# Introduction to Scattering Theory

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# Outline

1. Monday
   - Stationary scattering states and elastic-scattering cross section
   - Partial-wave expansion and notion of phase shift
   - PM : Exercise session on phase shifts
2. Tuesday
   - Low- and high-energy behaviour of phase shift
   - Effective-range expansion
   - Notion of resonance
3. Wednesday
   - $S$ Matrix
   - Coulomb scattering

1. Quantum Collisions

2. Notion of cross section
   - Definition
   - Theoretical framework
   - Stationary Scattering States
   - Theoretical scattering cross section

3. Partial-wave expansion and phase shift
   - Partial-wave expansion
   - Phase shift
   - Scattering cross section

# Quantum Collisions

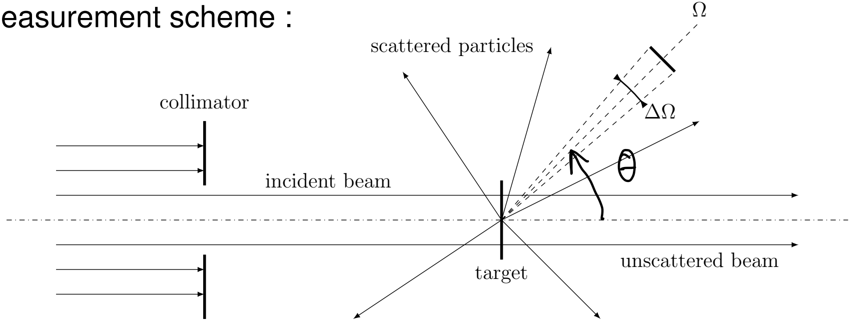
Quantum collisions used to

- study the interaction between particles/nuclei/atoms...

  (see Hans-Werner Hammer & Kai Hebeler's classes)
- analyse the structure of particles/nuclei/atoms...

  (see Sonia Bacca & Daniel Phillips' classes)
- measure reaction rates of particular interest
  (stars, nuclear reactors, production of radioactive isotopes...)

Measurement scheme :




a mere theorist's viewpoint

# Reaction types

Various reactions can happen :

1. $a + b \to a + b$ (elastic scattering ; this series of lectures)
2. $\to a + b^*$ (inelastic scattering)
3. $\to c + f + b$ (breakup ; see Week 3)
4. $\to d + e$ (rearrangement or transfer)

Examples :

1. ${}^{11}\mathrm{Be} + {}^{208}\mathrm{Pb} \to {}^{11}\mathrm{Be} + {}^{208}\mathrm{Pb}$ (elastic scattering)
2. $\to {}^{11}\mathrm{Be}^*(1/2^-) + {}^{208}\mathrm{Pb}$ (inelastic scattering)
3. $\to {}^{10}\mathrm{Be} + \mathrm{n} + {}^{208}\mathrm{Pb}$ (breakup) week 3.
4. $\to {}^{10}\mathrm{Be} + {}^{209}\mathrm{Pb}$ (transfer)

# Energy conservation

Total energy is conserved :
$m_a c^2 + m_b c^2$ + incident kinetic energy
= mass of products $c^2$ + kinetic energy

The $Q$ value is the energy “produced” by the reaction

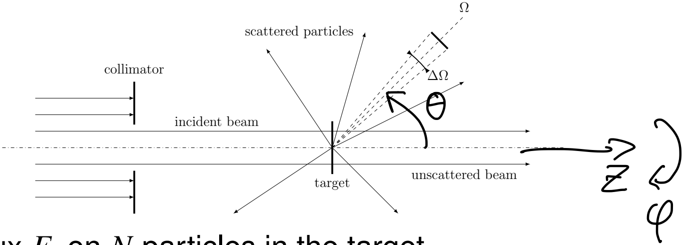
$$
\begin{aligned}
m_a c^2 + m_b c^2 &= \text{mass of products } c^2 + Q \\
\Leftrightarrow \quad Q &= m_a c^2 + m_b c^2 - \text{mass of products } c^2
\end{aligned}
$$

