are not negligible in second order. Show that these new contributions can be cancelled by appropriate shifts in the coefficients of the lower dimension operators in the Lagrangian. Thus the dimension-eight operator has no net effect on the results of the theory through order  $p^2/\Lambda^2$ .

This procedure can obviously be extended so as to correct the theory to any order in  $p/\Lambda$ , but the price of this improved accuracy is a more complicated Lagrangian. So why bother with cut-off field theory? Certainly in perturbation theory it seems desirable that the number of interactions be kept as few as possible so as to avoid an explosion in the number of diagrams. On the other hand the theory with no cutoff or with  $\Lambda \rightarrow \infty$  is ill-defined. Furthermore in practice it is essential to reduce the number of quantum degrees of freedom before one is able to solve a quantum field theory. A cutoff does just this. While cutoffs tend to appear only in intermediate steps of perturbative analyses, they are of central importance in most nonperturbative analyses. In numerical treatments, using lattices for example, the number of degrees of freedom must be finite since computers are finite.

The practical consequences of our analysis are important, but the most striking implication is that nonrenormalizable interactions make sense in the context

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<sup>(d)</sup>If the regulator breaks one of the symmetries of the theory then interactions that break the symmetry will also arise. These interactions serve to cancel the symmetry-breaking effects of the regulator. In lattice gauge theory, for example, the lattice regulator breaks the Lorentz symmetry and as a consequence interactions in such a theory can be Lorentz *non*-invariant.

of a cut-off field theory. Insofar as experiments can probe only a limited range of energies it is natural to analyze experimental results using cut-off field theories. Since nonrenormalizable interactions occur naturally in such theories it is probably wrong to ignore them in constructing theoretical models for currently accessible physics.

**Exercise:** The idea behind renormalization is rather generally applicable in quantum mechanics. Suppose we have a problem in which the states of most interest all have low energies, but couple through the Hamiltonian to states with much higher energies. To simplify the problem we want to truncate the state space, excluding all states but the (low-energy) ones of interest. Suppose we define a projection operator  $\mathcal{P}$  that projects onto this low-energy subspace; its complement is  $\mathcal{Q} \equiv 1 - \mathcal{P}$ . Then show that the full Hamiltonian  $H$  for the theory can be replaced by an effective Hamiltonian that acts only on the low-energy subspace:

$$H_{\text{eff}}(E) = H_{\mathcal{P}\mathcal{P}} + H_{\mathcal{P}\mathcal{Q}} \frac{1}{E - H_{\mathcal{Q}\mathcal{Q}}} H_{\mathcal{Q}\mathcal{P}} \quad (19)$$

where  $H_{\mathcal{P}\mathcal{Q}} \equiv \mathcal{P}H\mathcal{Q}$ . . . . In particular if  $|E\rangle$  is an eigenstate of the full Hamiltonian then its projection onto the low-energy subspace  $|E\rangle_{\mathcal{P}} \equiv \mathcal{P}|E\rangle$  satisfies

$$H_{\text{eff}}(E)|E\rangle_{\mathcal{P}} = E|E\rangle_{\mathcal{P}}. \quad (20)$$

Note that since the energies  $E$  in the  $\mathcal{P}$ -space are much smaller than the energies in the  $\mathcal{Q}$ -space, the second term in  $\mathcal{H}_{\text{eff}}$  can be expanded in a powers of  $E/H_{\mathcal{Q}\mathcal{Q}}$ . Thus the new interactions are polynomial in  $E$  and therefore “local” in time. In practice only a few terms might be needed in this expansion. Such truncations are made all the time in atomic physics. For example, if one is doing radio-frequency studies of the hyperfine structure of the ground state of hydrogen one usually wants to forget about all the radial excitations of the atom (i.e., optical frequencies).

## 2.4 Structure and Interpretation of Cut-off Field Theories

In the preceding sections we illustrated the nature of a cut-off field theory using QED perturbation theory. It should be emphasized that most of what we discovered is not tied to QED or to perturbation theory. The central notion, that the effects of high-energy states on low-energy processes can be accounted for through the introduction of local interactions, follows from the uncertainty principle and so is quite general. We can summarize the general ideas underlying cut-off field theories as follows:

- A finite cutoff  $\Lambda$  can be introduced into any field theory for the purposes of discarding high-energy states from the theory. The cut-off theory can then be used for processes involving momenta  $p$  much less than  $\Lambda$ .
- The effects of the discarded states can be retained in the theory by adjusting the existing couplings in the Lagrangian, and by adding new, local, nonrenormalizable interactions. These interactions are polynomial in the fields and derivatives of the fields. The nonrenormalizable couplings do not result in intractable infinities since the theory has a finite cutoff.

