

ANALYTIC THIN WALL DECAY RATE

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- ii) SETUP
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- vi) CONCLUSIONS & OUTLOOK

I.) INTRODUCTION & MOTIVATION

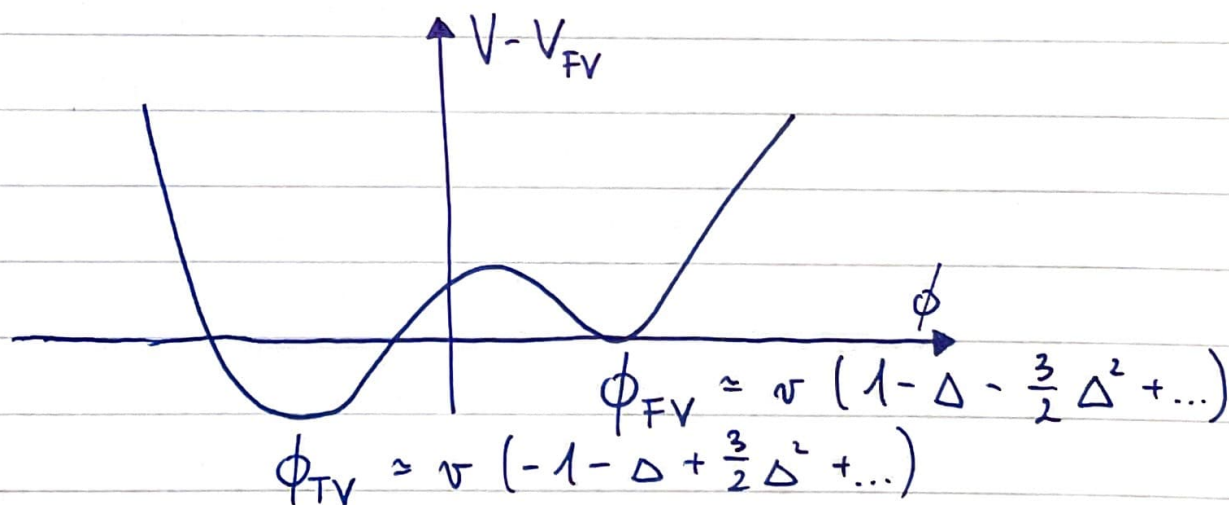
QFT / TFT ground states may not be stable.

Benchmark: real scalar field ϕ , singlet:

$$V = \frac{\lambda}{8} (\phi^2 - v^2)^2 + \lambda \Delta v^3 (\phi - v).$$

$\Delta = 0$: two degenerate minima $\langle \phi \rangle = \pm v$, no tunneling

$\Delta \geq 0$: thin wall (TW) limit, FV is on the right.



- at $\Delta = \frac{1}{3\sqrt{3}} \sim 0.19$, the FV disappears, no tunneling
- we assume $\lambda \ll 1$ & $\Delta \ll 1$ for perturbativity,
- we keep D general for as long as possible.

PHYSICS MOTIVATION

- stability at $T=0$: SM [Weinberg, Frampton ...
... Isidori, Rictorfi, Strumia,
de Grassi et al.,
Andreassen et al., ...]
- meta-stability in BSM, precise limits on hi-quatics,
- phase transitions at $T > 0$
 - production of : defects, BH, DM, GWs, B-fields, ...

HISTORY of $\Gamma = A e^{-B}$

- early works by Langer and Kobzarev et al.
- foundational works by Coleman et al.
 - Coleman on the bounce '77
 - Callan, Coleman on corrections '77
 - Coleman, De Luccia incl. gravity '79

FLAT space-time,
 $T=0$

II.) SETUP

Γ ... decay rate for nucleation

V ... volume

Coleman &

Callan

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi} \right)^{D/2} \left| \frac{\det' \sigma}{\det \sigma_{\text{FV}}} \right|^{-\frac{1}{2}} \left[-\frac{S}{\hbar} - S_{\text{ct}} \right]$$

• dominant semi-classical when $\hbar \rightarrow 0$

• S is the Euclidean bounce action

Minkowski $\mathcal{L} = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$

$\Downarrow$
EUCLIDEAN

$$S = \int d^D x_E \left(\frac{1}{2} \partial \phi^2 + V(\phi) \right)$$

$\Downarrow$
 $O(D)$ SYMMETRIC

$$S = \Omega \int_0^\infty dr r^{D-1} \left(\frac{1}{2} \dot{\phi}^2 + V - V_{\text{FV}} \right)$$

⤵ * proof by Coleman, Glaser & Martin '78

⤵ multifields by Blum et al. '16

prime = removed zeroes

action counterterms

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi} \right)^{D/2} \left| \frac{\det' \sigma}{\det \sigma_{\text{FV}}} \right|^{-1/2} e^{-\frac{S}{\hbar} - S_{\text{ct}}}$$

dimensionless correction from zero removal, collective coordinates

the functional determinant

- S is the bounce action, calculated with renormalized parameters that run $\lambda_R = \lambda_R(\mu)$.
- S_{ct} comes from integrating the counter-terms in dim-reg, we get the $\frac{1}{\epsilon}$ poles $\times$ # numbers.
- σ is the fluctuation operator, 2nd derivative of $S[\phi]$

$$S[\phi] = S[\bar{\phi}] + \frac{\delta S}{\delta \phi}[\bar{\phi}] \delta \phi + \frac{\delta^2 S}{\delta \phi^2}[\bar{\phi}] \delta \phi^2$$

$\uparrow$ bounce action $\bar{\phi}$ extremizes S , hence $\frac{dS}{d\phi} = 0$ $\uparrow$ first corrections, one loop

$$\sigma = -\partial_\mu \partial^\mu + V^{(2)} \quad , \quad V^{(n)} = \frac{d^n V}{d\phi^n}$$

- $\det \sigma \propto \prod \lambda_i$, $\sigma \psi_i = \lambda_i \psi_i$, each symmetry brings a zero λ (translations, internal global & local)

• The $\det' O$ will be infinite and μ -dependent.

