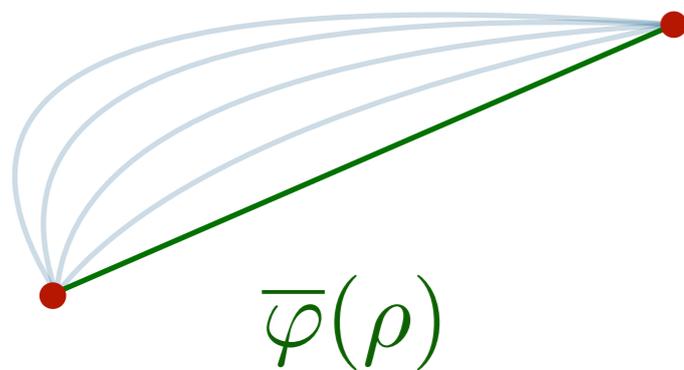


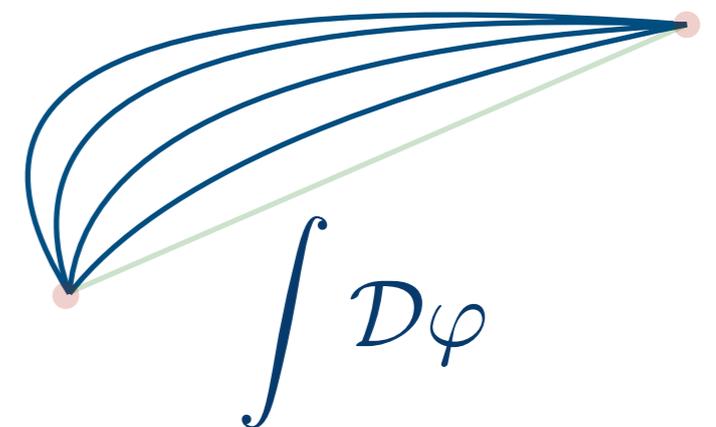


Universal Bounces and Thin Wall prefactors



$\bar{\varphi}(\rho)$

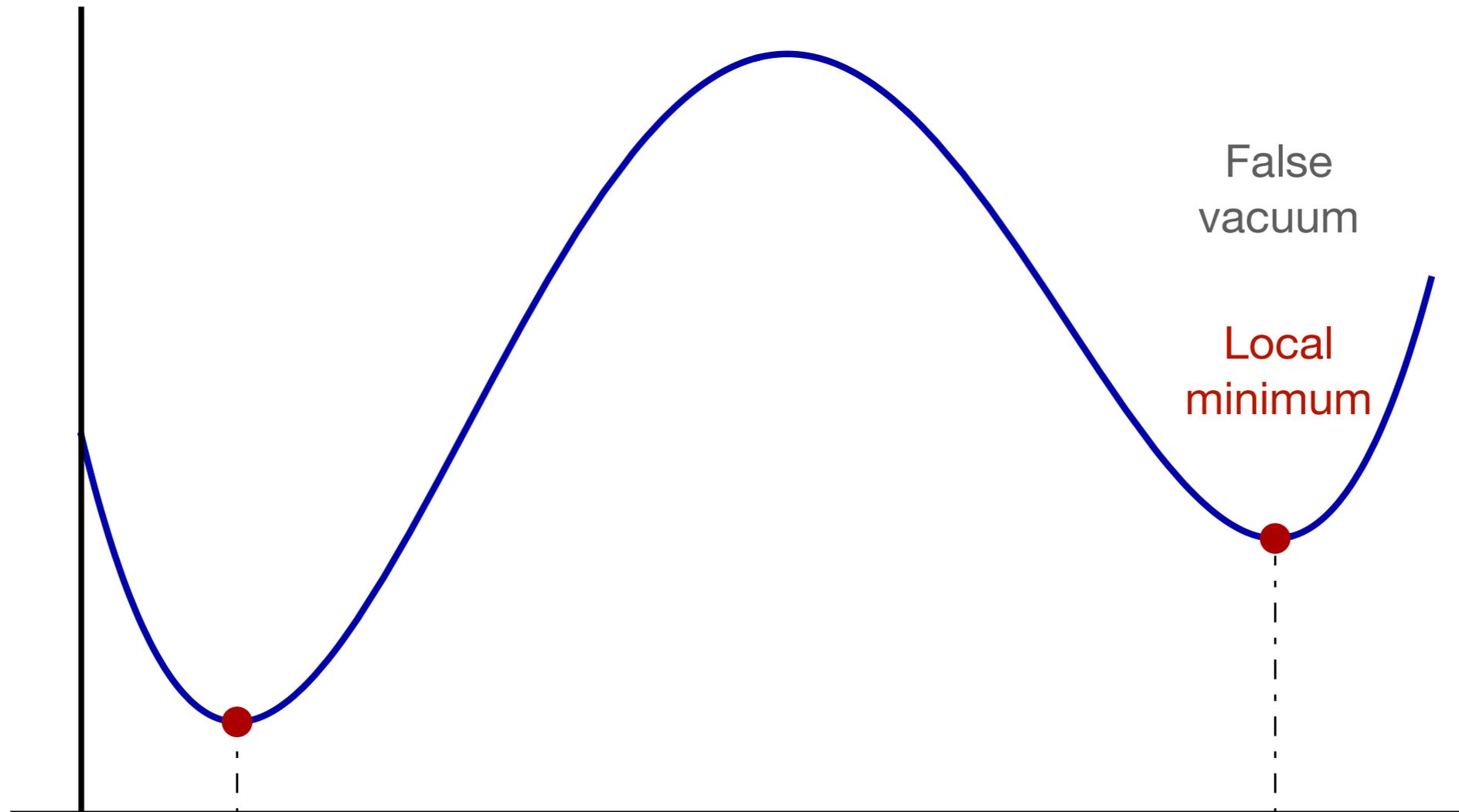
Miha Nemevšek
(IJS & FMF)