- Only a finite number of interactions is needed when working to a particular order in  $p/\Lambda$ , where  $p$  is a typical momentum in whatever process is under study. The cutoff usually sets the scale of the coefficient of an interaction operator: the coefficient of an operator with (energy) dimension  $n + 4$  is  $\mathcal{O}(1/\Lambda^n)$ , unless symmetries or approximate symmetries of the theory further suppress the interaction. Therefore one must include all interactions involving operators with dimension  $n + 4$  or less to achieve accuracy through order  $(p/\Lambda)^n$ .
- Conversely an operator of dimension  $n + 4$  can only affect results at order  $(p/\Lambda)^n$  or higher, and so may be dropped from the theory if such accuracy is unnecessary.

Note that the interactions in a cut-off field theory are completely specified by the requirement of locality, by the symmetries of the theory and regulator, and by the accuracy desired. The structure of the operators introduced into the Lagrangian does not depend upon the detailed dynamics of the high-energy states being discarded. It is only the numerical values of the coefficients of these operators that depend upon the high-energy dynamics. Thus while the high-energy states do have a strong effect on low-energy processes we need know very little about the high-energy sector in order to compute low-energy properties of the theory. The coupling constants of the cut-off theory— $e_\Lambda$ ,  $m_\Lambda$ ,  $c_1$ ,  $c_2$ ,  $d \dots$  in the previous sections—completely characterize the behavior of the discarded high-energy states for the purposes of low-energy analyses. In cases where we understand the dynamics of the discarded states, as in our analysis of QED, we can compute these coupling constants. In other situations we are compelled to measure them experimentally.

The expansion of a cut-off Lagrangian in powers of  $1/\Lambda$  is somewhat analogous to multipole expansions used in classical field theory. For example, the detailed charge distribution of a nucleus is of little importance to an electron in an atomic orbital. To compute the long range electrostatic field of the nucleus one need only know the charge of the nucleus, and, depending upon the level of accuracy desired, perhaps its dipole and quadrupole moments. Again, for all practical purposes the effects of short-range structure on long-range behavior can be expressed in terms of a finite number of numbers, the multipole moments, characterizing the short-range structure.

Armed with this new understanding of field theory, it is time to reexamine traditional theories in an effort to better understand why they are the way they are and how they might be changed to accommodate future experiments.

## 3 Applications

### 3.1 Why is QED renormalizable?

Having argued that nonrenormalizable interactions are admissible one has to wonder why QED is renormalizable after all. What do we learn from the fact

of its renormalizability? Our new attitude towards renormalization suggests that the key issue in addressing a theory like QED is not whether or not it is renormalizable, but rather how renormalizable it is—i.e., how large are the nonrenormalizable interactions in the theory.

It is quite likely that QED is a low-energy approximation to some complicated high-energy supertheory (a string theory?). Consequently there will be some large energy scale beyond which QED dynamics are insufficient to describe nature, where the supertheory will become necessary. Since we have yet to figure out what the supertheory is, it is natural in this scenario that we introduce a cutoff  $\Lambda$  equal to this energy scale so as to exclude the unknown physics. The supertheory still affects low-energy phenomena, but it does so only through the values of the coupling constants that appear in the cut-off Lagrangian:

$$\begin{aligned} \mathcal{L}_\Lambda = & \bar{\psi}(i\partial \cdot \gamma - e_\Lambda A \cdot \gamma - m_\Lambda)\psi - \frac{1}{2}(F^{\mu\nu})^2 + \\ & + \frac{e_\Lambda m_\Lambda c_1}{\Lambda^2} \bar{\psi} F^{\mu\nu} \sigma_{\mu\nu} \psi + \frac{e_\Lambda c_2}{\Lambda^2} \bar{\psi} i\partial_\mu F^{\mu\nu} \gamma_\nu \psi + \frac{d}{\Lambda^2} (\bar{\psi} \gamma_\mu \psi)^2 + \dots \end{aligned} \quad (21)$$

We cannot calculate the coupling constants in this Lagrangian until we discover and solve the supertheory; the couplings must be measured. The nonrenormalizable interactions are certainly present, but if  $\Lambda$  is large their affect on current physics will be very small, down by  $(p/\Lambda)^2$  or more. The fact that we haven't needed such terms to account for the data tells us that  $\Lambda$  is indeed large. This is the key to the significance of renormalizability and its origins: very low-energy approximations to *arbitrary* high-energy dynamics can be formulated in terms of renormalizable field theories since nonrenormalizable interactions would be suppressed in their effects by powers of  $p/\Lambda$  and therefore would be irrelevant for  $p \ll \Lambda$ . This analysis also tells us how to look for low-energy evidence of new high-energy dynamics: look for (small) effects caused by the leading nonrenormalizable interactions in the theory. Indeed we can estimate the energy scale  $\Lambda$  at which new physics must appear simply by measuring the strength of the nonrenormalizable interactions in a theory. The low-energy theory must fail and be replaced at energies of order  $\Lambda$ . Processes with  $p \sim \Lambda$  start to probe the detailed structure of the supertheory, and can no longer be described by the low-energy theory; the expansion of the Lagrangian in powers of  $1/\Lambda$  no longer converges.