$Q > 0$ : exoenergetic, always (energetically) possible
$Q < 0$ : endoenergetic, requires a minimal incident kinetic energy

A channel can be open if the incident kinetic energy $> -Q$
otherwise the channel is closed

The elastic channel is always open ($Q = 0$)

# Cross section




For an incident flux $F_i$ on $N$ particles in the target,
the number $\Delta n$ of events detected in direction $\Omega = (\theta, \varphi)$
per unit time within the solid angle $\Delta\Omega$ is proportional to $F_i$ and $N$ :

$$\Delta n = F_i \; N \; \Delta\sigma$$

The factor of proportionality $\sigma$ is the cross section
$[\Delta\sigma]$ = surface ; unit : barn $1 \text{ b} = 10^{-24}\text{cm}^2$
The differential cross section

$$\frac{d\sigma}{d\Omega} = \lim_{\Delta\Omega\to 0} \frac{\Delta\sigma}{\Delta\Omega} = \frac{\Delta n}{F_i N \Delta\Omega}$$

Taking $Z$ as the beam axis, $d\sigma/d\Omega$ depends only on $\theta$ by symmetry

# Theoretical framework

Let us consider particles $a$ and $b$ of mass $m_a$ and $m_b$
interacting through potential $V(\boldsymbol{r})$,
where $\boldsymbol{r} = \boldsymbol{r}_a - \boldsymbol{r}_b$ is the $a$-$b$ relative coordinate

The Hamiltonian reads

$$\mathcal{H} = T_a + T_b + V(\boldsymbol{r}),$$

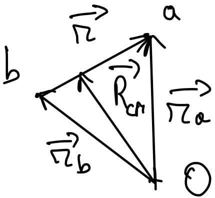
where

$$\begin{aligned} T_a &= \frac{p_a^2}{2m_a} = -\frac{\hbar^2 \Delta_{r_a}}{2m_a} \\ T_b &= \frac{p_b^2}{2m_b} = -\frac{\hbar^2 \Delta_{r_b}}{2m_b} \end{aligned}$$

# Change to cm and $a$-$b$ relative motion

The coordinate change

$$\begin{aligned}\boldsymbol{R}_{\rm cm} &= (m_a \boldsymbol{r}_a + m_b \boldsymbol{r}_b)/M \\ \boldsymbol{r} &= \boldsymbol{r}_a - \boldsymbol{r}_b\end{aligned}$$




with $M = m_a + m_b$, the total mass, leads to

$$\mathcal{H} = T_{\rm cm} + T_r + V(\boldsymbol{r}),$$

where

$$\begin{aligned}T_{\rm cm} &= \frac{P_{\rm cm}^2}{2M} = -\frac{\hbar^2 \Delta_{R_{\rm cm}}}{2M} \\ T_r &= \frac{p^2}{2\mu} = -\frac{\hbar^2 \Delta_r}{2\mu},\end{aligned}$$

with $\boldsymbol{P}_{\rm cm} = \boldsymbol{p}_a + \boldsymbol{p}_b$, $\boldsymbol{p} = (m_b \boldsymbol{p}_a - m_a \boldsymbol{p}_b)/M = \mu\, \boldsymbol{v}$,
and $\mu = m_a m_b/M$ the reduced mass of $a$ and $b$

$\mathcal{H}$ is then the sum of two Hamiltonians : $H_{cm}(\boldsymbol{R}_{cm}) + H(\boldsymbol{r})$
Hence the two-body wave function factorises

$$\Psi_{\rm tot}(\boldsymbol{r}_a, \boldsymbol{r}_b) = \Psi_{cm}(\boldsymbol{R}_{cm})\, \Psi(\boldsymbol{r})$$