The $\frac{1}{\epsilon}$ poles cancel with S_{ct} , the log terms cancel with $S_R(\mu) \Rightarrow \Gamma$ finite and μ -independent.

• $\bar{\Phi}$, the branes is $O(D)$ symmetric, only a function of the Euclidean radius ρ .

$\Downarrow$

O is symmetric, radially separable in O_e

$$d_e = \frac{(2l+D-2)(l+D-3)!}{l! (D-2)!}, \quad \begin{aligned} d_0 &= 1 \\ d_1 &= D \end{aligned}$$

$\uparrow$

degeneracy in l $l=0 \dots$ a single spherically symmetric mode (bubble)

$l=1 \dots$ D -translational modes that give zeroes and have to be removed

FINAL RESULT

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi} \frac{12}{e^{D-1}} \lambda v^2 \right)^{\frac{D}{2}} e^{-S - \frac{1}{\Delta^{D-1}}} \left\{ \begin{array}{l} \frac{30 + 9 \ln 3}{54}, D=3 \\ \frac{45 - 4\pi\sqrt{3}}{132}, D=4 \end{array} \right.$$

* This is done in the $\overline{MS}$ scheme with $\mu_0 = \sqrt{\lambda} v$.

III.) BOUNCE

III. i) Bounce action

III. ii) Higher orders

III. iii) Counterterms and running

III. i) Bounce action

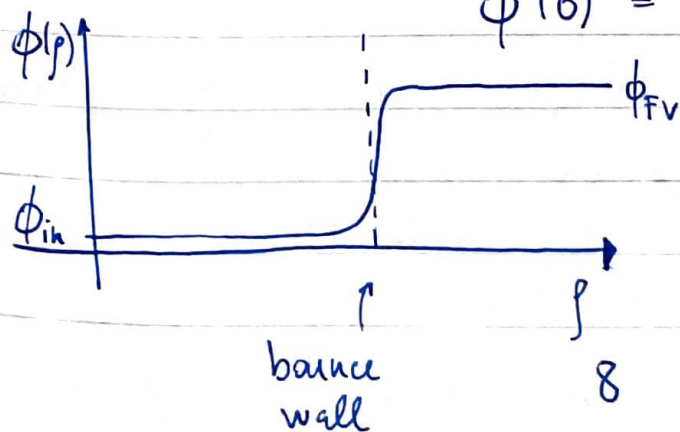
$$S = \int_D \frac{1}{2} \dot{\phi}^2 + V - V_{FV}, \quad \int_D = \Omega \int_0^f dr \int^{D-1}$$
$$\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

$$\cdot f^2 = t^2 + x_i^2$$

• Extremize $S \Rightarrow \bar{\phi}$ (I will call it ϕ)

↳ via E.L. $\Rightarrow \ddot{\phi} + \frac{D-1}{f} \dot{\phi} = \frac{dV}{d\phi}, \quad \dot{\phi}(0, \infty) = 0$

$$\phi(0) = \phi_{in}, \quad \phi(\infty) = \phi_{FV}$$



• wall $\rightarrow \infty$ when $\Delta \rightarrow 0$,
 $S \rightarrow \infty$, $\Gamma \sim e^{-S} \rightarrow 0$, $\tau \rightarrow \infty$
= NO TUNNELING.

THIN WALL EXPANSION

- We expand $\bar{\phi}$, S and $\det \sigma$ in powers of $\Delta \ll 1$.

This is a systematic approach for any $\mathcal{D}$ and

separates all orders Δ^n , e.g. cancellations of $\frac{1}{\epsilon}$, $\ln \mu$ happen $\forall \Delta^n$.

- Dimensionless fields: $\phi = v \varphi$,

$$\text{radius: } z = \sqrt{\lambda} \sigma \rho - r.$$

- [short sidenote on units]

@ tree level $[\phi] = [\sigma] = \frac{\mathcal{D}}{2} - 1$, $[\lambda] = 4 - \mathcal{D}$, $[\Delta] = 0$

then $[\sqrt{\lambda} \sigma] = 2 - \frac{\mathcal{D}}{2} + \frac{\mathcal{D}}{2} - 1 = 1$... mass parameter

actually $V''_{\text{FV}} = V^{(2)}(\phi_{\text{FV}}) \simeq \lambda v^2 (1 + \sigma(\Delta))$

- TW perturbation expansion

$$\varphi = \sum \varphi_n \Delta^n, \quad r = \frac{1}{\Delta} \sum r_n \Delta^n$$

LEADING ORDER

$$\varphi = \varphi_0, \quad r = \frac{1}{\Delta} r_0 \xrightarrow{\Delta \rightarrow 0} \infty$$

$$V_0 \approx \frac{\lambda}{8} (\phi^2 - v^2)^2$$

- The solution to the bounce equation at $u=0$ is

$$\varphi_0 = \tanh\left(\frac{z}{2}\right), \quad \text{symmetric in } z \in [-r, \infty]$$

$\downarrow \Delta \rightarrow 0$
 $-\infty$

- The position of the bounce r (or r_0) is still undefined. We need to include the $n=1$ term in the potential $V_0 + \Delta v^3 (\phi - v)$.