As we noted, the successes of renormalizable QED imply that the energy threshold for new physics in electrodynamics is rather high. For example, the  $\bar{\psi} \sigma \cdot F \psi$  interaction in  $\mathcal{L}_\Lambda$  would shift the  $g$ -factor in the electron's magnetic moment by an amount of order  $(m_e/\Lambda)^2$ . So the fact that renormalizable QED accounts for the  $g$ -factor to almost 12 digits implies that

$$\frac{m_e^2}{\Lambda^2} < 10^{-12} \quad (22)$$

from which we can conclude that  $\Lambda$  is probably larger than a TeV.

### 3.2 Strong Interactions: Pions or Quarks?

QED is spectacularly successful in explaining the magnetic moment of the electron. It's failure to explain the magnetic moment of the proton is equally spectacular:  $g_p$  is roughly three times larger than predicted by QED when the proton is treated as an elementary, point-like particle. Nonrenormalizable terms in  $\mathcal{L}_\Lambda$  make contributions of order unity,

$$\frac{m_p^2}{\Lambda^2} \sim 1, \tag{23}$$

indicating that the cutoff in proton QED must be of order the proton's mass  $m_p$ . Thus QED with an elementary proton must fail around 1 GeV, to be replaced by some other more fundamental theory. That new theory is quantum chromodynamics, of course, and the large shift in the proton's magnetic moment is a consequence of its being a composite state built of quarks. This example illustrates how the measured strength of nonrenormalizable interactions can be used to set the energy scale for the onset of new physics. Also it strongly suggests that one ought to use QCD in analyzing strong interaction physics above a GeV.

This result also suggests that one can and ought to treat the proton as a point-like particle in analyses of sub-GeV physics. In the hydrogen atom, for example, typical energies are of order a few eV. At such energies the proton's electromagnetic interactions are most efficiently described by a cut-off QED (or nonrelativistic QED) for an elementary proton. The cutoff  $\Lambda$  should be set below a GeV, and the anomalous magnetic moment, charge radius and other properties of the extended proton should be simulated by nonrenormalizable interactions of the sort we have discussed. This theory can be made arbitrarily precise by adding a sufficient number of interactions—an expansion in  $1/\Lambda$ —and by working to sufficiently high order in perturbation theory—an expansion in  $\alpha$ . With the cutoff in place, the nonrenormalizable interactions cause no particular problems in perturbation theory.

In the same spirit one expects very low-energy strong interactions to be explicable in terms of a theory of point-like mesons and baryons. Indeed these ideas account for the great success of PCAC and current algebra in describing pion-pion and pion-nucleon scattering at threshold. A cut-off theory of elementary pions, protons and other hadrons must work above threshold as well, but will ultimately fail somewhere around a GeV. It is still an open question as to just where the cutoff lies. In particular it is unclear whether or not nuclear physics can still be efficiently described by pion-nucleon theories at energies of a few hundred MeV, or whether the quark structure is already important at such energies. To evaluate the utility of the pion-nucleon theory one must treat it as a cut-off field theory, taking care to make consistent use of the cutoff throughout, and systematically enumerating the possible interactions, determining their couplings from experimental data. It also seems likely that some sort of (systematic) nonperturbative approach to the problem is essential. This is clearly a crucial issue for nuclear physics since a theory of point-like hadrons, if it works, should be far simpler to use than a theory of quarks and gluons.

### 3.3 Weak Interactions: Renormalization, Naturalness and the SSC

The Standard Model of strong, electromagnetic and weak interactions has been enormously successful in accounting for experimental data up to energies of order 100 GeV or more. Still there remain several unanswered questions, and the most pressing of these concerns the origins of the masses for the  $W$  and  $Z$  bosons. A whole range of possible mechanisms has been suggested—the Higgs mechanism, technicolor, a supersymmetric Higgs particle, composite vector bosons—and extensive experimental searches have been conducted. Yet we still do not have a definitive answer. What will it take to unravel this puzzle?

