Introduction to Scattering Theory II

Pierre Capel

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

26 July 2022

Exercise 1(a)

1. We consider the following square-well potential

$$V(r) = \begin{cases} -V_0 & r < r_0 \\ 0 & r \geq r_0 \end{cases} ,$$

where V_0 is a positive constant.
(a) We propose to solve the radial Schrödinger equation for a positive energy E in the s wave ($l = 0$). From this solution, deduce an analytical expression for the phase shift δ_0.

- What is the behaviour of δ_0 at low energy E?
  How can the different cases relate to the presence of a bound state hosted by the potential V in the s wave?
- What is the behaviour of δ_0 at high energy E?

$$\begin{cases} -u''_0(r) - \frac{2\mu}{\hbar^2} V_0 \, u_0(r) = k^2 u_0(r) & r \leq r_0 \\ -u''_0(r) = k^2 u_0(r) & \text{elsewhere} \end{cases}$$

$$\begin{cases} u''_0(r) = -k_0^2 u_0(r) & r \leq r_0 \\ u''_0(r) = -k^2 u_0(r) \end{cases}$$

$$E = \frac{\hbar^2 k^2}{2\mu} \qquad k_0^2 = k^2 + \frac{2\mu V_0}{\hbar^2}$$

$$\left(\begin{matrix} e^{-ik_0 r} \\ e^{ik_0 r} \end{matrix} \right)$$

$$u_0(r) = \begin{cases} A \sin k_0 r + \cancel{B}^{=0} \cos k_0 r & r \leq r_0 \\ A' \sin kr + B' \cos kr & r > r_0 \end{cases}$$

$$\boxed{u_0(0) = 0}$$

$$\Psi_{\ell m}(\vec{r}) = R_\ell(r) \, Y_\ell^m(\Omega)$$
$$= \frac{1}{r} u_\ell(r) \, Y_\ell^m(\Omega)$$

$$u_0(r) = \begin{cases} A \sin k_0 r & r \leqslant r_0 \\ C \sin(kr + \delta_0) & r > r_0 \end{cases}$$

$$\left.\frac{u_0'}{u_0}\right|_{r_0^-} = \left.\frac{u_0'}{u_0}\right|_{r_0^+}$$

$$\frac{\cancel{A}\, k_0 \cos k_0 r_0}{\cancel{A} \sin k_0 r_0} = \frac{\cancel{C}\, k \cos(k r_0 + \delta_0)}{\cancel{C} \sin(k r_0 + \delta_0)}$$

$$\Leftrightarrow \frac{1}{k_0} \operatorname{tg} k_0 r_0 = \frac{1}{k} \operatorname{tg}(k r_0 + \delta_0) = \frac{1}{k} \frac{\operatorname{tg} k r_0 + \operatorname{tg} \delta_0}{1 - \operatorname{tg} k r_0 \operatorname{tg} \delta_0}$$

$$\Rightarrow \delta_0 = \operatorname{Actg}\left(\frac{k}{k_0} \operatorname{tg} k_0 r_0\right) - k r_0 + n\pi \quad n \in \mathbb{Z}$$

or $\tan\delta_0 = \dfrac{k \tan k_0 r_0 - k_0 \tan k r_0}{k_0 + k \tan k_0 r_0 \tan k r_0}$

• **Low Energy Behaviour**

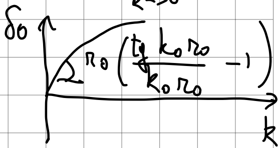
$$\tan\delta_0 \xrightarrow[k\to 0]{} k \, \frac{\tan k_0 r_0 - k_0 r_0}{k_0 + \cancel{k^2 r_0 \tan k_0 r_0}}$$

