

Exercise

Square well potential

We consider the attractive square-well potential:

$$V(r) = \begin{cases} -V_0 & r \leq r_0 \\ 0 & \text{elsewhere} \end{cases}$$

with $V_0 \geq 0$. We compute the phase shift in the s wave. The radial Schrödinger then becomes

$$\begin{cases} -u_0'(r) = \frac{2\mu}{\hbar^2} V_0 u_0(r) = k^2 u_0(r) & r \leq r_0 \\ -u_0''(r) = k^2 u_0(r) & r > r_0 \end{cases}$$

with $E = \frac{\hbar^2 k^2}{2\mu} > 0$ the (positive) energy in the continuum.

$$\Leftrightarrow \begin{cases} u_0''(r) = -k_0^2 u_0(r) & r \leq r_0 \\ u_0''(r) = -k^2 u_0(r) & r > r_0 \end{cases}$$

where $\frac{\hbar^2 k_0^2}{2\mu} = E + V_0 > 0 \quad \forall E, V_0$, which are both > 0 .

The solutions are

$$\begin{cases} A \sin k_0 r & r \leq r_0 \\ B \sin(kr + \delta_0) & r > r_0 \end{cases}$$

(no $\cos k_0 r$ possible for $r < r_0$ because $u_0(0) = 0$)

The logarithmic derivative must be continuous $\forall r$ so in r_0

$$\frac{A k_0 \cos k_0 r_0}{A \sin k_0 r_0} = \frac{B k \cos(kr_0 + \delta_0)}{B \sin(kr_0 + \delta_0)}$$

$$\Rightarrow \frac{1}{k_0} \tan k_0 r_0 = \frac{1}{k} \tan(k r_0 + \delta_0)$$

$$\Rightarrow \frac{k}{k_0} \tan k_0 r_0 = \frac{\tan k r_0 + \tan \delta_0}{1 - \tan k r_0 \tan \delta_0}$$

$$\Rightarrow \tan \delta_0 = \frac{\frac{k}{k_0} \tan k_0 r_0 - \tan k r_0}{1 + \frac{k}{k_0} \tan k_0 r_0 \tan k r_0}$$

$$\Rightarrow \tan \delta_0 = \frac{k \tan k_0 r_0 - k_0 \tan k r_0}{k_0 + k \tan k_0 r_0 \tan k r_0}$$

$$\xrightarrow[k \rightarrow 0]{} \frac{k (\tan k_0 r_0 - k_0 r_0)}{k_0 + k^2 r_0 \tan k_0 r_0} \approx \frac{k (\tan k_0 r_0 - k_0 r_0)}{k_0 \left(1 - \frac{k^2}{k_0} r_0 \tan k_0 r_0\right)}$$

From this, we see that $\delta_0 \xrightarrow[k \rightarrow 0]{} 0$ linearly ($\propto k$) (actually $n\pi, n \in \mathbb{Z}$)

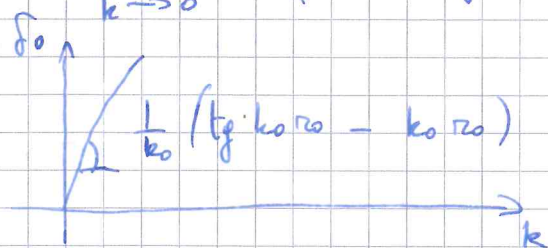
For small V_0 , $\tan k_0 r_0 - k_0 r_0 > 0$ (i.e. $k_0 r_0 < \frac{\pi}{2}$) $\Rightarrow \delta_0 \nearrow$ for $k \nearrow$

When $k_0 r_0 = \frac{\pi}{2}$, $\tan \delta_0 \xrightarrow[k \rightarrow 0]{} \infty \Rightarrow \delta_0 \xrightarrow[k \rightarrow 0]{} \frac{\pi}{2}$

For larger V_0 , i.e. $k_0 r_0 > \frac{\pi}{2}$, $\tan k_0 r_0 - k_0 r_0 < 0 \Rightarrow \delta_0 \searrow$ for $k \nearrow$

So,

- When $\tan k_0 r_0 - k_0 r_0 > 0$ (i.e. $k_0 r_0 < \frac{\pi}{2}$ so, "small" V_0)
 $\delta_0 \xrightarrow[k \rightarrow 0]{} 0$ (actually $n\pi; n \in \mathbb{Z}$) by positive values