# cm motion

The cm wave function is solution of

$$H_{cm}\,\Psi_{cm}(\boldsymbol{R}_{cm}) = E_{cm}\,\Psi_{cm}(\boldsymbol{R}_{cm}),$$

where

$$H_{cm} = \frac{P_{cm}^2}{2M}$$

is the Hamiltonian of a free particle of mass $M$

The cm motion is described by a plane wave

$$\Psi_{\boldsymbol{K}_{cm}}(\boldsymbol{R}_{cm}) = (2\pi)^{-3/2} e^{i\boldsymbol{K}_{cm}\cdot\boldsymbol{R}_{cm}}$$

with $E_{cm} = \hbar^2 K_{cm}^2/2M$

The factor $(2\pi)^{-3/2}$ is chosen such that

$$\langle\Psi_{\boldsymbol{K'}_{cm}}|\Psi_{\boldsymbol{K}_{cm}}\rangle = \delta(\boldsymbol{K}_{cm} - \boldsymbol{K'}_{cm})$$

# Stationary Scattering States

The Hamiltonian of the $a$-$b$ relative motion reads

$$H = T_r + V(\boldsymbol{r})$$

where $T_r = -\frac{\hbar^2 \Delta_r}{2\mu}$

This is the Hamiltonian for a virtual particle of mass $\mu = \frac{m_a m_b}{m_a + m_b}$ evolving within the external potential well $V$

A stationary scattering state $\Psi_{\boldsymbol{k}}(\boldsymbol{r})$ is solution of

$$H\,\Psi_{\boldsymbol{k}} = E\,\Psi_{\boldsymbol{k}}$$

where $E = \hbar^2 k^2/2\mu$
with the asymptotic behaviour

$$\Psi_{k\hat{\boldsymbol{z}}}(\boldsymbol{r}) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left[e^{ikz} + f_k(\theta)\,\frac{e^{ikr}}{r}\right],$$

with $\hat{z}$ chosen as the beam axis
$f_k$ is the scattering amplitude

# Physical interpretation

To interpret the stationary scattering state

$$\Psi_{k\hat{z}}(\boldsymbol{r}) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left[e^{ikz} + f_k(\theta)\,\frac{e^{ikr}}{r}\right],$$

let us recall the probability current

$$\boldsymbol{j}(\boldsymbol{r}) = \frac{1}{\mu}\Re\left[\Psi^*(\boldsymbol{r})\,\boldsymbol{p}\,\Psi(\boldsymbol{r})\right]$$

$$\vec{j}_i = \frac{1}{\mu}\, Re\left\{(2\pi)^{-3}\, e^{-ikz} \left(-i\hbar\vec{\nabla}\right) e^{ikz}\right\}$$

$$= \frac{\hbar (2\pi)^{-3}}{\mu} Re\left\{\cancel{e^{-ikz}}\, k\, \cancel{e^{ikz}}\, \vec{1}_z\right\}$$

$$= \frac{\hbar k}{\mu}(2\pi)^{-3}\,\vec{1}_z = (2\pi)^{-3}\,\vec{v}_i \propto F_i$$

$$\vec{j}_s = \frac{(2\pi)^{-3}}{\mu} \mathrm{Re} \left\{ \frac{e^{-ikr}}{r} f^*(\theta) \left(-i\hbar\right) \vec{\nabla} f(\theta) \frac{e^{ikr}}{r} \right\}$$

$$= \frac{\hbar (2\pi)^{-3}}{\mu} \mathrm{Re} \left\{ \frac{\cancel{e^{-ikr}}}{r} f^*(\theta) \cancel{(-i)} f(\theta) \cancel{i} k \frac{\cancel{e^{ikr}}}{r} \vec{1}_r \right\}$$

$$= \frac{\hbar k}{\mu} (2\pi)^{-3} \left| f(\theta) \right|^2 \frac{\vec{1}_r}{r^2} + O\left(\frac{1}{r^3}\right) + O\left(\frac{1}{r^3}\right)$$

$$= (2\pi)^{-3} \left| f(\theta) \right|^2 \frac{v}{r^2} \vec{1}_r + O\left(\frac{1}{r^3}\right)$$

# Physical interpretation

To interpret the stationary scattering state

$$\Psi_{k\hat{\boldsymbol{z}}}(\boldsymbol{r}) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left[e^{ikz} + f_k(\theta)\,\frac{e^{ikr}}{r}\right],$$

let us recall the probability current

$$\boldsymbol{j}(\boldsymbol{r}) = \frac{1}{\mu}\Re[\Psi^*(\boldsymbol{r})\,\boldsymbol{p}\,\Psi(\boldsymbol{r})]$$