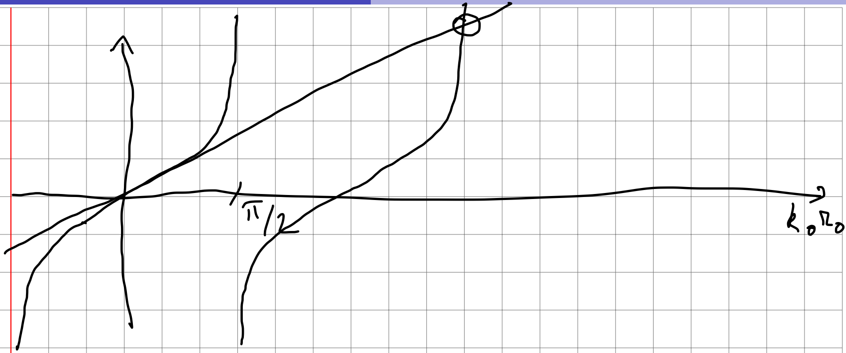
$$\simeq k \, r_0 \left(\frac{\tan k_0 r_0}{k_0 r_0} - 1 \right) \xrightarrow[k\to 0]{} 0$$

$\Rightarrow \delta_0 \xrightarrow[k\to 0]{} n\pi$

• $\delta_0 \longrightarrow 0^+$ if $\tan k_0 r_0 > k_0 r_0$

$\Leftrightarrow k_0 r_0 < \dfrac{\pi}{2}$





- $\delta_0 \to 0^- \ (\text{or } \pi^-)$ if $\frac{\pi}{2} < k_0 r_0 < \frac{3\pi}{2}$

$\lg k_0 r_0 \longrightarrow \infty \Rightarrow \delta_0 = \frac{\pi}{2} \; (+ n\pi)$

Bound state in the well

$$\begin{cases} -u_0''(r) - \frac{2\mu}{\hbar^2} V_0 \, u_0(r) = -k^2 u_0(r) & r \leq r_0 \\ \cancel{+} u_0''(r) = \cancel{+} k^2 u_0(r) & r > r_0 \end{cases}$$

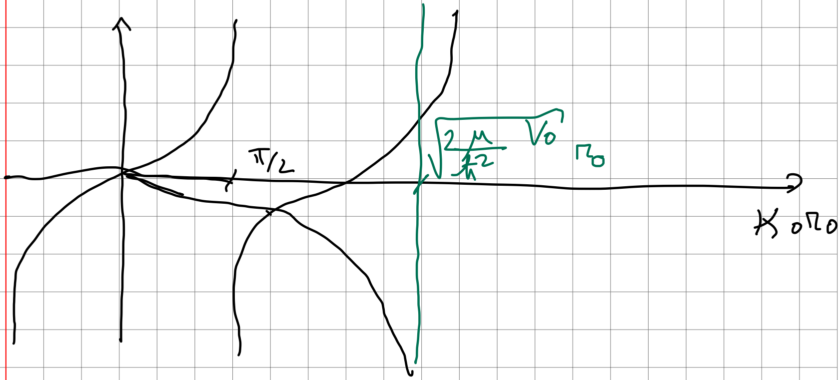
with $-\frac{\hbar^2 k^2}{2\mu} = E_0 \quad \frac{\hbar^2 k_0^2}{2\mu} = V_0 + E_0$

$$u_0(r) = \begin{cases} A \sin(k_0 r) & r \leq r_0 \\ \underbrace{C}\, e^{-kr} & r > r_0 \end{cases}$$

ANC = Asymptotic Normalisation Const

$$\frac{k_0 \cos k_0 r_0}{\sin k_0 r_0} = -k \, \frac{\cancel{e^{-k r_0}}}{\cancel{e^{-k r_0}}}$$

$$\Leftrightarrow \quad \operatorname{tg} k_0 r_0 = -\frac{k_0}{k} = -\frac{k_0}{\sqrt{\frac{2\mu V_0}{\hbar^2} - k_0^2}}$$



• Behaviour at high energy

$$\operatorname{tg}\delta_0 = \frac{k \operatorname{tg} k_0 r_0 - k_0 \operatorname{tg} k r_0}{k_0 + k \operatorname{tg} k_0 r_0 \operatorname{tg} k r_0}$$

$$\xrightarrow[k \to \infty]{} \operatorname{tg}(k_0 - k) r_0 \longrightarrow 0$$

$$\Rightarrow \delta_0 \xrightarrow[k \to \infty]{} n\pi = 0$$

Exercise 1(b)

1. We consider the following square-well potential

$$V(r) = \begin{cases} -V_0 & r < r_0 \\ 0 & r \geq r_0 \end{cases} ,$$

where V_0 is a positive constant.
(b) Extend the previous discussion to any partial wave $l > 0$.