This creates a tension between surface and volume terms and fixes

$$r_0 = \frac{D-1}{3}$$

- At this order, the action is given by

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \left(\frac{D-1}{3} \right)^{D-1} \frac{2}{3D}$$

- It satisfies Derrick's theorem: $(D-2)\tilde{T} = -DV$.

III. ii) HIGHER ORDER CORRECTIONS

• To proceed to higher $\Delta^{u \rightarrow 0}$ orders, we have to expand the bounce equation and the action S .

• In a sense, TW is a double expansion:

a) it expands $\frac{dV}{d\phi}$ in $\phi = \phi_0 + \Delta\phi_1 + \Delta^2\phi_2 + \dots$

b) it expands $\frac{D-1}{f} \frac{d\phi}{d\rho}$ when $\sqrt{\lambda} \rho f = z + r$
 $= z + \frac{1}{\Delta}(r_0 + r_1 \Delta + \dots)$

a) is a field expansion of $V^{(n)}$ and b) is the radius expansion with "friction" corrections

• Going further to higher orders, we get

$$\phi_1 = -1 \quad (\text{even in } z), \quad r_1 = 0$$

$$\phi_2 = \frac{3}{4(D-1)} \frac{1}{dz^2 \frac{z}{2}} \left((2-D-2(4+dz)) \ln(1+e^z) dz - z(D-e^z(4+dz)) + 3(\text{Li}_2(\bar{e}^z) - \text{Li}_2(e^{-z})) \right)$$

(odd in z)

• and the Δ^2 correction to the radius is

$$r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D-1)}$$

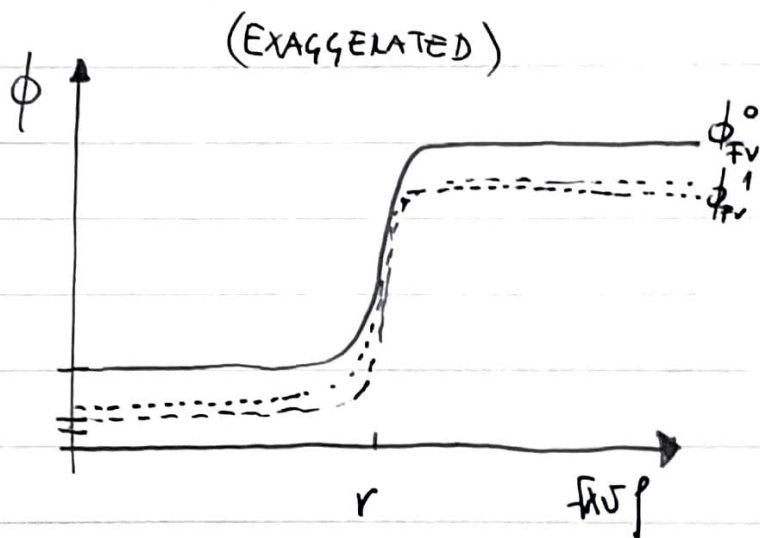
• This construction of the bounce matches the TV / FV at the ϕ extrema

$$\phi_0(z \rightarrow \pm\infty) = \pm 1,$$

$$\phi_1(z \rightarrow \pm\infty) = -1,$$

$$\phi_2(z \rightarrow \pm\infty) = \mp \frac{3}{2}.$$

$\Rightarrow$



• With Δ corrections, the bounce shifts (ϕ_1) and changes shape (ϕ_2). Its position remains the same at $u=1$, but shifts with r_2 .

• Derrick's theorem still holds at $u=2$

$$(D-2) (\mathcal{T}_0 + \Delta^2 \mathcal{T}_2 + \dots) = -D (V_0 + \Delta^2 V_2 + \dots)$$

- The Euclidean action gets corrected @ Δ^2 :

NEW!!

$$S = S_0 \left(1 + \Delta^2 \left(\frac{1 + D(25 - 8D - 3\pi^2)}{2(D-1)} \right) \right)$$

(turns out v_2 and φ_2 don't really enter)

- Subsequent correction is $\mathcal{O}(\Delta^4)$, so this result is already quite precise; $\Delta \lesssim \frac{1}{3\sqrt{5}} \approx 0.19$ anyways.
- It also serves as an estimate for the upper bound on Δ . For large D : $1 - \Delta^2 4D$; $\Delta \lesssim \frac{1}{2\sqrt{D}}$.

III iii) Counter-terms and running ($D=4$ here)

- At one loop, we renormalize with c.t.s

$$V_{ct} = \frac{1}{2} \delta m^2 \phi^2 + \frac{1}{4} \delta \lambda \phi^4, \quad V^{(4)} = \frac{d^4 V}{d\phi^4} (\langle \phi \rangle),$$

4P:  +  = 0 $\Rightarrow \delta \lambda = \frac{1}{(4\pi)^2 2\varepsilon} V^{(4)2}$

The $\delta \lambda$ removes the $\frac{1}{\varepsilon}$ pole of the 3P as well:

 +  = 0 $\Leftrightarrow V^{(3)} = \langle \phi \rangle V^{(4)}$,
automatic in $\lambda \phi^4$

- The δm^2 is fixed by the tadpole 1P

 +  = 0 $\delta m^2 = \frac{1}{(4\pi)^2 \varepsilon} V^{(4)} (V^{(2)} - \frac{1}{2} V^{(4)} \langle \phi \rangle^2)$,

and works to remove the 2P pole

 +  +  = 0. $\checkmark$

- All of this works independently of Δ , $\delta_\Delta = 0$ and the Δ coupling does not run!