• When $\lim_{k \rightarrow 0} k_0 r_0 - k_0 r_0 = \infty$, so $k_0 r_0 = \frac{\pi}{2} (+n\pi)$

$$\delta_0 \xrightarrow{k \rightarrow 0} \frac{\pi}{2} (+n\pi; n \in \mathbb{Z})$$

The low- k behaviour can be obtained from

$$\frac{1}{k_0} \tan k_0 r_0 = \frac{1}{k} \tan(k r_0 + \delta_0)$$

Remembering that $k_0 = \sqrt{k^2 + \frac{2\mu}{\hbar^2} V_0}$

$$k_0 r_0 = r_0 \sqrt{\frac{2\mu V_0}{\hbar^2}} \sqrt{1 + \frac{\hbar^2 k^2}{2\mu V_0}}$$

$$= \frac{\pi}{2} = \sqrt{1 + \frac{E}{V_0}}$$

$$\xrightarrow{k \rightarrow 0} \frac{\pi}{2} \left(1 + \frac{1}{2} \frac{E}{V_0} \right)$$

$$= \frac{\pi}{2} + \frac{k^2 r_0^2}{\pi}$$

$$\tan k_0 r_0 \xrightarrow{k \rightarrow 0} \tan \left(\frac{\pi}{2} + \frac{k^2 r_0^2}{\pi} \right)$$

$$= \cot \left(\frac{k^2 r_0^2}{\pi} \right) \xrightarrow{k \rightarrow 0} -\frac{\pi}{k^2 r_0^2}$$

$$\Rightarrow \frac{1}{k_0} \tan k_0 r_0 \xrightarrow{k \rightarrow 0} -\frac{2}{k^2 r_0}$$

$$\Rightarrow \frac{1}{k} \tan(k r_0 + \delta_0) = -\frac{2}{k^2 r_0}$$

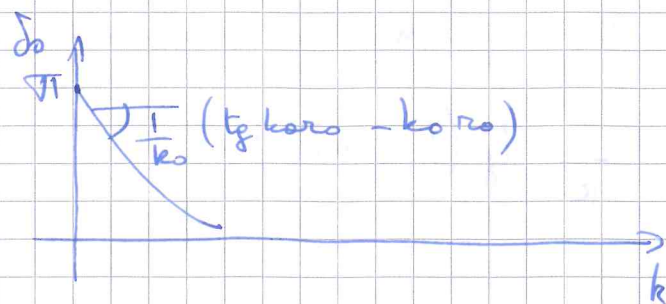
$$\Rightarrow \tan(k r_0 + \delta_0) = -\frac{2}{k r_0}$$

$$\Rightarrow k r_0 + \delta_0 \xrightarrow{k \rightarrow 0} \frac{\pi}{2} + \frac{k r_0}{2} (+n\pi)$$

$$\Rightarrow \delta_0 \xrightarrow{k \rightarrow 0} \frac{\pi}{2} - \frac{k r_0}{2} (+n\pi)$$

- When $\tan k_0 r_0 - k_0 r_0 < 0$ ($\Rightarrow k_0 r_0 > \frac{\pi}{2}$ so, "large" V_0)

$\delta_0 \rightarrow 0 (+n\pi; n \in \mathbb{Z})$ by negative values



So what corresponds this value $k_0 r_0 = \frac{\pi}{2}$?

A square-well potential will have a bound state under the following conditions:

$$\begin{cases} -u_0''(r) - \frac{2\mu}{\hbar^2} V_0 u_0(r) = -K^2 u_0(r) & r \leq r_0 \\ -u_0''(r) = -K^2 u_0(r) & r > r_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} u_0''(r) = -K_0^2 u_0(r) & r \leq r_0 \\ u_0''(r) = K^2 u_0(r) & r > 0 \end{cases} \quad \begin{aligned} &\text{with } \frac{\hbar^2 K_0^2}{2\mu} = V_0 + E > 0 \\ &\text{and } E = -\frac{\hbar^2 K^2}{2\mu} < 0 \end{aligned}$$

which have for solution

$$\begin{cases} u_0(r) = A \sin K_0 r & r \leq r_0 \\ u_0(r) = B e^{-Kr} & r > r_0 \end{cases}$$