The plane wave describes the incoming current

$$\boldsymbol{j}_i(\boldsymbol{r}) = (2\pi)^{-3}\frac{\hbar k}{\mu}\hat{\boldsymbol{z}} = (2\pi)^{-3}\,v\,\hat{\boldsymbol{z}}$$

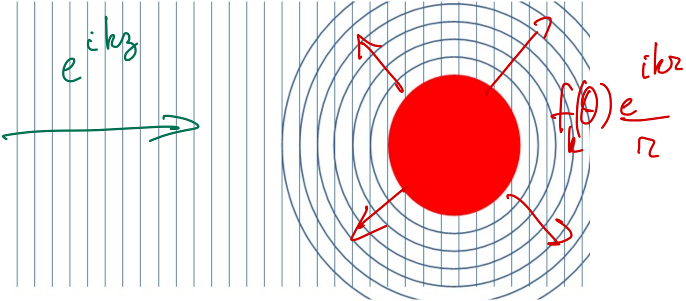
where $v$ is the $a$-$b$ relative velocity

The spherical wave $f_k(\theta)\frac{e^{ikr}}{r}$ describes the scattered current

$$\boldsymbol{j}_s(\boldsymbol{r}) = (2\pi)^{-3}\,v\,|f_k(\theta)|^2\frac{1}{r^2}\hat{\boldsymbol{r}} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

is purely radial at $R\to\infty$ ; directed outwards ; $\propto v$ but varies with $\theta$

# Physical interpretation




Incoming wave :

$$e^{ikz} \to \boldsymbol{j}_i(\boldsymbol{r}) \propto v\,\hat{\boldsymbol{z}}$$

Scattered wave :

$$f_k(\theta)\,\frac{e^{ikr}}{r} \to \boldsymbol{j}_s(\boldsymbol{r}) \propto v\,|f_k(\theta)|^2\,\frac{1}{r^2}\hat{\boldsymbol{r}}$$

# Theoretical scattering cross section

We can assume the incoming flux

$$F_i = C\ j_i$$

The scattered flux in direction $\Omega$ is then

$$F_s(\Omega) = C\ j_s(\Omega)$$

For one scattering nucleus,
the number of events per unit time detected in direction $\Omega$ reads

$$\Delta n = F_i \underbrace{N}_{=1} \Delta\sigma$$

$$\Leftrightarrow \Delta\sigma = \frac{\Delta n}{F_i\ N}$$

$$\begin{aligned} dn = F_s(\Omega)\, dS &= C\ j_s(\Omega)\ r^2 d\Omega \\ \Rightarrow \frac{d\sigma}{d\Omega} &= \frac{dn}{F_i\, d\Omega} \\ &= \frac{r^2\, j_s(\Omega)}{j_i} \\ &= |f_k(\theta)|^2 \end{aligned}$$

The scattering amplitude $f_k$ contains all information about $V$

# Partial-wave expansion

If the potential does not depend on $\Omega$, i.e. $V(r)$,

$$\begin{aligned}[H, L^2] &= 0 \\ [H, L_Z] &= 0\end{aligned}$$

the angular motion is described by spherical harmonics

$$\Psi(\boldsymbol{r}) = \frac{1}{r}\sum_{l,m} u_{kl}(r)\; Y_l^m(\Omega),$$

where $u_{kl}$ is solution of the radial equation

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}V(r) + k^2\right)u_{kl}(r) = 0$$

with $\frac{\hbar^2 k^2}{2\mu} = E$

# Free wave

Let's start with no interaction : $V(r) = 0 \quad \forall r$
In that case $u_{kl}$ is solution of

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right) u_{kl}(r) = 0$$

If we pose $u_{kl}(r) = x\, f_l(x)$, with $x = kr$, the equation becomes

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + 1 - \frac{l(l+1)}{x^2}\right) f_l(x) = 0$$

which is a spherical Bessel equation (aka Helmholtz equation) whose solutions are the spherical Bessel functions $j_l$ and $n_l$