COUNTERTERM FOR THE ACTION : S_{ct}

- Remember $\frac{P}{V}$ contains the S_{ct} , since the $\frac{1}{\epsilon}$ enter only in the potential, we have

$$S_{ct} = \int_{D=4} V_{ct} = \int_0^\infty dp \, p^3 \left(\frac{1}{2} \bar{\phi}^2 \delta_{m^2} + \frac{1}{4} \bar{\phi}^4 \delta_\lambda - FV \right)$$

in $D=4$, for our potential : $\delta_{m^2} = -\frac{3\lambda^2 v^4}{(4\pi)^2 2\epsilon}$, $\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2 2\epsilon}$,

and we use $\bar{\phi} = v (\varphi_0 + \Delta\varphi_1) = v \left(\ln \frac{\epsilon}{2} - \Delta \right)$

$$S_{ct} = \frac{3\lambda^2}{8(4\pi)^2 \epsilon} \int_{D=4} 3(\bar{\phi}^4 - \phi_{FV}^4) - 2v^2(\bar{\phi}^2 - \phi_{FV}^2) \approx -\frac{3}{16 \epsilon \Delta^3}.$$

RUNNING OF THE ACTION

$$\beta_\lambda = \frac{d\lambda}{d \ln \mu} = \frac{9\lambda^2}{(4\pi)^2} \Rightarrow \lambda(\mu) = \overset{\lambda_0}{\lambda(\mu_0)} + \frac{9\lambda_0^2}{(4\pi)^2} \ln\left(\frac{\mu}{\mu_0}\right)$$

COMBINE : $S_R + S_{ct} = S \left(1 - \frac{9\lambda_0}{(4\pi)^2} \left(\underset{\substack{\uparrow \\ \text{pole}}}{\frac{1}{\epsilon}} + \ln \frac{\mu}{\mu_0} \right) \right).$

arbitrary μ

IV. FLUCTUATIONS

IV. i.) Low- l multipoles

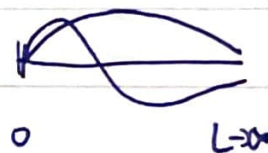
IV. ii.) High- l multipoles

IV. i.) Low- l

$$\frac{\det O}{\det O_{FV}} = \prod_i \frac{\lambda_i}{\lambda_{iFV}} \xrightarrow{\text{reorganize}} \prod_e \left(\frac{\lambda_e}{\lambda_{eFV}} \right)^{d_e}$$

"DIRICHLET"
APPROACH

where $\psi_e(p=0, \infty) = 0$



Gel'fand - Yaglom: $O_e \psi_e = 0$, $O_e^{FV} \psi_e^{FV} = 0$,

$$R_e = \frac{\psi_e}{\psi_{eFV}}, \quad R_e(0) = 1, \quad \dot{R}_e(0) = 0.$$

$$\frac{\det O}{\det O_{FV}} = \prod_e (R_e(\infty))^{d_e}$$

l -dependence only here

where: $\ddot{R}_e + 2 \left(\frac{\dot{\psi}_{eFV}}{\psi_{eFV}} \right) \dot{R}_e = (V^{(2)} - V_{FV}^{(2)}) R_e$

normalization of ψ_{eFV} cancels out

makes it stable when $p \rightarrow \infty$ $V^{(1)} \rightarrow V_{FV}^{(2)}$

Re solution at low l in TW

- First we solve for $\psi_{\text{EFV}} \rightarrow r^{\frac{D-1}{2}} \psi_{\text{EFV}}$,

$$\frac{\psi_{\text{EFV}}''}{\psi_{\text{EFV}}} = \frac{V^2 - 1/4}{(z+r)^2} + \tilde{V}_{\text{FV}}^{(L)}, \quad V = l + \frac{D}{2} - 1,$$

$$\psi_{\text{EFV}} \approx c_{\text{FV}} e^{\underbrace{\left(1 - \frac{3}{2}\Delta + \left(\frac{V^2 - 1/4}{2r_0^L} - \frac{21}{8}\right)\Delta^2\right)z}},$$

$\uparrow$ cancels out enters into the Re equation, l -dependence comes in @ Δ^2 -level, no z -dependence.

- Multiplicative expansion and $x = e^z \in [e^{-r}, e^{\infty}] \approx [0, \infty]$

$$\psi_{\text{EFV}} = \prod_{n=0} \psi_{\text{EFVn}}^{\Delta^n}, \quad R_e = \prod_{n=0} R_{en}^{\Delta^n} = R_{e0} R_{e1}^{\Delta} R_{e2}^{\Delta^2} \dots$$

$\nwarrow$ expand for small Δ

$$R_e = \underline{R_{e0}} \left(1 + \Delta \ln R_{e1} + \Delta^2 \left(\frac{1}{2} \ln^2 R_{e1} + \ln R_{e2} \right) + \dots \right)$$

- The final results are:

$$n=0 : R_{e0} = \frac{1}{(1+x)^2} \rightarrow 0,$$

$$n=1 : \ln R_{e1} = 3(r + \ln x), \text{ still } R_e = 0$$

$$n=2 : \ln R_{e2} = \frac{3}{4} \frac{(l-1)(l+D-1)}{(D-1)^2} \underbrace{x^2}_{\text{this beats the } \frac{1}{x^2} \text{ at } n=0} \dots$$

this beats the $\frac{1}{x^2}$ at $n=0$.