In fact new physics, connected with the  $Z$  mass, must appear at energies of order a few TeV or lower. Current data indicates that the vector bosons are massive and that their interactions are those of nonabelian gauge bosons, at least insofar as the quarks are concerned. This suggests that a minimal Lagrangian for the vector bosons that accounts for current data would be the standard Yang-Mills Lagrangian for nonabelian gauge fields plus simple mass terms for the  $W$ 's and  $Z$ . Traditionally such a theory is rejected immediately since the mass terms spoil gauge invariance and ruin renormalizability. However from our new perspective, the appearance of nonrenormalizable terms in the minimal theory poses no problems; it indicates that this theory is necessarily a cut-off field theory with a *finite* cutoff. Thus the theory with minimal particle content becomes inadequate at a finite energy of order the cutoff, and new physics is inevitable. By examining the nonrenormalizable interactions one finds that the value of the cutoff is set by the vector-boson mass  $M$ :<sup>(e)</sup>

$$\Lambda \sim \frac{\sqrt{4\pi}M}{\sqrt{\alpha}} \quad (24)$$

This cutoff is of order a few TeV, and it represents the threshold energy for new physics.

This analysis is the basis for one of the most important scientific arguments for building an accelerator, like the SSC, that can probe few-TeV physics; the secret behind the  $W$  and  $Z$  masses almost certainly lies in this energy range. One potential problem with this argument is the possibility that the Higgs particle exists and has a mass well below a few TeV, in which case the interesting physics is at too low an energy for an SSC. In fact the argument survives because a theory with a low-mass Higgs particle is “unnatural” unless there is new

<sup>(e)</sup>It is the longitudinal degrees-of-freedom in a massive Yang-Mills theory that spoil the renormalization. The longitudinal part of the action can be isolated through gauge transformations and shown to be equivalent to a nonlinear sigma model, which is nonrenormalizable. To see this simply, one can examine the standard theory with a Higgs particle. One arranges the scalar couplings of this theory in such a way that the mass of the Higgs becomes infinite, while keeping the vector-boson masses constant. What remains, once the Higgs particle is removed in this fashion, is just a theory of massive gauge bosons. The part of the Lagrangian dealing with the Higgs particle becomes a nonlinear sigma model in this limit, with the magnitude of the complex scalar field frozen at its vacuum expectation value  $v \approx 200$  GeV. The sigma model is nonrenormalizable but makes sense as a cut-off theory provided the cutoff is less than  $\mathcal{O}(4\pi v)$ , which is equivalent to the limit given in the text.

structure, again at energies of order a few TeV. If we regard the standard model with a Higgs field as a cut-off field theory, approximating some more complex high-energy theory, then we expect that the scale of dimensionful couplings in the cut-off Lagrangian is set by the cutoff  $\Lambda$ . In particular the bare mass of the scalar boson is of order the cutoff, and thus the renormalized mass of the Higgs particle ought to be of this order as well. Since this theory is technically renormalizable, one can make the Higgs mass much smaller than the cutoff by fine tuning the bare mass to cancel almost exactly the large mass generated by quantum fluctuations; but such tuning is highly contrived, and as such is unlikely to occur in the low-energy approximation to any more complex theory. Thus the Higgs theory is sensible only when the cutoff  $\Lambda$  is finite. To estimate how large a natural cutoff might be, we compare the bare mass with the mass renormalization due to quantum fluctuations. Barring (unlikely) accidental cancellations, one expects the physical and bare masses to have roughly the same order of magnitude; that is, we expect

$$m_H^2 \equiv \mu^2 + \delta m^2 \sim |\mu^2|, \quad (25)$$

where  $\delta m^2$  is the mass renormalization, and  $m_H^2$  and  $\mu^2$  are the squares of the physical and bare masses respectively. The mass renormalization is quadratically sensitive to the cutoff,

$$\delta m^2 \sim \lambda \Lambda^2 \quad (26)$$

where  $\lambda$  is the  $\phi^4$  coupling constant. Therefore the bare and physical masses can be comparable only if the cutoff is less than

$$\Lambda^2 \sim \frac{|\mu^2|}{\lambda}, \quad (27)$$

which turns out to be the same limit we obtained above for the nonrenormalizable massive Yang-Mills theory (Eq. (24)).<sup>(f)</sup> Therefore new physics is expected in the few-TeV region whether or not there is a Higgs particle at lower energies.

The Yang-Mills theory coupled to a Higgs doublet gained widespread acceptance because it was renormalizable. However, our new understanding of cut-off field theories suggests that naturalness, rather than renormalizability, is the key property of a physical theory. And from this perspective, the renormalizable theory with a Higgs doublet is neither better nor worse than the nonrenormalizable massive Yang-Mills theory. One might even argue that the latter model is the more attractive given that it is minimal. Both theories are predictive in the 100 GeV region and below; neither theory can survive without modification much beyond a few TeV.