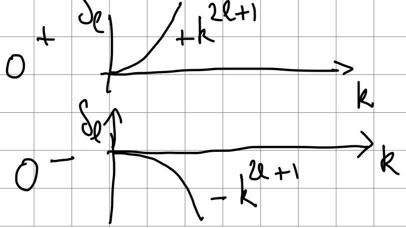
- What is the behaviour of δ_l at low energy E?
- What is the behaviour of δ_l at high energy E?

$$\operatorname{tg}\delta_\ell = \frac{k\, j_\ell(k_0 r_0)\, j_\ell'(k r_0) - k_0\, j_\ell'(k_0 r_0)\, j_\ell(k r_0)}{k_0\, j_\ell'(k_0 r_0)\, n_\ell(k r_0) - k\, j_\ell(k_0 r_0)\, n_\ell'(k r_0)}$$

$$\xrightarrow[k \to 0]{} k^{2\ell+1}$$

$$\Rightarrow \delta_\ell \xrightarrow[k \to 0]{} 0 + n\pi$$

$k_0 r_0\, j_{\ell-1}(k_0 r_0) = 0$ is the limit between



Exercise 2

2. An experimental cross section is well described by the following formula

$$\frac{d\sigma}{d\Omega} = a + b\cos\theta + c\cos^2\theta,$$

where a, b, and c are numerical parameters fitted to the data. This parametrisation remains valid at lower beam energies.

- What can be deduced about the phaseshifts at that energy ?
- Propose a way to test this hypothesis.
- What is the total scattering cross section ?
  Find the lowest number of measurements needed to infer it from the differential scattering cross section (assuming a perfect measurement without any uncertainty).
- Show that the formula remains valid when $\delta_0 \gg \delta_1 \gg \delta_2$, while δ_2 is not negligible.
  In that case, find a way to infer an experimental value for δ_0 and δ_1.

- $\delta_\ell \sim 0 \quad \forall \ell > 1$
- δ_0, δ_1 but a, b, c $\Rightarrow$ a, b and c not independent

$$b = \frac{2}{3} \underbrace{\overset{[L^{-2}]}{k^2}\, \overset{[L]}{a}\, \overset{[L^2]}{c}}_{[L^2]} \pm \underbrace{\sqrt{a - a^2 k^2}\ \sqrt{9c - c^2k^2}}_{[L^2]}$$

$$\sqrt{a - a^2 k^2} : [L] \qquad \sqrt{9c - c^2k^2}: \underbrace{\overset{[L^2]}{9c} - \overset{[L^4]}{c^2}\overset{[L^{-2}]}{k^2}}_{[L^2]} \to [L]$$

- $\sigma_{tot} = 4\pi \left(a + \frac{c}{3} \right)^{[L]}$

$$2' \quad \cos\theta = \pm \frac{\sqrt{3}'}{3}$$

$$\Rightarrow \sigma_{tot} = 2\pi \left(\left.\frac{d\sigma}{d\Omega}\right|_{A\cos\frac{1}{\sqrt{3}}} + \left.\frac{d\sigma}{d\Omega}\right|_{A\cos\frac{-1}{\sqrt{3}}} \right)$$