Low- l : $R_e \approx \Delta^2 e^{\frac{D-1}{4}} \frac{(l-1)(l+D-1)}{(D-1)^2}$.

* starts at Δ^2 , as expected,

* contains a non-trivial $e^{\frac{D-1}{4}}$ correction from $u=1$ (checked numerically, not there in list. Dirichlet),

* gives us a negative value at $l=0$,

* gives us D zeroes when $l=1$, translations,

* does not go to 1 when $l \gg 1$ $R_{e \gg 1}(\infty) \rightarrow 1$,

because we took $(\Delta v)^2$ to be $O(\Delta^2)$, thus

this is valid only for $v < \frac{1}{\Delta}$.

* We can use this R_e for zero removal and get

$$R'_1 = \frac{e^{\frac{D-1}{4}}}{12} \frac{1}{\lambda v^2}.$$

(ask me later how to derive this, if you are interested, does not affect the leading order in $\Delta^{\frac{D-1}{4}}$, but it gives our Γ the 18 right dimension).

IV. ii) High- l , or generic, multipoles

• We will count $\Delta V = O(1)$, not Δ^1 , so we include the multipole dependence already at $u=0$, not $u=2$.

• We now have ψ_{VFV} and R_V :

$$\psi_{\text{VFV}} \approx e^{k_V z}, \quad k_V^2 = 1 + \left(\frac{\Delta V}{r_0}\right)^2,$$

This enters directly into the eq. for R_V

$$R_V'' + (2k_V + 1) \frac{1}{x} R_V' = \frac{1}{x^2} (\tilde{V}^{(2)} - \tilde{V}_{\text{VFV}}^{(2)}) R_V$$

The solution for R_V is easy to get at $u=0$,

$$u=0 : \quad R_{V0}(\infty) = \frac{(k_V - 1)(2k_V - 1)}{(k_V + 1)(2k_V + 1)}$$

$\begin{matrix} \nearrow \Delta \rightarrow 0, & k_V \rightarrow 1 & \rightarrow 0 \\ \searrow \Delta \gg 1, & k_V \gg 1 & \rightarrow 1 \end{matrix}$

✓

but hard to get the Δ^0 corrections

$$U = 3r_0 (k_V - \sqrt{k_V^2 - 1})$$

$$R_V = R_{V0} e^u$$

final result for generic (V) !

$$v = l + \frac{D}{2} - 1, \quad r_0 = \frac{D-1}{3},$$

$$k_v^2 = 1 + \left(\frac{\Delta v}{v_0}\right)^2$$

Let us summarize and comment on this result:

$$\ln R_v = \ln \frac{(k_v-1)(2k_v-1)}{(k_v+1)(2k_v+1)} + 3r_0 (k_v - \sqrt{k_v^2-1}).$$

- * l -dependence enters at $u=0$. $k_v \sim 1$ reproduces the $R_e \rightarrow 0$ result, but: we only included Ψ_0 and Ψ_1 , no Ψ_2 or other Δ^2 corrections. These are subleading and won't enter in the final expression.
- * The U -factor $U(k_v \rightarrow 1) = 3r_0 = D-1$ reproduces the low- l expression e^{D-1} .
- * e^u does not affect the leading order of $R_v \xrightarrow{k_v \gg 1} 1$ but it will give sub-leading $\frac{1}{k_v}$ terms, which are crucial for renormalization. Without these, the $\frac{1}{\epsilon}$ poles and μ -dependence would not cancel.

V.) RENORMALIZED DETERMINANT

V. i) Renormalized rates

V. ii) Finite sum in $D=3$

V. iii) Finite sum in $D=4$

V. iv) Generic D s

V. i) RENORMALIZED RATES

To compute the determinant, we need to multiply all R_e in T , or sum over l in hT . This quantity is divergent, but we can regulate the divergence, translate it into $\frac{1}{\epsilon}$ & log parts within a consistent scheme. Approaches are:

1) EFFECTIVE ACTION / FEYNMAN DIAGRAMS,

2) WKB / ζ -FUNCTION FORMALISM.

• Here, we will adopt the ζ -function. Let's see how it works. First, the infinities in $l \gg 1$

$$\ln \left(\frac{\det \sigma}{\det O_{FV}} \right) = \sum_{\nu=\frac{D}{2}-1}^{\infty} d_{\nu} \ln R_{\nu}$$

$\swarrow$ degeneracy factor $\searrow \nu \gg 1$

$$\ln \left(1 + \frac{C-1}{\nu} + \frac{C-3}{\nu^3} + \dots \right)$$

large ν , exact for $D=2,3,4$.

$$d_{\nu} = \frac{2\nu(\nu + \frac{D}{2} - 2)!}{(D-2)!(\nu - \frac{D}{2} + 1)!} \simeq \frac{2}{(D-2)!} \nu^{D-2}$$

* at large ν : $\sum_{\nu \gg 1} d_{\nu} \ln R_{\nu} \sim - \frac{3r_0(2-r_0)}{(D-2)! \Delta} \sum_{\nu} \nu^{D-2} \left(\frac{1}{\nu} - \frac{1}{\nu^3} \right) \times \left(\frac{r_0}{2\Delta} \right)^2$

$D \geq 4$

from $\frac{1}{\nu} \Rightarrow \sum_{\nu} \frac{\nu^2}{\nu} \sim \nu^2$ divergence,

$\frac{1}{\nu^3} \Rightarrow \sum_{\nu} \frac{\nu^2}{\nu^3} \sim \ln \nu$ divergence.