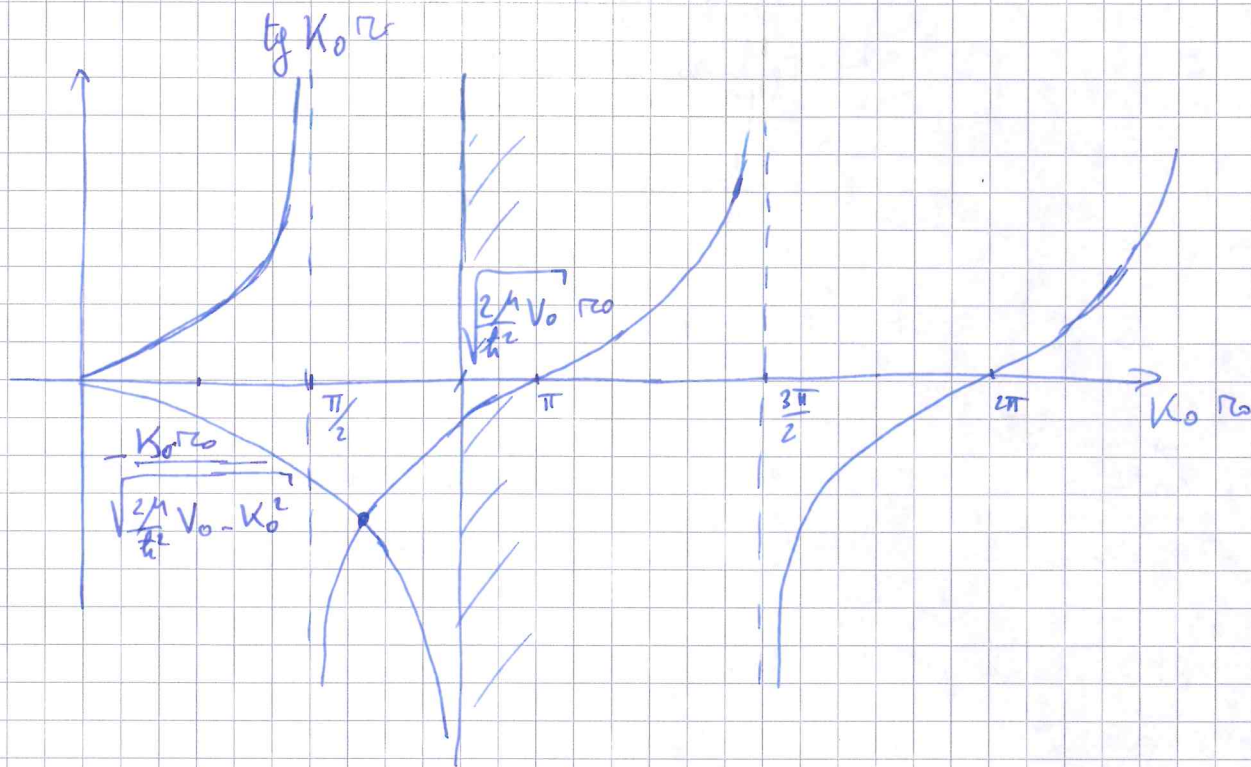
no $\cos K_0 r$ in the first solution because $u_0(0) = 0$

and no e^{Kr} in the second one because the function has to be L^2 .

In r_0 the logarithmic derivative must be continuous, so

$$K_0 \frac{\cos K_0 r_0}{\sin K_0 r_0} = -K \frac{e^{-Kr_0}}{e^{-Kr_0}}$$

$$\begin{aligned} \textcircled{2} \quad \tan K_0 r_0 &= - \frac{K_0}{k} \\ &= - \frac{K_0}{\sqrt{\frac{2\mu}{\hbar^2} V_0 - K_0^2}} \end{aligned}$$



Thus the square-well potential hosts a bound state only if $\sqrt{\frac{2\mu}{\hbar^2} V_0} \geq \frac{\pi}{2r_0}$

$$\Leftrightarrow \sqrt{\frac{2\mu}{\hbar^2} V_0} \cdot r_0 \geq \frac{\pi}{2}$$

$$\Leftrightarrow \lim_{k \rightarrow 0} k r_0 \geq \frac{\pi}{2}$$

When $k r_0 = \frac{\pi}{2}$ ($@ k \rightarrow 0$), the square well hosts 1st bound state @ $E = 0$. This is known as a virtual state.