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<sup>(f)</sup>Recall that the complex scalar field generates a mass for the gauge bosons by acquiring a vacuum expectation value. Replacing  $i\partial_\mu$  by  $i\partial_\mu - gA_\mu$  in the Lagrangian for a free scalar boson leads to an interaction term  $g^2 A^2 \phi^2/2$ . This becomes a mass term for the  $A$  field if the  $\phi$  field has nonzero vacuum expectation value  $v$ , the mass being  $gv$ . Thus  $v$  is of order  $M/\sqrt{\alpha}$ . This  $v$  arises from the competition between the bare scalar mass term and the  $\phi^4$  interaction, with the result that  $v^2$  is also of order  $|\mu^2|/\lambda$ , which in turn is of order the cutoff squared in a natural theory. Combining these two expressions for  $v$  gives the limit in Eq. (24).



Our analysis of the Higgs sector of the standard weak interaction theory illustrates a general feature of physically relevant scalar field theories: the renormalized mass of the field is most likely as large as the bare mass, and both are at least as large as the mass renormalization due to quantum fluctuations. In strongly interacting theories this means that the renormalized mass is of order the cutoff; scalars with masses small compared to  $\Lambda$  are unnatural. This result should be contrasted with the situation for spin-1/2 particles and for gauge bosons. In the case of vector bosons like the photon, the bare mass in the Lagrangian's mass term,  $M_\Lambda^2 A^2/2$ , would indeed be of order the cutoff were it not for gauge invariance, which requires  $M_\Lambda = 0$ . Similarly for fermions like the electron, chiral symmetry implies that all mass renormalizations must vanish if the bare mass does. Consequently one has  $\delta m_e \sim m_e$  even though the cutoff is much larger. In general one expects low-mass particles in a theory only when there is a symmetry, like gauge invariance or chiral symmetry, that protects the low mass. No such symmetry exists for scalar bosons, unless they are tied to fermionic partners via a supersymmetry. This may explain why all existing experimental data in high-energy physics can be understood in terms of elementary particles that are either spin-1/2 fermions or spin-1 gauge bosons.

### 3.4 Nonrelativistic Field Theories

Nonrelativistic systems, such as positronium or the  $\Upsilon$  and  $\psi$  mesons, play an important role in several areas of elementary particle physics. Being nonrelativistic these systems are generally weakly coupled, and as a result typically involve only a single channel. Thus, for example, the  $\Upsilon$  is predominantly a bound state of a  $b$  quark and antiquark. It has some probability for being in a state comprised of a  $b\bar{b}$  pair and a gluon, or a  $b\bar{b}$  pair and a  $u\bar{u}$  pair, etc., but the probabilities are small and therefore these channels have only a small effect on the gross physics of the meson. This is an enormous simplification relative to relativistic systems where many channels may be important, and it is this that accounts for the prominence of such systems in fundamental studies of electromagnetic and strong interactions. Nevertheless there are significant technical problems connected with the study of nonrelativistic systems. Central among these is the problem of too many energy scales. Typically a nonrelativistic system has three important energy scales: the masses  $m$  of the particles involved, their three-momenta  $p \sim mv$ , and their kinetic energies  $KE \sim mv^2$ . These scales are widely different in a nonrelativistic system since  $v \ll 1$  (where the speed of light  $c = 1$ ), and this greatly complicates any analysis of such a system. In this section we will see how renormalization can be used as a tool to dramatically simplify studies of this sort.

One can appreciate the problems that arise when analyzing these systems by considering a lattice simulation of the  $\Upsilon$  meson. The space-time grid used in such a simulation must accommodate wavelengths covering all of the of scales in the meson, ranging from  $1/mv^2$  down to  $1/m$ . Given that  $v^2 \sim 0.1$  in the  $\Upsilon$ , one might easily need a lattice as large as 100 sites on side to do a good job. Such a lattice could be three times larger than the largest wavelength, with a grid

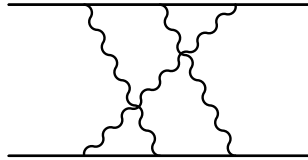


Figure 4: A two-loop kernel contributing to the Bethe-Salpeter potential for positronium.

spacing three times smaller than the smallest wavelength, thereby limiting the errors caused by the grid. This is a fantastically large lattice by contemporary standards and is quite impractical.