* The renormalization prescription has to remove those and make the sum finite. The removed part is then added back in the Γ_R . Let's see how this works in $D=3$.

V. ii) FINITE SUM IN $D=3$

* Renormalized det :

[DUNNE & KIRSTEN,
DUNNE & HIN]

$$\ln \left(\frac{\det \sigma}{\det \sigma_{\text{FV}}} \right) = \sum_v d_v \left(\ln R_v - \underbrace{\frac{I_1}{2v}}_{\ln R_v^a} \right)$$

* The I_1 is derived from WKB/ ξ and is given by

$$I_1 = \int_0^\infty dp \int (V^{(2)} - V_{\text{FV}}^{(2)}) \approx -3(2-r_0) \left(\frac{r_0}{\Delta} \right).$$

When multiplied by $\left(-\frac{dv}{2v} \right)$, we see that it

precisely removes the $\sum \frac{v^{D-2}}{v} = \sum = \text{lin. divergence}$.

* There are no $\frac{1}{2}$ poles or μ in odd D .

* Now we just have to sum up, using Euler-Maclaurin

$$y = \frac{\Delta V}{r_0}, \quad r_0 = \frac{2}{3}, \quad k_r = \sqrt{1+y^2}, \quad \ln R_v^a = -\frac{2}{y}$$

$$\sum_3^f \approx 2 \left(\frac{r_0}{\Delta} \right)^2 \int_{y_0}^\infty dy y \left(\ln R_v + \frac{2}{y} \right) = \frac{1}{\Delta^2} \frac{20 + 9 \ln 3}{27}.$$

* THIS IS Π ! The same expression as the one we stated in the intro/summary.

* Let us comment a bit on this result.

* The $\int_{y_0}^{\infty} dy$ does not depend on the lower boundary

$y_0 = \frac{\Delta V_0}{V_0}$. We could send it to zero and

thus neglect / ignore the low multipoles. This

justifies why we "dropped" the low- l part R_e ,

we only need R_e when $l \sim \frac{1}{\Delta}$, which is

precisely where R_v is valid.

* The EotWac procedure / approx. also has corrections:

$$\sum_D = \sum_{v=v_0}^{\infty} \sigma_D = \sum_{v=v_0}^{\infty} d_v (\ln R_v - \ln R_v^a)$$

" $\frac{D}{2}-1$ ", i.e. $l=0$

$$\approx \sum_D^S + \sum_D^{\text{bud}} + R_p^{\text{remainder}}$$

$$\int_{v_0}^{\infty} dv \sigma_D \quad \frac{1}{2} \sigma_D(v_0) - \sum_{j=1}^{\lfloor \frac{D}{2} \rfloor} \frac{B_{2j}}{(2j)!} \sigma_D^{(2j-1)}(v_0).$$

* In $D=3$, $\sigma_3 \approx \frac{8}{3\Delta}$, which is $< \mathcal{O}(\frac{1}{\Delta^2})$ from the $\sum_3^S$, so we can safely drop it.

⌘ Thermal Field Theory [Laine, Vovriuen, ...]

$$\frac{\Gamma}{V} = \left(\frac{\lambda_-}{2\pi} \right) \left(\frac{S_3}{2\pi T} \right)^{3/2} \sqrt{\frac{\det \delta_{FV}}{\det' \delta}} e^{-S_3/T}$$

Here : $S_3 = \frac{1}{\Delta^2} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1 \right) \Delta^2 \right)$.

NEW!

$[v]=1$, so $\frac{S_3}{T}$ is dimensionless

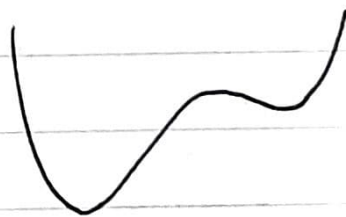
AND : $\ln \sqrt{\frac{\det \delta_{FV}}{\det' \delta}} \approx -\frac{1}{2} \sum_3$ nucleation T, $P(\text{bubble}) \sim 1$

* a good approximation when $T_N \sim T_c$

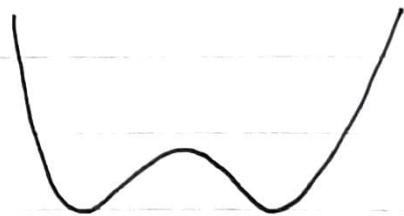
High T



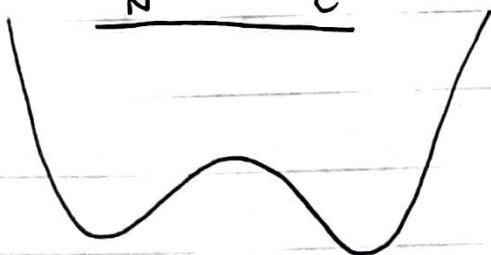
$T \gtrsim T_c$



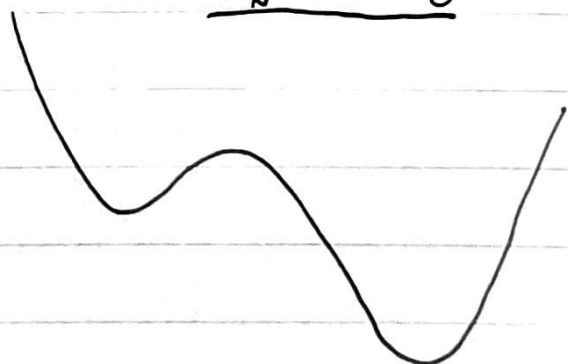
CRITICAL T



$T_N \approx T_c$



$T_N \ll T_c$



TW applies HERE! ↗

SUPER-COOLED TRANSITIONS

V.iii) FINITE SUM IN D=4

* Again we start with the renormalized $\ln \det$ calculated in WKB / ζ [DK, DM, ...]

$$\ln \left(\frac{\det \sigma}{\det \sigma_{\text{FV}}} \right) = \sum_{\nu} d_{\nu} \left(\ln R_{\nu} - \underbrace{\frac{1}{2\nu} I_1 + \frac{1}{8\nu^3} I_2}_{\ln R_{\nu}^a} - \frac{1}{8} \tilde{I}_2 \right)$$

quadratic div. $\downarrow$
 $\downarrow$ ln div.