Relation between phase shift and potential

Let us consider two scattering potentials V and $\bar{V}$:

$$\begin{aligned}\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}V(r)\right]u_{kl}(r) &= 0 \\ \left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}\bar{V}(r)\right]\bar{u}_{kl}(r) &= 0\end{aligned}$$

whose solutions exhibit the following asymptotic behaviour

$$\begin{aligned}u_{kl}(r) &\underset{r\to\infty}{\longrightarrow} \frac{1}{k}\left[\cos\delta_l \sin(kr - l\pi/2) + \sin\delta_l \cos(kr - l\pi/2)\right] \\ \bar{u}_{kl}(r) &\underset{r\to\infty}{\longrightarrow} \frac{1}{k}\left[\cos\bar{\delta}_l \sin(kr - l\pi/2) + \sin\bar{\delta}_l \cos(kr - l\pi/2)\right]\end{aligned}$$

We consider the Wronskian of these two solutions

$$W(u_{kl}, \bar{u}_{kl}) = u_{kl}\bar{u}'_{kl} - u'_{kl}\bar{u}_{kl}$$

$$\frac{dW}{dr} = \cancel{u'_{kl}\bar{u}'_{kl}} + u_{kl}\bar{u}''_{kl} - u''_{kl}\bar{u}_{kl} - \cancel{u'_{kl}\bar{u}'_{kl}}$$

$$= -\frac{2\mu}{\hbar^2}\left[V(r) - \bar{V}(r)\right] u_{kl}\bar{u}_{kl}$$

$$\int_0^\infty \frac{dW}{dr}\,dr = -\frac{2\mu}{\hbar^2}\int_0^\infty \bar{u}_{kl}\left[V(r) - \bar{V}(r)\right] u_{kl}\,dr$$

$$= \left[W\right]_0^\infty \qquad u_{kl}(0) = \bar{u}_{kl}(0) = 0$$

$$= \lim_{r\to\infty}\left[u_{kl}(r)\bar{u}'_{kl}(r) - u'_{kl}(r)\bar{u}_{kl}(r)\right]$$

$$\vdots$$

$$= \frac{1}{k}\sin\left(\delta_l - \bar{\delta}_l\right)$$

$$\Rightarrow \sin\left(\delta_l - \bar{\delta}_l\right) = -\frac{2\mu}{\hbar^2}k\int_0^\infty \bar{u}_{kl}\left[V(r) - \bar{V}(r)\right] u_{kl}\,dr$$

1) $\bar{V} = 0$

$$\sin(\delta_\ell) = -\frac{2\mu}{\hbar^2} k \int_0^\infty j_\ell(kr)\, V(r)\, R_\ell(r)\, r^2\, dr$$

2) $V(\lambda, r)$ $\qquad \lambda \equiv$ strength parameter

$\bar{V}(\bar{\lambda}, r)$ $\qquad \bar{\lambda} = \lambda + d\lambda$

$$\frac{d\delta_\ell}{d\lambda} = -\frac{2\mu}{\hbar^2} k \int_0^\infty u_{k\ell}^2(\lambda, r) \frac{\partial V(\lambda, r)}{\partial \lambda}\, dr$$

$$\Delta \delta_\ell \propto - \Delta V$$

Phase shift for a general potential

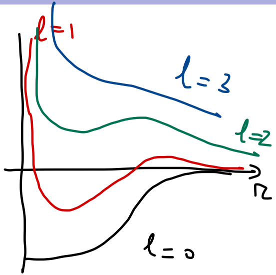
We consider a potential with a finite range r_0
(or negligible beyond that range)
The solution R_{kl} in the internal region $(r < r_0)$
must be C^1 with the asymptotic expression

$$R_{kl}^{\text{as}}(r) = c_l \left[\cos\delta_l\, j_l(kr) + \sin\delta_l n_l(kr)\right]$$