(change
denote)

We can now look @ the behavior of the phase shift
@ high energy.

$$\begin{aligned} \tan \delta_0 &\xrightarrow{k \rightarrow \infty} \frac{\tan k_0 r_0 - \tan k r_0}{1 + \tan k_0 r_0 \tan k r_0} \quad (\text{because } k_0 \xrightarrow{k \rightarrow \infty} k) \\ &= \frac{0}{1 + \tan^2 k r_0} \\ &\approx \tan(k_0 - k)r_0 \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

otherwise

$$\begin{aligned} \delta_0 &= \arctan\left(\frac{k}{k_0} \tan k_0 r_0\right) - k r_0 \\ &\xrightarrow{k \rightarrow \infty} \arctan(\tan k_0 r_0) - k r_0 \\ &= (k_0 - k)r_0 \rightarrow 0 \quad \underline{\text{OK}} \end{aligned}$$

$\Rightarrow \delta_0 \rightarrow 0$ when $E \rightarrow \infty$ (at least towards $n\pi$)

Note that since $k_0 > k$ $\left[\begin{array}{l} \frac{\hbar^2 k_0^2}{2\mu} = V_0 + E \quad (V_0 > 0) \\ \frac{\hbar^2 k^2}{2\mu} = E \end{array} \right]$
 $\tan \delta_0 \xrightarrow{k \rightarrow \infty} 0^+$
so $\delta_0 \rightarrow 0 (+n\pi)$ by upper (>0) values.

If $V_0 < 0$ (repulsive potential), $\delta_0 \xrightarrow{k \rightarrow \infty} 0^-$ ($\delta_0 < 0$)

This confirms that $\left\{ \begin{array}{l} \delta_\ell > 0 \text{ when } V(r) < 0 \quad \forall r \\ \delta_\ell < 0 \text{ when } V(r) > 0 \quad \forall r \end{array} \right.$

This can be generalised for higher partial waves, i.e. $l > 0$. In those cases, the radial Schrödinger equation reads:

$$-u_l''(r) + \left[\frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2} V(r) \right] u_l(r) = k^2 u_l(r)$$

For a square well potential, we get

$$u_l''(r) - \frac{l(l+1)}{r^2} u_l(r) = -k_0^2 u_l(r) \quad r \leq r_0$$

$$u_l''(r) - \frac{l(l+1)}{r^2} u_l(r) = -k^2 u_l(r) \quad r > r_0$$

with again $\frac{\hbar^2 k^2}{2\mu} = E$ and $\frac{\hbar^2 k_0^2}{2\mu} = E + V_0$

Or, assuming the wave function to read

$$Y_l^m(\vec{r}) = R_l(r) Y_l^m(r)$$

the radial Schrödinger equation reads

$$-\frac{1}{r} \frac{\partial^2}{\partial r^2} [r R_l(r)] + \frac{l(l+1)}{r^2} R_l(r) + \frac{2\mu}{\hbar^2} V(r) R_l(r) = k^2 R_l(r)$$

$$\Leftrightarrow -\frac{1}{r} \frac{\partial}{\partial r} [R_l(r) + r R_l'(r)] + \frac{l(l+1)}{r^2} R_l(r) + \frac{2\mu}{\hbar^2} V(r) R_l(r) = k^2 R_l(r)$$

$$\Leftrightarrow -\frac{2}{r} R_l'(r) - R_l''(r) + \frac{l(l+1)}{r^2} R_l(r) + \frac{2\mu}{\hbar^2} V(r) R_l(r) = k^2 R_l(r)$$

$$\Leftrightarrow r^2 R_l''(r) + 2r R_l'(r) + \left[k^2 r^2 - \frac{2\mu}{\hbar^2} V(r) r^2 - l(l+1) \right] R_l(r) = 0$$

$$\Leftrightarrow \begin{cases} r^2 R_l''(r) + 2r R_l'(r) + [k_0^2 r^2 - l(l+1)] R_l(r) = 0 & r \leq r_0 \\ r^2 R_l''(r) + 2r R_l'(r) + [k^2 r^2 - l(l+1)] R_l(r) = 0 & r > r_0 \end{cases}$$