$\uparrow$ renormalized part
asymptotic, subtraction part

* In D=4, $d_{\nu} = \nu^2$ and I_1 is the same as before.

$$I_2 = \int_0^{\infty} dp \, p^3 \left(V^{(2)2} - V_{\text{FV}}^{(2)2} \right) \stackrel{\text{TW}}{=} -3(2-r_0) \left(\frac{r_0}{\Delta} \right)^3, \quad r_0=1$$

$$\tilde{I}_2 = \int dp \, p^3 \left(V^{(2)2} - V_{\text{FV}}^{(2)2} \right) \left(\underbrace{\frac{1}{\epsilon} + \gamma_{\epsilon} + 1}_{\text{FINITE PART}} + \ln \frac{\mu p}{2} \right)$$

POLES $\nearrow$
 $\nwarrow$ ln μ PART

$$\tilde{I}_2 \stackrel{\text{TW}}{=} I_2 \left(\frac{1}{\epsilon} + \gamma_{\epsilon} + 1 + \ln \left(\frac{\mu r_0}{2\sqrt{\lambda} v \Delta} \right) \right).$$

* We have $\tilde{I}_2$, now let's do the finite sum.

* Again: $\Sigma_4 = \Sigma_4^S + \Sigma_4^{\text{bud}}$

$$y = \frac{\Delta V}{r_0}, \quad \ln R_V^a = -\frac{3}{2y} + \frac{3}{8y^3},$$

$$\Sigma_4^S = \frac{1}{\Delta^3} \int_{y_0}^{\infty} dy y^2 \left(\ln R_V + \frac{3}{2y} - \frac{3}{8y^3} \right) \quad \star$$

$$\underline{\text{TW}} = \frac{3}{8\Delta^3} \left(\frac{9-4\sqrt{3}\pi}{36} + \ln 2y_0 \right).$$

* The TW limit gives us $\frac{1}{\Delta^3}$, just like the action, so Δ^{-D+1} . The λ is gone, it's a λ connection of the $\frac{1}{\lambda}$ piece, which cancels out.

* There is a residual y_0 dependence. This is an artefact of the ζ scheme, FD don't have it. It comes only from the last term in $\star$, that integrates into $\ln y$. It will be cancelled by the "bud" part of the EuHac approximation.

* Let us focus on $\sum_4^{\text{bud}}$, the boundary terms of the Euler approximation.

Here: $V_0 = O(1)$ & we have $\Delta V_0 = O(\Delta)$, so we can expand in small Δ :

$$\Gamma_4^{(j)}(V_0) = \frac{3}{8} \frac{(-1)^{j+1} j!}{\Delta^3 V_0^{j+1}}. \quad \star$$

$V_0 \sim O(1)$ and $j!$ takes over $\Rightarrow$ diverges.

* Solution: split the sum in two pieces

$$\sum_4 = \sum_4^{\text{low}} + \sum_4^{\text{high}} = \sum_{V=V_0}^{V_1 \sim \frac{1}{\Delta}} \Gamma_4 + \sum_{V=V_1+1}^{\infty} \Gamma_4.$$

* The high part is $\sim \sum_4^S$ and $\star$ vanishes.

* The low part

$$\sum_4^{\text{low}} = -\frac{3}{8\Delta^3} \sum_{V=1}^{V_1} \frac{1}{V} = -\frac{3}{8\Delta^3} H(V_1) \approx -\frac{3}{8\Delta^3} (\ln V_1 + \gamma_E)$$

$V_0 \text{ in } D=4$

$$\sum_4^{\text{high}} = \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{28} + \ln 2y_1 \right)$$

ΔV_1

V_1 cancels out

* The dependence on the (arbitrary) point of separation at r_1 is reassuringly gone.

$$\Sigma_4 = \frac{3}{8\Delta^3} \left(\frac{9-4\sqrt{3}\pi}{36} - \gamma_E + \ln 2\Delta \right)$$

RENORMALIZED RATE

$$\ln \left(\frac{\det \sigma}{\det \sigma_{\text{FV}}} \right) = \Sigma_4 - \frac{1}{8} \tilde{I}_2 \quad \bullet \quad \text{and finally:}$$

$$\begin{aligned} \ln \frac{\Gamma}{V} &\ni - \left(S_R + S_{\text{ct}} + \frac{1}{2} \left(\Sigma_4 - \frac{1}{8} \tilde{I}_2 \right) \right) \\ &= -S - \frac{1}{\Delta^3} \frac{45-4\pi\sqrt{3}}{192}. \end{aligned}$$

* no μ -dependence, no $\frac{1}{\varepsilon}$ poles,

* μ_0 also canceled out with $\sqrt{\chi r}$ in $\tilde{I}_2$,

* the $\gamma_E + \ln 2\Delta$ from Σ_4 cancels with $\tilde{I}_2$.