To determine δ_l, the easiest is to consider the logarithmic derivative

$$\begin{aligned}
\gamma_l &= \left.\frac{R'_{kl}}{R_{kl}}\right|_{r_0} \\
&= k\frac{\cos\delta_l\, j'_l(kr_0) + \sin\delta_l\, n'_l(kr_0)}{\cos\delta_l\, j_l(kr_0) + \sin\delta_l\, n_l(kr_0)} \\
&= k\frac{j'_l(kr_0) + \tan\delta_l\, n'_l(kr_0)}{j_l(kr_0) + \tan\delta_l\, n_l(kr_0)} \\
\Leftrightarrow \tan\delta_l &= \frac{k\, j'_l(kr_0) - \gamma_l\, j_l(kr_0)}{\gamma_l n_l(kr_0) - kn'_l(kr_0)}
\end{aligned}$$

Phase shift at low-energy



$$j_l(x) \xrightarrow[x\to 0]{} \frac{x^l}{(2l+1)!!}$$
$$n_l(x) \xrightarrow[x\to 0]{} \frac{(2l-1)!!}{x^{l+1}}$$

Therefore

$$\tan\delta_l = -\frac{j_l(kr_0)}{n_l(kr_0)} \frac{\frac{kj_l'(kr_0)}{j_l(kr_0)} - \gamma_l}{\frac{kn_l'(kr_0)}{n_l(kr_0)} - \gamma_l}$$
$$\xrightarrow[k\to 0]{} \frac{(kr_0)^{2l+1}}{(2l+1)!!(2l-1)!!} \frac{l - r_0\gamma_l}{(l+1) + r_0\gamma_l}$$
$$\Rightarrow \quad \delta_l \xrightarrow[k\to 0]{} 0 \ \ (+n\pi; n \in \mathbb{Z}) \text{ as } k^{2l+1}$$

(unless $(l+1) + r_0\gamma_l = 0$)

Notion of scattering length

In the s wave, we have

$$\tan\delta_0 \quad \underset{k\to 0}{\longrightarrow} \quad 0 \text{ as } k,$$

The factor of proportionality is defined as the scattering length

$$a \quad = \quad -\lim_{k\to 0}\frac{\tan\delta_0}{k}$$

The contribution of each partial wave to the cross section is

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \quad \underset{k\to 0}{\longrightarrow} k^{4l+2}$$

At very low energy, only the s wave contributes and $\sigma_{\text{tot}} = \sigma_0 = 4\pi a^2$

Actually $\frac{d\sigma}{d\Omega} \underset{k\to 0}{\longrightarrow} a^2$: the cross section is isotropic

For the square well potential,

$$a = r_0\left(1 - \frac{\tan k_0 r_0}{k_0 r_0}\right)$$

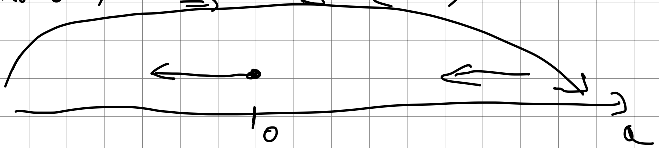
$$k_0 r_0 \ll 1 \Rightarrow a < 0$$

$$k_0 r_0 \nearrow \Rightarrow -a \nearrow$$

$$k_0 r_0 \to \frac{\pi}{2} \Rightarrow a \to -\infty$$

$$k_0 r_0 = \frac{\pi}{2}^+ \Rightarrow a \to +\infty$$

$$k_0 r_0 \nearrow \Rightarrow a \searrow \quad (>0)$$



Bibliography

The following books are good references for more details on low-energy nuclear-reaction theory :

- C. J. Joachain *Quantum Collision Theory* (North-Holland, Amsterdam, 1975)
- C. Cohen-Tannoudji, B. Diu & F. Laloë *Quantum Mechanics*, Vol.2 (John Wiley & Sons, Paris, 1977)
- C. A. Bertulani & P. Danielewicz *Introduction to Nuclear Reations* (Institute of Physics, London, 2004)
- I. J. Thompson & F. M. Nunes *Nuclear Reactions for Astrophysics : Principles, Calculation and Applications of Low-Energy Reactions* (Cambridge University Press, 2009)
- J. R. Taylor *Scattering Theory : The Quantum Theory of Nonrelativistic Collisions* (Dover, New York, 1972)