Posing $x = \begin{cases} k_0 r & r \leq r_0 \\ k r & r > r_0 \end{cases}$, we obtain a Helmholtz

equation in spherical coordinate:

$$x^2 R_l''(x) + 2x R_l'(x) + [x^2 - l(l+1)] R_l(x) = 0$$

whose solutions are the spherical Bessel functions

$$j_l\left(\frac{kr}{r_0}\right) \quad \text{and} \quad n_l\left(\frac{kr}{r_0}\right)$$

The former (regular spherical Bessel function) remains bound $\forall x$, while the latter (irregular spherical Bessel function) diverges for $x \rightarrow 0$ ($\sim x^{-(l+1)}$)

n_l is the Neumann spherical fct of order l

$$n_l = -y_l$$

In the internal part ($r < r_0$) the solution is thus

$$R_l(r) = A j_l(kr)$$

whereas, in the external part, the solution is the linear combination

$$R_l(r) = B j_l(kr) + C n_l(kr)$$

$$\text{Since } j_l(x) \xrightarrow{x \rightarrow \infty} \frac{1}{x} \sin\left(x - \frac{l\pi}{2}\right)$$

$$n_l(x) \xrightarrow{x \rightarrow \infty} \frac{1}{x} \cos\left(x - \frac{l\pi}{2}\right)$$

$$\text{it makes sense to pose } \begin{cases} B = A' \cos \delta_l \\ C = A' \sin \delta_l \end{cases}$$

$$\text{so that } R_l(kr) \xrightarrow{r \rightarrow \infty} \frac{A'}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$$

where δ_l is the phase shift.

reduced

In r_0 , the internal and external ∇ radial wave functions connect C' so

$$\frac{R_l'(r_0)}{R_l(r_0)} = \frac{R_l'(r_0^+)}{R_l(r_0^+)}$$

$$\Rightarrow \frac{A k_0 j_l'(k_0 r_0)}{A j_l(k_0 r_0)} = \frac{A' k \left[\cos \delta_l j_l'(k r_0) + \sin \delta_l n_l'(k r_0) \right]}{A' \left[\cos \delta_l j_l(k r_0) + \sin \delta_l n_l(k r_0) \right]}$$

$$\begin{aligned} \Rightarrow \tan \delta_l &= \frac{k_0 j_l(kr_0) j_l'(kr_0) - k j_l'(kr_0) j_l(kr_0)}{k n_l'(kr_0) j_l(kr_0) - k_0 n_l(kr_0) j_l'(kr_0)} \\ &= \frac{k j_l(kr_0) j_l'(kr_0) - k_0 j_l'(kr_0) j_l(kr_0)}{k_0 j_l'(kr_0) n_l(kr_0) - k j_l(kr_0) n_l'(kr_0)} \end{aligned}$$

For $l=0$, $j_0(x) = \frac{\sin x}{x}$ and $n_0(x) = \frac{\cos x}{x}$ ($= -y_0(x)$)

$$\begin{cases} j_0'(x) = \frac{x \cos x - \sin x}{x^2} \\ n_0'(x) = -\frac{x \sin x + \cos x}{x^2} \end{cases}$$

So we get

$$\begin{aligned} \tan \delta_0 &= \frac{\sin kr_0 (kr_0 \cos kr_0 - \sin kr_0) - (kr_0 \cos kr_0 - \sin kr_0) \times \sin kr_0}{-(kr_0 \cos kr_0 - \sin kr_0) \cos kr_0 - \sin kr_0 (kr_0 \sin kr_0 + \cos kr_0)} \\ &= \frac{kr_0 \cos kr_0 \sin kr_0 - kr_0 \cos kr_0 \sin kr_0}{kr_0 \cos kr_0 \cos kr_0 + kr_0 \sin kr_0 \sin kr_0} \\ &= \frac{k \tan kr_0 - k_0 \tan k r_0}{k_0 + k \tan k r_0 \tan k_0 r_0} \\ &= \tan \left[\tan^{-1} \left(\frac{k}{k_0} \tan k_0 r_0 \right) - k r_0 \right] \end{aligned}$$

which is what we have obtained earlier.