The range of length scales also causes problems in analytic analyses. As an example, consider a traditional Bethe-Salpeter calculation of the energy levels of positronium. The potential in the Bethe-Salpeter equation is given by a sum of two-particle irreducible Feynman amplitudes. One generally solves the problem for some approximate potential and then incorporates corrections using time-independent perturbation theory. Unfortunately, perturbation theory for a bound state is far more complicated than perturbation theory for, say, the electron's  $g$ -factor. In the latter case a diagram with three photons contributes only in order  $\alpha^3$ . In positronium a kernel involving the exchange of three photons (e.g., Fig. 4) can also contribute to order  $\alpha^3$ , but the same kernel will contribute to all higher orders as well:<sup>(g)</sup>

$$\langle K_3 \rangle = \alpha^3 m \{ a_0 + a_1 \alpha + a_2 \alpha^2 + \dots \}. \quad (28)$$

So in the bound state calculation there is no simple correlation between the importance of an amplitude and the number of photons in it. Such behavior is at the root of the complexities in high-precision analyses of positronium or other QED bound states, and it is a direct consequence of the multiple scales in the problem. Any expectation value like that in Eq. (28) will be some complicated function of ratios of the three scales in the atom:

$$\langle K_3 \rangle = \alpha^3 m f(\langle p \rangle / m, \langle KE \rangle / m). \quad (29)$$

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<sup>(g)</sup>The situation is actually even worse. The contribution from a particular kernel is highly dependent upon gauge. For example, the coefficients  $a_0$  and  $a_1$  vanish in Coulomb gauge but not in Feynman gauge. The Feynman gauge result is roughly  $10^4$  times larger, and it is spurious: the large contribution comes from unphysical retardation effects in the Coulomb interaction that cancel when an infinite number of other diagrams is included. The Coulomb interaction is instantaneous in Coulomb gauge and so this gauge generally does a better job describing the fields created by slowly moving charged particles. On the other hand contributions coming from relativistic momenta are more naturally handled in a covariant gauge like Feynman gauge; in particular renormalization is far simpler in Feynman gauge than it is in Coulomb gauge. Unfortunately most Bethe-Salpeter kernels have contributions coming from both nonrelativistic and relativistic momenta, and so there is no optimal choice of gauge. This is again a problem due to the multiple scales in the system.

Since  $\langle p \rangle / m \sim \alpha$  and  $\langle KE \rangle / m \sim \alpha^2$ , a Taylor expansion of  $f$  in powers of these ratios generates an infinite series of contributions just as in Eq. (28). Similar series do not occur in the  $g$ -factor calculation because there is but one scale in that problem, the mass of the electron.

Traditional methods for analyzing either of these systems fail to take advantage of the nonrelativistic character of the systems. One way to capitalize on this feature is to introduce a cutoff of order the constituent's mass or smaller into the field theory. The cutoff here can be thought of as the boundary between relativistic and nonrelativistic physics. Since the gross dynamics in the problems of interest is nonrelativistic, relativistic physics is well simulated by local interactions in the cut-off Lagrangian.

The utility of such a cutoff is greatly enhanced if one also transforms the Dirac field so as to decouple its upper components from its lower components. This is called the Foldy-Wouthuysen transformation, and it transforms the Dirac Lagrangian into a nonrelativistic Lagrangian. In QED one obtains

$$\begin{aligned} \bar{\Psi}(iD \cdot \gamma - m)\Psi \quad \rightarrow \quad & \psi^\dagger iD_0 \psi + \psi^\dagger \frac{\vec{D}^2}{2m} \psi - \psi^\dagger \frac{\vec{D}^4}{8m^3} \psi \\ & - \frac{e}{2m} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi - \frac{e}{8m^2} \psi^\dagger \nabla \cdot \vec{E} \psi + \dots \end{aligned} \quad (30)$$

where  $D_\mu = \partial_\mu + ieA_\mu$  is the gauge-covariant derivative,  $\vec{E}$  and  $\vec{B}$  are the electric and magnetic fields, and  $\psi$  is a two-component Pauli spinor representing the electron part, or upper components, of the original Dirac field. The lower components of the Dirac field lead to analogous terms that specify the electromagnetic interactions of positrons. The Foldy-Wouthuysen transformation generates an infinite expansion of the action in powers of  $1/m$ . As an ordinary  $\Lambda \rightarrow \infty$  field theory this expansion is a disaster; the renormalizability of the theory is completely disguised, requiring a delicate conspiracy involving terms of all orders in  $1/m$ . However, setting  $\Lambda \sim m$  implies that the Foldy-Wouthuysen expansion is an expansion in  $1/\Lambda$ , and from our general rules for cut-off theories we know that only a finite number of terms need be retained in the expansion if we want to work to some finite order in  $p/\Lambda \sim p/m \sim v$ . Thus, to study positronium through order  $v^2 \sim \alpha^2$ , we can replace QED by a nonrelativistic QED (NRQED) with the Lagrangian