with Alessio Maiezza, Victor Guada,
Aleksandar Ivanov, Lorenzo Ubaldi and Marco Matteini

Phase transitions

Free energy



Global
minimum

True
vacuum

Order parameter

False
vacuum

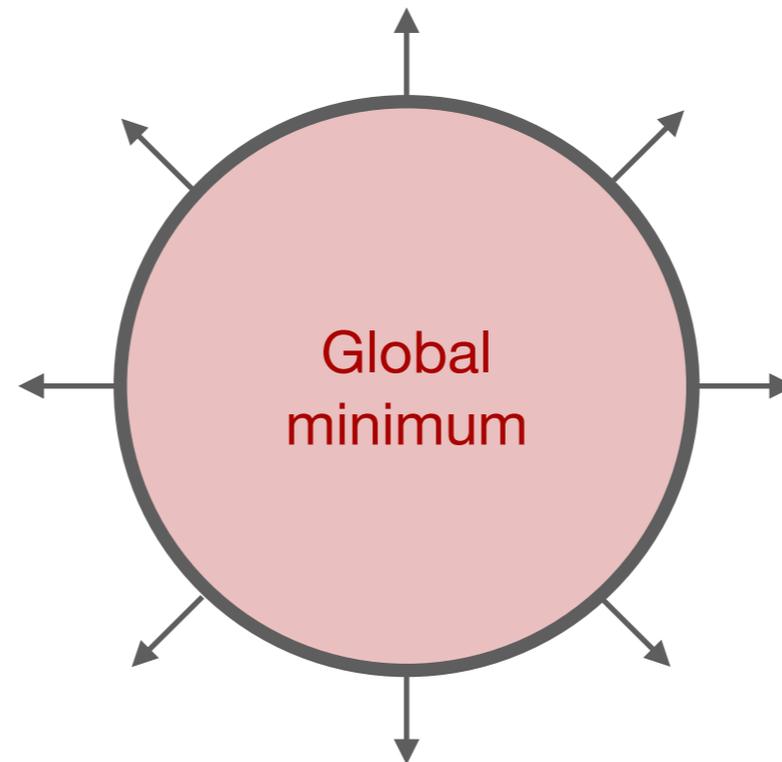