At low energy, the behaviour is

$$\begin{aligned} \tan \delta_l \xrightarrow{k \rightarrow 0} & \frac{k j_l(kr_0) \frac{l(kr_0)^{l-1}}{(2l+1)!!} - k_0 j_l'(kr_0) \frac{(kr_0)^l}{(2l+1)!!}}{k_0 j_l'(kr_0) \frac{(2l-1)!!}{(kr_0)^{l+1}} + k j_l(kr_0) \frac{(2l-1)!! (l+1)}{(kr_0)^{l+2}}} \\ &= \frac{1}{(2l+1)!! (2l-1)!!} k^{2l+1} \frac{l j_l(kr_0) r_0^{2l+1} - k_0 j_l'(kr_0) r_0^{2l+2}}{k_0 j_l'(kr_0) r_0 + (l+1) j_l(kr_0)} \end{aligned}$$

$$\Rightarrow j_l \delta_l \xrightarrow{k \rightarrow 0} (k \rho_0)^{2l+1} \frac{l j_l(k \rho_0) - k \rho_0 j_l'(k \rho_0)}{(2l+1)!! (2l-1)!! k \rho_0 j_l'(k \rho_0) + (l+1) j_l(k \rho_0)}$$

$$\Rightarrow \delta_l \xrightarrow{k \rightarrow 0} k^{2l+1} \xrightarrow{k \rightarrow 0} 0$$

The way how δ_l goes to 0 depends on the relative sign of the last factor.

This has been obtained from the small-argument behaviour of

$$j_l(x) \xrightarrow{x \rightarrow 0} \frac{x^l}{(2l+1)!!}$$

and

$$n_l(x) \xrightarrow{x \rightarrow 0} \frac{(2l-1)!!}{x^{l+1}}$$

Since

$$\left(\frac{1}{x} \frac{d}{dx} \right) (x^{l+1} j_l(x)) = x^l j_{l-1}(x)$$

$$\Leftrightarrow (l+1) x^l j_l(x) + x^{l+1} j_l'(x) = x^{l+1} j_{l-1}(x)$$

$$\Leftrightarrow (l+1) j_l(x) + x j_l'(x) = x j_{l-1}(x)$$

$$\Rightarrow k \rho_0 j_l'(k \rho_0) + (l+1) j_l(k \rho_0) = k \rho_0 j_{l-1}(k \rho_0)$$

And since

$$\frac{1}{x} \frac{d}{dx} x^{-l} j_l(x) = -x^{-(l+1)} j_{l+1}(x)$$

$$\Leftrightarrow -l x^{-(l+1)} j_l(x) + x^{-l} j_l'(x) = -x^{-(l+1)} j_{l+1}(x)$$

$$\Leftrightarrow x j_l'(x) - l j_l(x) = -x j_{l+1}(x)$$

$$\Rightarrow l j_l(k \rho_0) - k \rho_0 j_l'(k \rho_0) = k \rho_0 j_{l+1}(k \rho_0)$$

$$\Rightarrow \lim_{k \rightarrow 0} \tan \delta_l = \frac{(k r_0)^{2l+1}}{(2l+1)!! (2l-1)!!} \frac{k r_0 j_{l+1}(k r_0)}{k r_0 j_{l-1}(k r_0)}$$

Since all zeros of j_{l-1} are single valued there is no possibility that this expression takes an infinite value for $k \rightarrow 0$ for $l > 0$ because the factor k^{2l+1} will always force $\tan \delta_l \xrightarrow[k \rightarrow 0]{} 0$, but for the exception of $l=0$.

For a small, though finite value of k , we can have $\tan \delta_l \rightarrow \infty \Leftrightarrow \delta_l = \frac{\pi}{2} (+n\pi)$ i.e. we can have a resonance @ low energy if $k r_0 j_{l-1}(k r_0) = 0$

For $k \rightarrow 0$, this corresponds to the appearance of a bound state of energy 0 in the partial wave l .

It means that once again the slope of the phase shift will change from >0 to <0 when a bound-state appears in the partial wave.