DONE!

This is what was quoted in the summary.

V. iv) GENERIC DS

- The $\ln R_V^a$ in the ξ approach was defined by successive subtractions in $\frac{1}{V}$, $\frac{1}{V^3}$, $\frac{1}{V^5}$, ... etc.
- The general formula for T/V in any D is not known (to me). However, we do know the minimal subtraction $\ln R_V^a$ from expanding $\ln R_V$ in large v and extracting the $\frac{1}{v^p}$ terms.
- These asymptotic terms are given by ($y = \frac{\Delta v}{v_0}$)

$$\ln R_{V_0}^a = -2 \sum_{j=1}^u \frac{1 + 2^{-2j+1}}{2j-1} \sum_{p=0}^{u-j} \binom{-j+\frac{1}{2}}{p} y^{-2(j+p)+1},$$

$$U^a = -3v_0 \sum_{j=1}^u \binom{\frac{1}{2}}{j} (-1)^j y^{-2j+1} \sum_{p=0}^{u-j} \binom{-j+\frac{1}{2}}{p} y^{-2p}.$$

where $D = 2u$ for even and $D = 2u+1$ for odd.

* Now we simply repeat the EuHac steps.

* It is slightly easier to do it for odd D s :

$$\begin{aligned}\sum_{2m+1} &= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-1)^{m+1}}{m \binom{m-1/2}{m}} \left(\frac{1}{m} + {}_2F_1\left(1, \frac{1}{2}-m; \frac{3}{2}; \frac{1}{4}\right) - \frac{3}{2}r_0\right) \\ &= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \begin{cases} \frac{5}{6} + \frac{3}{8} \ln 3, & D=3 \\ -\frac{37}{240} - \frac{9}{64} \ln 3, & D=5 \\ -\frac{3}{224} + \frac{9}{128} \ln 3, & D=7, \dots \end{cases}\end{aligned}$$

* The even D 's are a bit more complicated. We get the dependence on $\ln V_1$ again, by splitting the sum. However, it cancels out between the low and high l parts :

$$\sum_D^{\text{low}} = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \begin{cases} \frac{5}{2} (\gamma_E + \ln V_1), & D=2 \\ -\frac{3}{8} (\gamma_E + \ln V_1), & D=4 \\ \frac{1}{10} (\gamma_E - \zeta(3) + \ln V_1), & D=6, \dots \end{cases}$$

$$\sum_D^{\text{high}} = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \begin{cases} \frac{\pi}{2\sqrt{3}} - \frac{11}{4} - \frac{5}{2} \ln 2 \gamma_1, & 2 \\ -\frac{\pi}{8\sqrt{3}} + \frac{3}{32} + \frac{3}{8} \ln 2 \gamma_1, & 4 \\ \frac{3\sqrt{3}\pi}{160} - \frac{211}{4800} - \frac{1}{10} \ln 2 \gamma_1, & 6 \end{cases}$$

VI. SUMMARY & OUTLOOK

* We solved the 45-year old problem, ^{or "left"} "posed" by Callan & Coleman and found an analytic closed form solution for:

- the bounce action up to $\mathcal{O}(\Delta^4)$,
- the functional determinant,
- explicit $\frac{\Gamma}{V}$ for $D=3,4$ in $\overline{HS}$.

* We also derived generic expression for the finite $\ln \det \delta$ for any D .

* Outlook :
a) $D\varphi \rightarrow D\varphi_{\perp}, D\varphi, DA$,
b) $\overline{\varphi} \rightarrow \varphi_{PB}$, $D\varphi$ analytically,
c) $\mathcal{P}_{new} = \mathcal{P}_{HTrs}, GWs, baryogenesis, \dots$

APPENDIX : ZERO REMOVAL

- * At $l=1$ we have D degenerate zeros that are multiplied with other non-zero λ 's with $l=1$.
- * We remove them perturbatively by off-setting $V^{(2)}$ inside $\sigma_{l=1}$ by a small dimensional μ_ε^2

$$\sigma_1 \psi_1 = 0, \quad \psi_1(\infty) = 0 \Rightarrow (\sigma_1 + \mu_\varepsilon^2) \psi_1^\varepsilon = 0.$$

- * We can solve for ψ_1^ε by $V^{(2)} \rightarrow V^{(2)} + \Delta^2 \mu_\varepsilon^2$, because the l -dependence and the $l=1$ zero starts at Δ^2 .

$$\text{Then :} \quad \underset{l=2 \nearrow \sigma(\Delta^2)}{\text{lu}} R_{12}^\varepsilon = \frac{1}{12} \frac{\mu_\varepsilon^2}{\lambda v^2} X^2$$

$$\& \quad R_1'(\infty) = R_{l0}(\infty) \underset{\frac{1}{X^2}}{l^{D-1}} \lim_{\mu_\varepsilon^2 \rightarrow 0} \frac{1}{\Delta^2 \mu_\varepsilon^2} \Delta^2 \text{lu } R_{21}^\varepsilon = \frac{e^{D-1}}{12} \times \frac{1}{\lambda v^2}.$$

Why? Because:

$$R_1^\varepsilon(\infty) = \frac{(\mu_\varepsilon^2 + \gamma_1) \prod_{n=2}^{\infty} \gamma_n}{\prod_{n=1}^{\infty} \gamma_n^{\text{FV}}} = \frac{\psi_1^\varepsilon(\infty)}{\psi_1^{\text{FV}}(\infty)} = \mu_\varepsilon^2 R_1'(\infty)$$