Local
minimum

First order phase transition

Decay rate

$$\Gamma = A e^{-\mathcal{S}}$$

Local
minimum

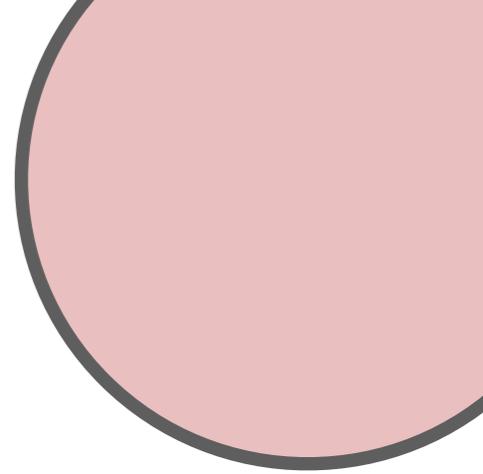
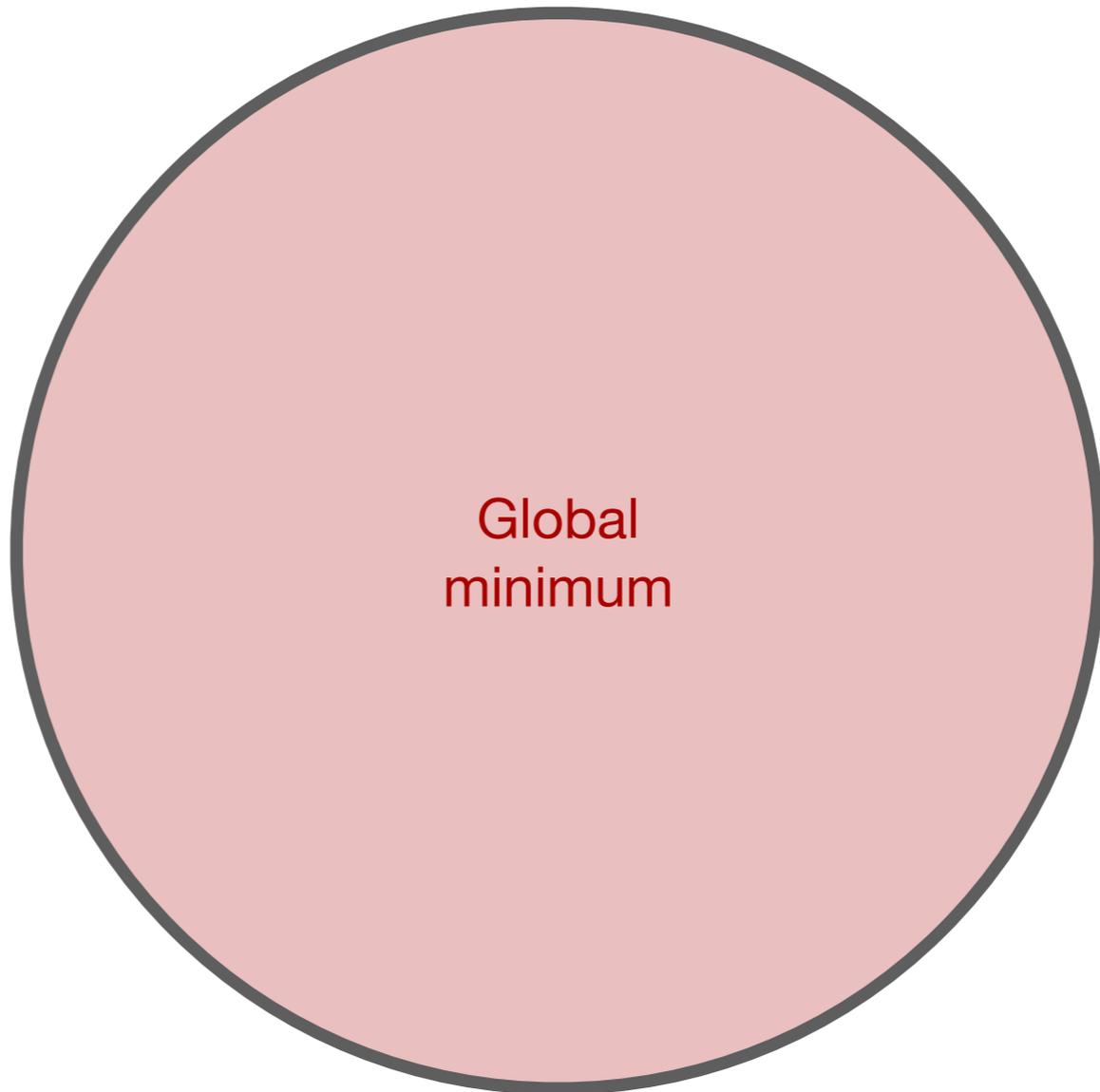