Bound - state in a square - well for $l > 0$

The radial Schrödinger equation then reads

$$\begin{cases} r^2 R''(r) + 2r R'(r) + [K_0^2 - l(l+1)] R(r) = 0 & r \leq r_0 \\ r^2 R''(r) + 2r R'(r) - [K^2 + l(l+1)] R(r) = 0 & r > r_0 \end{cases}$$

where $-\frac{\hbar^2 K^2}{2\mu} = E < 0$ (bound state)

$\frac{\hbar^2 K_0^2}{2\mu} = V_0 + E > 0$ ("free wave" inside the well)

$$R(r) = \begin{cases} A j_l(K_0 r) & r \leq r_0 \\ B h_l^{(1)}(iKr) & r > r_0 \end{cases}$$

That function must be C^1 at $r = r_0$, hence in r_0 .
 $\Rightarrow$ the logarithmic derivative of R has to be continuous in r_0 .

$$\frac{A K_0 j_l'(K_0 r_0)}{A j_l(K_0 r_0)} = \frac{i B K h_l^{(1)'}(iK r_0)}{B h_l^{(1)}(iK r_0)}$$

$$\Rightarrow \frac{j_l'(K_0 r_0)}{j_l(K_0 r_0)} \frac{h_l^{(1)'}(iK r_0)}{i h_l^{(1)}(iK r_0)} = \frac{K}{K_0}$$

We look for the case in which the state appears, so when it is barely bound: $K \rightarrow 0$

$$\begin{aligned} h_l^{(1)'}(iK r_0) &= j_l(iK r_0) - i n_l(iK r_0) \xrightarrow{r_0 \rightarrow \infty} \frac{1}{iK r_0} \\ &\xrightarrow{K \rightarrow 0} \frac{(iK r_0)^l}{(2l+1)!!} - i \frac{(2l-1)!!}{(iK r_0)^{l+1}} \xrightarrow{K \rightarrow 0} -i \frac{(2l-1)!!}{(iK r_0)^{l+1}} \\ h_l^{(1)'}(iK r_0) &\xrightarrow{K \rightarrow 0} l \frac{(iK r_0)^{l-1}}{(2l+1)!!} + (l+1)i \frac{(2l-1)!!}{(iK r_0)^{l+2}} \xrightarrow{K \rightarrow 0} i(l+1) \frac{(2l-1)!!}{(iK r_0)^{l+2}} \end{aligned}$$

$$h_l^{(1)}(ikr_0) \xrightarrow{k \rightarrow 0} \frac{l+1}{ikr_0} \cdot \frac{(l-1)!!}{(ikr_0)^{l+1}}$$

$$\Rightarrow \frac{h_l^{(1)}(ikr_0)}{h_l^{(1)'}(ikr_0)} \xrightarrow{k \rightarrow 0} \frac{-i(2l-1)!!}{(ikr_0)^l} \cdot \frac{i(l+1)(2l-1)!!}{ikr_0 (ikr_0)^{l+1}} = -i \frac{k r_0}{l+1}$$

$$\Rightarrow i k \frac{h_l^{(1)'}(ikr_0)}{h_l^{(1)}(ikr_0)} \xrightarrow{k \rightarrow 0} -\frac{(l+1)}{r_0}$$

If instead we assume

$$R_l(kr) \xrightarrow{k \rightarrow 0} B \frac{e^{-kr}}{r^{l+1}}$$

absorbing the k, l and i dependence of the denominator inside the ANC B .

$$R_l'(kr) \xrightarrow{k \rightarrow 0} B \left[-k \frac{e^{-kr}}{r^{l+1}} - (l+1) \frac{e^{-kr}}{r^{l+2}} \right]$$

$$\frac{R_l'}{R_l} \Big|_{r=r_0} \xrightarrow{k \rightarrow 0} -k - \frac{(l+1)}{r_0} \xrightarrow{k \rightarrow 0} -\frac{(l+1)}{r_0} \quad (\text{same result})$$