$$\begin{aligned} \mathcal{L}_{NRQED} = & -\frac{(F^{\mu\nu})^2}{2} + \psi^\dagger \left\{ i\partial_t - e_0 A_0 + \frac{\vec{D}^2}{2m_0} \right. \\ & - c_1 \frac{e_0}{2m_0} \vec{\sigma} \cdot \vec{B} - c_2 \frac{e_0}{8m_0^2} \nabla \cdot \vec{E} \\ & \left. - c_3 \left( \frac{ie_0}{8m_0^2} \vec{\sigma} \cdot \nabla \times \vec{E} - \frac{ie_0}{4m_0^2} \vec{\sigma} \cdot \vec{E} \times \nabla \right) \right\} \psi \\ & + \frac{d_1}{m_0^2} (\psi^\dagger \psi)^2 + \frac{d_2}{m_0^2} (\psi^\dagger \vec{\sigma} \psi)^2 \\ & + \text{positron and positron-electron terms.} \end{aligned} \quad (31)$$

The coupling constants  $e_0$ ,  $m_0$ ,  $c_1 \dots$  are all specified for a cutoff of  $\Lambda \sim m$ . Renormalization theory tells us that there exists a choice for the coupling constants in this theory such that NRQED reproduces all of the results of QED up to corrections of order  $(p/m)^3$ .

NRQED is far simpler to use than the original theory when studying non-relativistic atoms like positronium. The analysis falls into two parts. First one must determine the coupling constants in the NRQED Lagrangian. This is easily done by computing simple scattering amplitudes in both QED and NRQED, and then by adjusting the NRQED coupling constants so that the two theories agree through some order in  $\alpha$  and  $v$ . The coupling constants are functions of  $\alpha$  and the mass of the electron; to leading order one finds that the  $d_i$ 's vanish while all the  $c_i$ 's equal one. As the couplings contain the relativistic physics, this part of the calculation involves only scales of order  $m$ ; it is similar in character to a calculation of the  $g$ -factor. Furthermore there is no need to deal with complicated bound states at this stage. Having solved the high-energy part of QED by computing  $\mathcal{L}_{NRQED}$ , one goes on to solve NRQED for any nonrelativistic process or system. To study positronium one uses the Bethe-Salpeter equation for this theory, which is just the Schrödinger equation, and ordinary time-independent perturbation theory. One of the main virtues of this approach is that it builds directly on the simple results of nonrelativistic quantum mechanics, leaving our intuition intact. Even more important for high-precision calculations is that only two dynamical scales remain in the problem, the momentum and the kinetic energy, and these are easily separated on a diagram-by-diagram basis. As a result infinite series in  $\alpha$  can be avoided in calculating the contributions due to individual diagrams, and thus it is trivial to separate, say,  $\mathcal{O}(\alpha^6)$  contributions from  $\mathcal{O}(\alpha^5)$  contributions.

Obviously NRQED is useful in analyzing electromagnetic interactions in any nonrelativistic system. In particular it provides an elegant framework for incorporating relativistic effects into analyses of many-electron atoms and solids in general.

For heavy-quark mesons one can replace QCD by a nonrelativistic theory (NRQCD) whose Lagrangian has basically the same structure as  $\mathcal{L}_{NRQED}$ . Since the coupling constants relate to physics at scales of order  $m$  and above, they can be computed perturbatively if the quark mass is large enough. The lattice used in simulating NRQCD can be much coarser than that required for ordinary QCD since structure at wavelengths of order  $1/m$  has been removed from the theory. For example,  $v \sim 1/3$  for the  $\Upsilon$  and thus the lattice spacing can be roughly three times larger in NRQCD, leading to a lattice with  $3^4$  fewer sites. Furthermore, by decoupling the quark from the antiquark degrees of freedom we convert the numerical problem of computing quark propagators from a boundary-value problem for a four-spinor into an initial-value problem for a two-spinor, resulting in very significant gains in efficiency. The move from QCD to NRQCD makes the  $\Upsilon$  one of the easiest mesons to simulate numerically.