$n_l$ is sometimes called the spherical Neumann function

# Spherical Bessel functions

$$
\begin{aligned}
j_l(x) &= (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} & \qquad n_l(x) &= (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x} \\
j_0(x) &= \frac{\sin x}{x} & n_0(x) &= \frac{\cos x}{x} \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} & n_1(x) &= \frac{\cos x}{x^2} + \frac{\sin x}{x} \\
j_l(x) &\xrightarrow[x\to 0]{} \frac{x^l}{(2l+1)!!} & n_l(x) &\xrightarrow[x\to 0]{} \frac{(2l-1)!!}{x^{l+1}} \\
j_l(x) &\xrightarrow[x\to \infty]{} \frac{1}{x}\sin(x - l\frac{\pi}{2}) & n_l(x) &\xrightarrow[x\to \infty]{} \frac{1}{x}\cos(x - l\frac{\pi}{2})
\end{aligned}
$$

Recurrence relations : for $f_l = j_l$ or $n_l$,

$$
\begin{aligned}
x f_l'(x) &= x f_{l-1}(x) - (l+1) f_l(x) \\
x f_l'(x) &= l f_l(x) - x f_{l+1}(x) \\
(2l+1) x^{-1} f_l(x) &= f_{l+1}(x) + f_{l-1}(x)
\end{aligned}
$$

# Plane waves and spherical Bessel functions

When $V(r) = 0 \quad \forall r$, the plane wave $e^{ikz}$ (so with $f_k(\theta) = 0$) is also a solution of the Schrödinger equation, so

$$e^{ikz} = \sum_{l=0}^{\infty} \left[a_l \; j_l(kr) + \cancel{b_l} \; n_l(kr)\right] Y_l^0(\Omega)$$

$b_l = 0 \;\; \forall l$ since $n_l(kr) \underset{r\to 0}{\longrightarrow} \frac{(2l-1)!!}{(kr)^{l+1}}$

For $a_l$, one gets

$$a_l = i^l \sqrt{4\pi(2l+1)}$$

and

$$\begin{aligned} e^{ikz} &= \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} \; j_l(kr) \; Y_l^0(\Omega) \\ &\underset{r\to\infty}{\longrightarrow} \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} \frac{1}{kr} \sin(kr - l\pi/2) \; Y_l^0(\Omega) \end{aligned}$$

# Distorted wave

Switching on the $a$-$b$ interaction ($V \neq 0$),
we have to solve the radial equation

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}V(r) + k^2\right)u_{kl}(r) = 0$$

with $\frac{\hbar^2 k^2}{2\mu} = E$
But for a “few” cases (see exercises this afternoon)
it is done numerically (see exercises new week with N. Barnea and S. Bacca)

To compute the cross section from the solution $u_{kl}$
we need the scattering amplitude $f_k(\theta)$
which is deduced from the asymptotics of the wave function

If we assume $r^2 V(r) \underset{r\to\infty}{\longrightarrow} 0$, $u_{kl}(r) \underset{r\to\infty}{\longrightarrow} u_{kl}^{\mathrm{as}}(r)$, which is solution of

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right)u_{kl}^{\mathrm{as}}(r) = 0$$

# Phase shift $\delta_l$

The solutions of

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right) u_{kl}^{\mathrm{as}}(r) = 0$$

are once again spherical Bessel functions

$$\begin{aligned} u_{kl}^{\mathrm{as}}(r) &= a_l\, kr\ j_l(kr) + b_l\, kr\, n_l(kr) \\ &\underset{r\to\infty}{\longrightarrow} a_l\ \sin(kr - l\pi/2) + b_l\ \cos(kr - l\pi/2) \end{aligned}$$

Posing $a_l = c_l \cos\delta_l$ and $b_l = c_l \sin\delta_l$

$$u_{kl}^{\mathrm{as}}(r) \underset{r\to\infty}{\longrightarrow} c_l\ \sin(kr - l\pi/2 + \delta_l)$$

$c_l$ is just a normalisation factor
$\delta_l$ is the phaseshift : it contains all information on the influence of $V$