First order phase transition

Shape of the bubble $\varphi(t, x)$

Local
minimum

Global
minimum

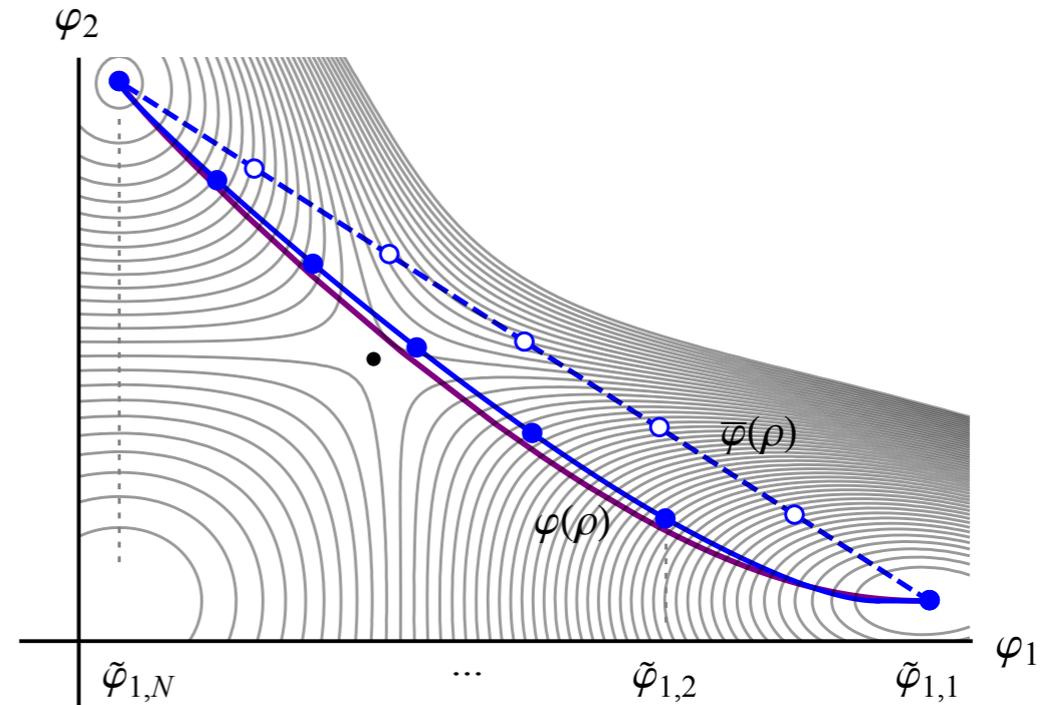


Physics

Quantum fluctuations

SM lifetime vs. m_t , α_s , m_h

BSM stability - m_S , λ_i
selection of vacuua (landscape)



Thermal fluctuations

Gravitational waves, primordial B -fields

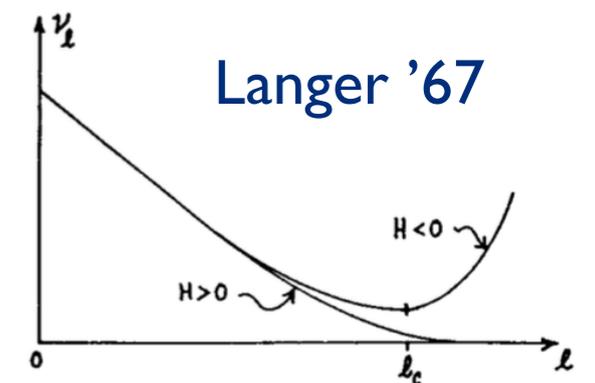
Baryogenesis / bubble wall fermion conversion

Inflation, dark energy

a bit of History

Statistical physics

Theory of condensation point, droplet model, ferromagnets



QFT / Cosmology

Bubbles in Metastable Vacuum

Voloshin, Kobzarev, Okun '74

Fate of the false vacuum: semiclassical theory

Coleman '77

Fate of the false vacuum. II. First quantum corrections

Callan, Coleman '77

Gravitational effects on and of vacuum decay

Coleman, de Luccia '79

also T.D. Lee, G.C. Wick, I. Affleck, A. Linde, S. Hawking, ...

Decay rate

Coleman '77

$$\mathcal{L} = \frac{1}{2} \partial\varphi^2 - V(\varphi) \quad \Rightarrow \quad \mathcal{L}_E = \frac{1}{2} \partial_E \varphi^2 + V(\varphi)$$

$$\frac{\Gamma}{\mathcal{V}} = \frac{\text{Im} \int \mathcal{D}\varphi e^{-S[\varphi]}}{\int \mathcal{D}\varphi e^{-S[\varphi_{\text{FV}}]}}$$

$$S[\varphi] \simeq S[\bar{\varphi}] + \left. \frac{\delta S}{\delta\varphi} \right|_{\bar{\varphi}} + \frac{1}{2} \left. \frac{\delta^2 S}{\delta\varphi^2} \right|_{\bar{\varphi}} + \dots$$