## 4 Conclusion

In this lecture we have seen that renormalizability is not miraculous; on the contrary, approximate renormalizability is quite natural in a theory that is a low-energy approximation to some more complex high-energy supertheory. Furthermore one expects symmetries like gauge invariance or chiral symmetry in the low-energy theory; low-mass particles are unnatural without such symmetries. In such a picture the ultraviolet cutoff, originally an artifice required to give meaning to divergent integrals, acquires physical significance as the threshold energy for the appearance of new physics associated with the supertheory; it is the boundary between what we do and do not know. With the cutoff in place it is quite natural to have small nonrenormalizable interactions in the low-energy theory, the size of these interactions being intimately related to the energy range over which the low-energy approximation is valid: the smaller the coupling constants for nonrenormalizable interactions, the larger the range of validity for the theory. Thus in studying weak, electromagnetic and strong processes we should be on the lookout for evidence indicating such interactions since this will give us clues as to where new physics might be found. Finally, we saw that renormalization and cut-off Lagrangians are powerful tools that can be used to separate energy scales in a problem, allowing us to deal with one scale at a time. In light of these developments we need no longer apologize for renormalization.

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## Bibliography

Cut-off field theories have played an important role in the modern development of the renormalization group as a tool for studying both quantum and statistical field theories. An elementary discussion of this subject, with lots of references to the original literature, can be found in Wilson's Nobel lecture:

K.G. Wilson, *Rev. Mod. Phys.* **55**, 583(1983); see also K.G. Wilson, *Sci. Am.* **241**, 140 (August, 1979).

The effects of a finite cutoff are naturally important in numerical simulations of lattice QCD, where it is hard enough to even get the lattice spacing small, let alone vanishingly small. The use of nonrenormalizable interactions to correct for a finite lattice spacing has been explored extensively; see, for example,

K. Symanzik, *Nucl. Phys.* **B226**, 187(1983).

The use of Monte Carlo techniques in renormalization-group analyses was discussed by R. Gupta at this Summer School.

Another area in which cut-off Lagrangians have come to the fore is in applications of current algebra to low-energy strong interactions. Nonlinear chiral

models provide the natural starting point for any attempt to model such physics in terms of elementary meson and baryon fields. A good introduction to the general ideas underlying these theories is given in Georgi's book:

H. Georgi, *Weak Interactions and Modern Particle Theory*, Benjamin/Cummings, Menlo Park, 1984.

Note that Georgi refers to cut-off theories as effective theories. I have avoided this usage to prevent confusion between cut-off Lagrangians and effective classical Lagrangians, in which all loops are incorporated into the action. Serious attempts have been made to carry chiral theories beyond tree level using perturbation theory; see, for example,

J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142(1984).

Unfortunately, the convergence of perturbation theory is not always particularly impressive here. Pion-nucleon theories are analyzed extensively in the literature of nuclear physics. However most of these analyses are inconsistent in the application of cutoffs and the design of the cut-off Lagrangian.

Experimental limits on possible deviations from the standard model have been studied extensively in preparation for the SSC and other multi-TeV accelerators. For example, a study of the limits on nonrenormalizable interactions in QED, QCD, etc. can be found in the proceedings of the first Snowmass workshop on SSC physics:

M. Abolins *et al.*, in *Proceedings of the 1982 DPF Summer Study on Elementary Particle Physics and Future Facilities*, edited by R. Donaldson *et al.*.

The significance of an extra anomaly in the electron's magnetic moment, beyond what is predicted by QED, was first discussed in

R. Barbieri, L. Maiani and R. Petronzio, *Phys. Lett.* **96B**, 63(1980); and S.J. Brodsky and S.D. Drell, *Phys. Rev.* **D22**, 2236(1980).

The proceedings of the Snowmass workshops and similar workshops contain extensive discussions of the various scenarios for the weak interactions at TeV energies. See also

M.S. Chanowitz, *Ann. Rev. Nucl. Part. Sci.* **38**, 323(1988)

where among other things, the phenomenology of the (nonrenormalizable) massive Yang-Mills version of weak interactions is discussed. The naturalness of field theories became an issue with the advent of early theories of the grand unification of strong, electromagnetic and weak interactions, these GUT's being renormalizable but decidedly unnatural. For a brief introduction to such theories see Quigg's book:

C. Quigg, *Gauge Theories of Strong, Weak, and Electromagnetic Interactions*, Benjamin/Cummings, Reading, 1983.

Nonrelativistic QED is practically as old as quantum mechanics, but the idea of using this theory as a cut-off field theory, thereby permitting rigorous calculations beyond tree level, originates with the (too) short paper

W.E. Caswell and G.P. Lepage, *Phys. Lett.* **167B**, 437(1986).

Here the theory is applied in state-of-the-art bound-state calculations for positronium and muonium. The application of these ideas to heavy-quark physics is discussed in

G.P. Lepage and B.A. Thacker, in *Field Theory on the Lattice*, edited by A. Billoire *et al.*, *Nucl. Phys. (Proc. Suppl.)* **4** (1988); B.A. Thacker, Ph.D. Thesis, Cornell University (September 1989); B.A. Thacker and G.P. Lepage, Cornell preprint (December, 1989).