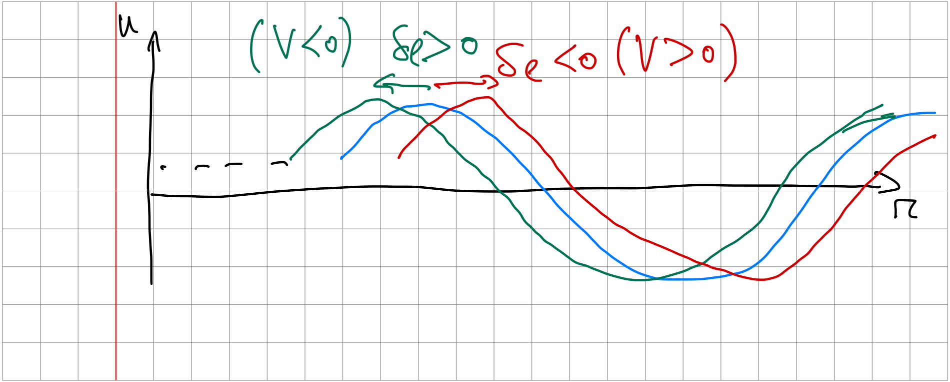
# Phase shift : Physical interpretation

The asymptotics of the distorted wave $(V \neq 0)$

$$u_{kl}^{\text{as}}(r) \underset{r\to\infty}{\longrightarrow} c_l \; \sin(kr - l\pi/2 + \delta_l)$$

is phase shifted compared to the plane wave $(V = 0)$

$$a_l \, kr \; j_{kl}(r) \underset{r\to\infty}{\longrightarrow} a_l \; \sin(kr - l\pi/2)$$



# Phase shift : Physical interpretation

The asymptotics of the distorted wave

$$u^{\mathrm{as}}_{kl}(r) \underset{r\to\infty}{\longrightarrow} c_l \ \sin(kr - l\pi/2 + \delta_l)$$

is phase shifted compared to the plane wave

$$a_l\, kr\ j_{kl}(r) \underset{r\to\infty}{\longrightarrow} a_l \ \sin(kr - l\pi/2)$$

- $V(r) = 0 \quad \forall r \ \Rightarrow \ \delta_l = 0$ : no shift
  the colliding particles don't see each other
- $V(r) < 0 \quad \forall r \ \Rightarrow \ \delta_l > 0$ : wave attracted by the field
  the colliding particles tend to stick together
- $V(r) > 0 \quad \forall r \ \Rightarrow \ \delta_l < 0$ : wave repulsed by the field
  the colliding particles repulse each other

(see exercises this afternoon)

# Scattering matrix $S_l$

Another way of seeing this is to interpret

$$u_{kl}^{\mathrm{as}}(r) \underset{r\to\infty}{\longrightarrow} c_l \sin(kr - l\pi/2 + \delta_l)$$

in terms of incoming and outgoing (spherical) waves :

$$u_{kl}(r) \quad \underset{r\to\infty}{\longrightarrow} \quad i\; c_l\; \frac{e^{-i\delta_l}}{2} \left[ e^{-i(kr - l\pi/2)} - S_l\; e^{i(kr - l\pi/2)} \right]$$

where

$$S_l \quad = \quad e^{2i\delta_l}$$

is the scattering matrix or $S$ matrix

The outgoing wave is shifted from the incoming wave by $2\delta_l$
due to the effect of $V \Rightarrow$ used to compute the scattering amplitude $f_k$

# Scattering amplitude

The scattering wave function can be expanded in partial waves

$$
\begin{aligned}
(2\pi)^{3/2}\Psi_{k\hat{\boldsymbol{z}}}(\boldsymbol{r}) &= \frac{1}{kr}\sum_{l=0}^{\infty} u_{kr}(r)\; Y_l^0(\Omega) \\
&\xrightarrow[r\to\infty]{} e^{ikz} + f_k(\theta)\frac{e^{ikr}}{r}
\end{aligned}
$$