= 0

bounce

extremize

fluctuations

$O(4)$ Coleman, Glaser, Martin '78

$$\rho^2 = t^2 + \sum x_i^2$$

Euclidean time = radius of the bubble

Decay rate

Coleman '77

$$\mathcal{L} = \frac{1}{2} \partial \varphi^2 - V(\varphi) \quad \Rightarrow \quad \mathcal{L}_E = \frac{1}{2} \partial_E \varphi^2 + V(\varphi)$$

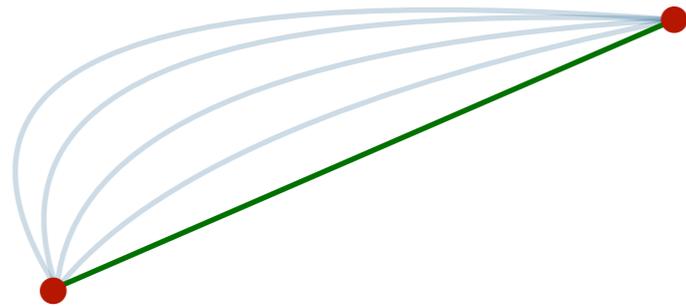
$$\frac{\Gamma}{\mathcal{V}} = \frac{\text{Im} \int \mathcal{D}\varphi e^{-S[\varphi]}}{\int \mathcal{D}\varphi e^{-S[\varphi_{\text{FV}}]}}$$

$$S[\varphi] \simeq S[\bar{\varphi}] + \underbrace{\frac{\delta S}{\delta \varphi} \Big|_{\bar{\varphi}}}_{=0} + \frac{1}{2} \frac{\delta^2 S}{\delta \varphi^2} \Big|_{\bar{\varphi}} + \dots$$

“...there always exists an $O(4)$ -invariant bounce and it always has strictly lower action than any non- $O(4)$ invariant bounce. The rigor of our proof is matched only by its tedium; I wouldn't lecture on it to my worst enemy.” Coleman, Erice lectures '77

multi-fields

Blum, Honda, Sato,
Takimoto, Tobioka '16



$$\delta S = 0 \rightarrow \bar{\varphi}(\rho)$$

Bounce

- * Thin wall Ivanov, Matteini, MN, Ubaldi '22
- * Polygonal bounces Guada, Maiezza, MN '18
- * FindBounce Guada, MN, Pintar '20



The bounce D dimensional symmetric Euclidean action

$$S = \Omega \int_0^\infty d\rho \rho^{D-1} \left(\frac{1}{2} \dot{\phi}^2 + V - V_{\text{FV}} \right), \quad \Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

bounce equation

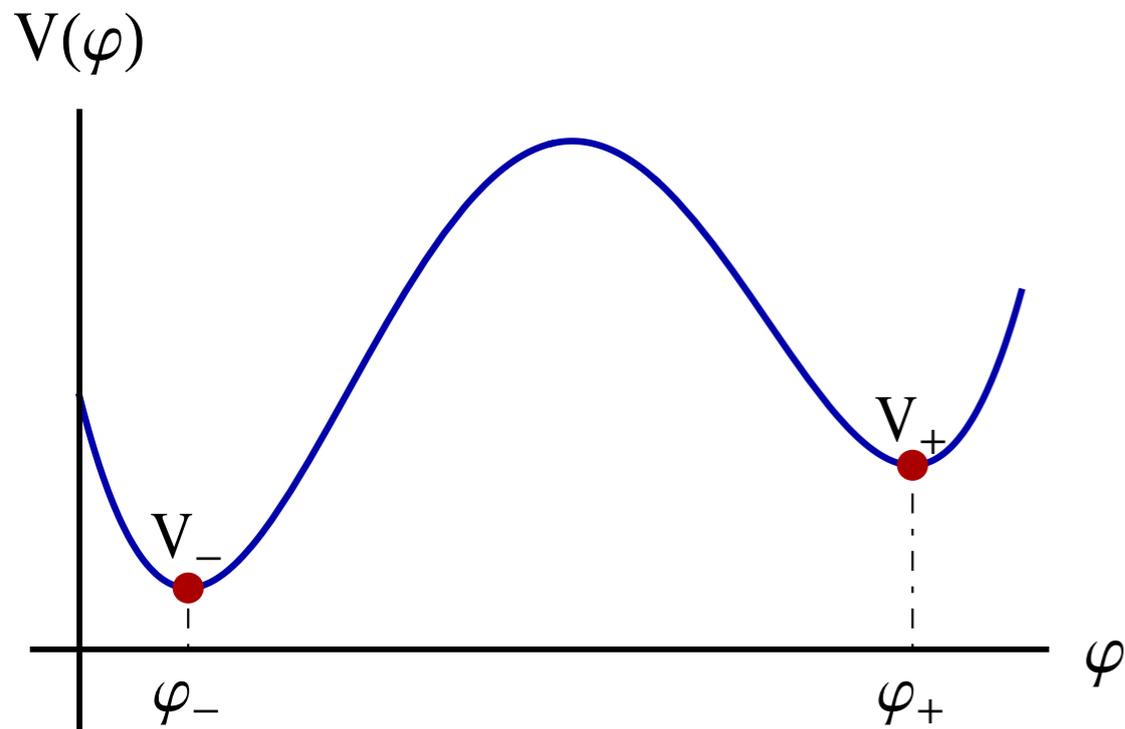
$$\ddot{\phi} + \frac{D-1}{\rho} \dot{\phi} = \frac{dV}{d\phi}$$