We also know that

$$j_l'(x) = j_{l-1}(x) - \frac{(l+1)}{x} j_l(x)$$

$$\Rightarrow \frac{k_0 j_l'(k_0 r_0)}{j_l(k_0 r_0)} = k_0 \frac{j_{l-1}(k_0 r_0)}{j_l(k_0 r_0)} - \frac{(l+1)}{r_0}$$

$$\frac{k_0 j_l'(k_0 r_0)}{j_l(k_0 r_0)} \xrightarrow{k \rightarrow 0} i k \frac{h_l^{(1)}(ikr_0)}{h_l^{(1)'}(ikr_0)}$$

$$k_0 \frac{j_{l-1}(k_0 r_0)}{j_l(k_0 r_0)} = \cancel{\frac{l+1}{r_0}} = -\cancel{\frac{l+1}{r_0}}$$

$$\Rightarrow k_0 \frac{j_{l-1}(k_0 r_0)}{j_l(k_0 r_0)} = 0$$

Since $j_l(x)$ is always bounded, the condition to have a state bound by $E=0$ is to have $j_{l-1}(k_0 r_0) = 0$

$\Rightarrow$ from a barely unbound state ($E \gtrsim 0$) to a barely bound state ($E \lesssim 0$) if δl goes from $-\infty$ to $+\infty$.

At high energy

$$\tan \delta_l = \frac{k j_l(k r_0) j_l'(k r_0) - k_0 j_l'(k_0 r_0) j_l(k r_0)}{k_0 j_l'(k_0 r_0) n_l(k r_0) - k j_l(k r_0) n_l'(k r_0)}$$

$$\text{Since } j_l(x) \xrightarrow{x \rightarrow \infty} \frac{1}{x} \sin\left(x - l\frac{\pi}{2}\right)$$

$$n_l(x) \xrightarrow{x \rightarrow \infty} \frac{1}{x} \cos\left(x - l\frac{\pi}{2}\right)$$

We have

$$j_l'(x) \xrightarrow{x \rightarrow \infty} \frac{1}{x} \cos\left(x - l\frac{\pi}{2}\right) - \frac{1}{x^2} \sin\left(x - l\frac{\pi}{2}\right)$$

$$n_l'(x) \xrightarrow{x \rightarrow \infty} -\frac{1}{x} \sin\left(x - l\frac{\pi}{2}\right) - \frac{1}{x^2} \cos\left(x - l\frac{\pi}{2}\right)$$

$$\Rightarrow \tan \delta_l \xrightarrow{k \rightarrow \infty} \frac{\frac{1}{k_0 r_0^2} \sin(k_0 r_0 - l\frac{\pi}{2}) \cos(k r_0 - l\frac{\pi}{2}) - \frac{1}{k r_0^2} \cos(k_0 r_0 - l\frac{\pi}{2}) \sin(k r_0 - l\frac{\pi}{2})}{\frac{1}{k r_0^2} \cos(k_0 r_0 - l\frac{\pi}{2}) \cos(k r_0 - l\frac{\pi}{2}) + \frac{1}{k_0 r_0^2} \sin(k_0 r_0 - l\frac{\pi}{2}) \sin(k r_0 - l\frac{\pi}{2})}$$

$$\xrightarrow{k \rightarrow \infty} \frac{\left(\sin \frac{k}{k_0} \xrightarrow{k \rightarrow \infty} 1\right) \sin[(k_0 - k) r_0]}{\cos[(k_0 - k) r_0]} = \tan[(k_0 - k) r_0]$$

$$\text{Since } k_0 = \sqrt{k^2 + \frac{2\mu}{\hbar^2} V_0} = k \sqrt{1 + \frac{2\mu}{\hbar^2} \frac{V_0}{k^2}} \approx k \left(1 + \frac{\mu}{\hbar^2} \frac{V_0}{k^2}\right)$$

$$\Rightarrow k_0 - k \approx \frac{\mu}{\hbar^2} \frac{V_0}{k} \propto \frac{1}{\sqrt{E}}$$

$$\Rightarrow \tan \delta_e \xrightarrow{k \rightarrow \infty} \tan \frac{\mu V_0 R_0}{\hbar^2 k}$$

$$\Rightarrow \delta_e \xrightarrow{k \rightarrow \infty} \frac{1}{\sqrt{E}} \xrightarrow{k \rightarrow \infty} 0 \quad (+n\pi)$$

$\Rightarrow \delta_e$ goes to zero ($+n\pi$) at high energy, which is expected since then the potential becomes negligible and $\delta_e \rightarrow 0$.