Since $e^{ikz} \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)}\frac{i}{2kr}\left[e^{-i(kr-l\pi/2)} - e^{i(kr-l\pi/2)}\right] Y_l^0(\Omega)$

and $\quad u_{kl} \xrightarrow[r\to\infty]{} i\; c_l\; \frac{e^{-i\delta_l}}{2}\left[e^{-i(kr-l\pi/2)} - S_l\; e^{i(kr-l\pi/2)}\right]$

comparing the incoming waves we obtain $c_l = i^l\sqrt{4\pi(2l+1)}\; e^{i\delta_l}$
and deduce the scattering amplitude from $S_l = e^{2i\delta_l}$

$$
f_k(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}(S_l - 1)Y_l^0(\Omega) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l - 1)P_l(\cos\theta)
$$

# Scattering cross section

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2$$

$$= \left| \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(S_l - 1) P_l(\cos\theta) \right|^2$$

After integration over $\Omega$ the total scattering cross section reads

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

$$\sigma = \int_{4\pi\,\mathrm{str.}} d\Omega \, \frac{1}{4k^2} \sum_{\ell\ell'} (2\ell+1)(2\ell'+1)(S_\ell - 1)(S_{\ell'}^* - 1) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$= \frac{2\pi}{4k^2} \sum_{\ell\cancel{\ell'}} \cancel{(2\ell+1)}(2\ell'+1)(S_\ell - 1)(S_\ell - 1)^* \frac{2\cancel{\delta_{\ell\ell'}}}{\cancel{2\ell+1}}$$

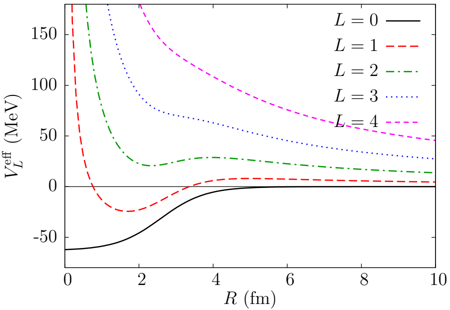
# Contribution of partial waves

Each partial wave contributes to $\sigma$ but with variable importance

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

Centrifugal barrier $\frac{l(l+1)}{r^2}$ ensures $\delta_l \underset{l\to\infty}{\longrightarrow} 0$

$\Rightarrow$ limited sum




At very low $E$, only $l = 0$ contributes and $\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0$

At large $E$ many partial waves must be included to reach convergence ($\Rightarrow$ low-energy method)

# How to compute $\delta_l$ ?

For short-range potentials, space can be divided in two regions :

- External one, where $V(r) \approx 0 \quad \forall r > r_0$, where $u_{kl}(r)$ is

$$u_{kl}^{\text{as}}(r) = c_l \left[\cos\delta_l \; kr \; j_l(kr) + \sin\delta_l \; kr \; n_l(kr)\right] \underset{r\to\infty}{\longrightarrow} c_l \sin(kr - l\pi/2 + \delta_l)$$

- Internal one ($r < r_0$), where $V$ is non-negligible and where we have to solve the radial equation

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2} V(r) + k^2\right) u_{kl}^{\text{int}}(r) = 0$$

e.g. numerically (see exercises next week with N. Barnea and S. Bacca)

At the boundary

$$u_{kl}^{\text{int}}(r_0) = u_{kl}^{\text{as}}(r_0) \quad \text{and} \quad u_{kl}^{\text{int}\prime}(r_0) = u_{kl}^{\text{as}\prime}(r_0)$$

$$\text{or} \quad \left.\frac{u_{kl}^{\text{int}\prime}}{u_{kl}^{\text{int}}}\right|_{r_0} = \left.\frac{u_{kl}^{\text{as}\prime}}{u_{kl}^{\text{as}}}\right|_{r_0} = \frac{k\cos(kr_0 - l\pi/2 + \delta_l)}{\sin(kr_0 - l\pi/2 + \delta_l)},$$

which gives access to the phase shift $\delta_l$ (see exercises this afternoon)

# How to infer $\delta_l$ from data ?

Because the elastic-scattering cross section is a function of $\delta_l$

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(S_l - 1) P_l(\cos\theta) \right|^2,$$

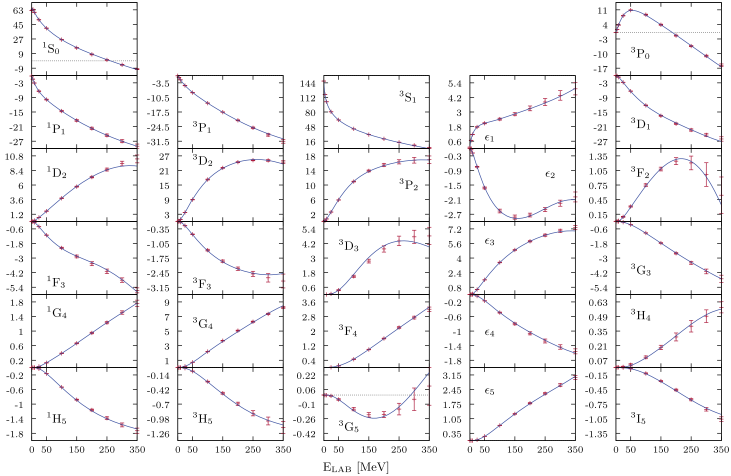
experimental phase shifts can be obtained from data

A development of the differential cross section
in Legendre polynomials $P_l$ provides $\delta_l$ “experimentally”

(see exercises this afternoon)

# Example : p-n phaseshifts

Notation : ${}^{2S+1}L_J$




[Navarro Pérez, Amaro, Ruiz Arriola, PLB 724 138 (2013)]

Data used to fit the various terms of $V_{NN}$ (see lectures of Kai Hebeler)

# Summary

- Collisions used to study interaction and structure of “particles”
- Notion of cross section used to characterise a process in quantum collision theory

$$\frac{d\sigma}{d\Omega} = \lim_{\Delta\Omega\to 0} \frac{\Delta n}{F_i N \Delta\Omega}$$

- Computing stationary scattering state

$$\Psi_{k\hat{\boldsymbol{z}}}(\boldsymbol{r}) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left[e^{ikz} + f_k(\theta)\frac{e^{ikr}}{r}\right],$$

gives the scattering cross section

$$\frac{d\sigma}{d\Omega} \quad = \quad |f_k(\theta)|^2$$

# Summary (2)

- In partial-wave expansion, defining phaseshifts

$$u_{kl}^{\mathrm{as}}(r) \underset{r\to\infty}{\longrightarrow} c_l \sin(kr - l\pi/2 + \delta_l)$$

the scattering cross section reads

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)(S_l - 1) P_l(\cos\theta) \right|^2$$

- Phase shifts $\delta_l$ obtained by solving the radial equation for $r < r_0$

$$\left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2} V(r) + k^2 \right) u_{kl}^{\mathrm{int}}(r) = 0$$

and matching the solution with the external one $u_{kl}^{\mathrm{as}}$ :

$$\left. \frac{u_{kl}^{\mathrm{int}\prime}}{u_{kl}^{\mathrm{int}}} \right|_{r_0} = \left. \frac{u_{kl}^{\mathrm{as}\prime}}{u_{kl}^{\mathrm{as}}} \right|_{r_0}$$

# Bibliography

The following books are good references for more details on low-energy nuclear-reaction theory :


- C. J. Joachain *Quantum Collision Theory* (North-Holland, Amsterdam, 1975)
- C. Cohen-Tannoudji, B. Diu & F. Laloë *Quantum Mechanics*, Vol.2 (John Wiley & Sons, Paris, 1977)
- C. A. Bertulani & P. Danielewicz *Introduction to Nuclear Reations* (Institute of Physics, London, 2004)
- I. J. Thompson & F. M. Nunes *Nuclear Reactions for Astrophysics : Principles, Calculation and Applications of Low-Energy Reactions* (Cambridge University Press, 2009)
- J. R. Taylor *Scattering Theory : The Quantum Theory of Nonrelativistic Collisions* (Dover, New York, 1972)