friction

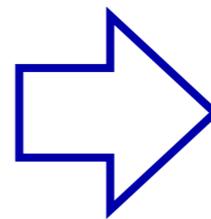
boundary conditions

$$\dot{\phi}(0) = \dot{\phi}(\infty) = 0,$$

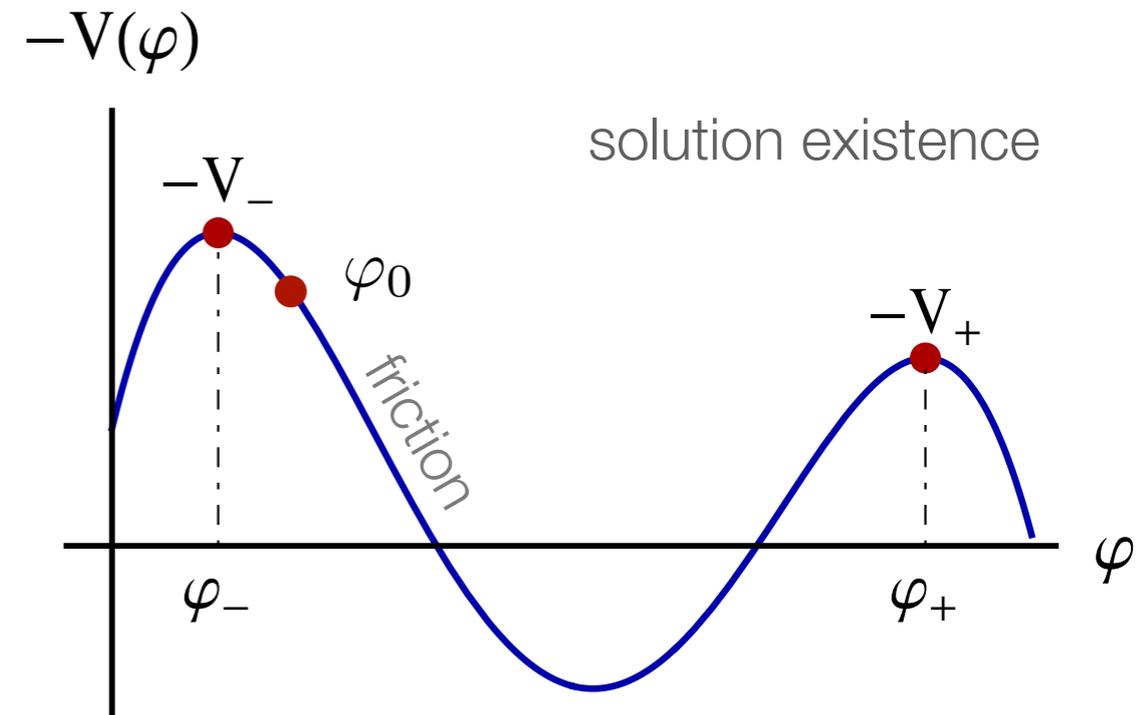
$$\phi(0) = \phi_{\text{in}}, \quad \phi(\infty) = \phi_{\text{FV}}$$



particle analogy



inverted potential



solution existence

Thin wall and beyond

Thin Wall

Ivanov, Matteini, MN, Ubaldi '22

$$V = \frac{\lambda}{8} (\phi^2 - v^2)^2 + \lambda \Delta v^3 (\phi - v)$$

Overall coupling

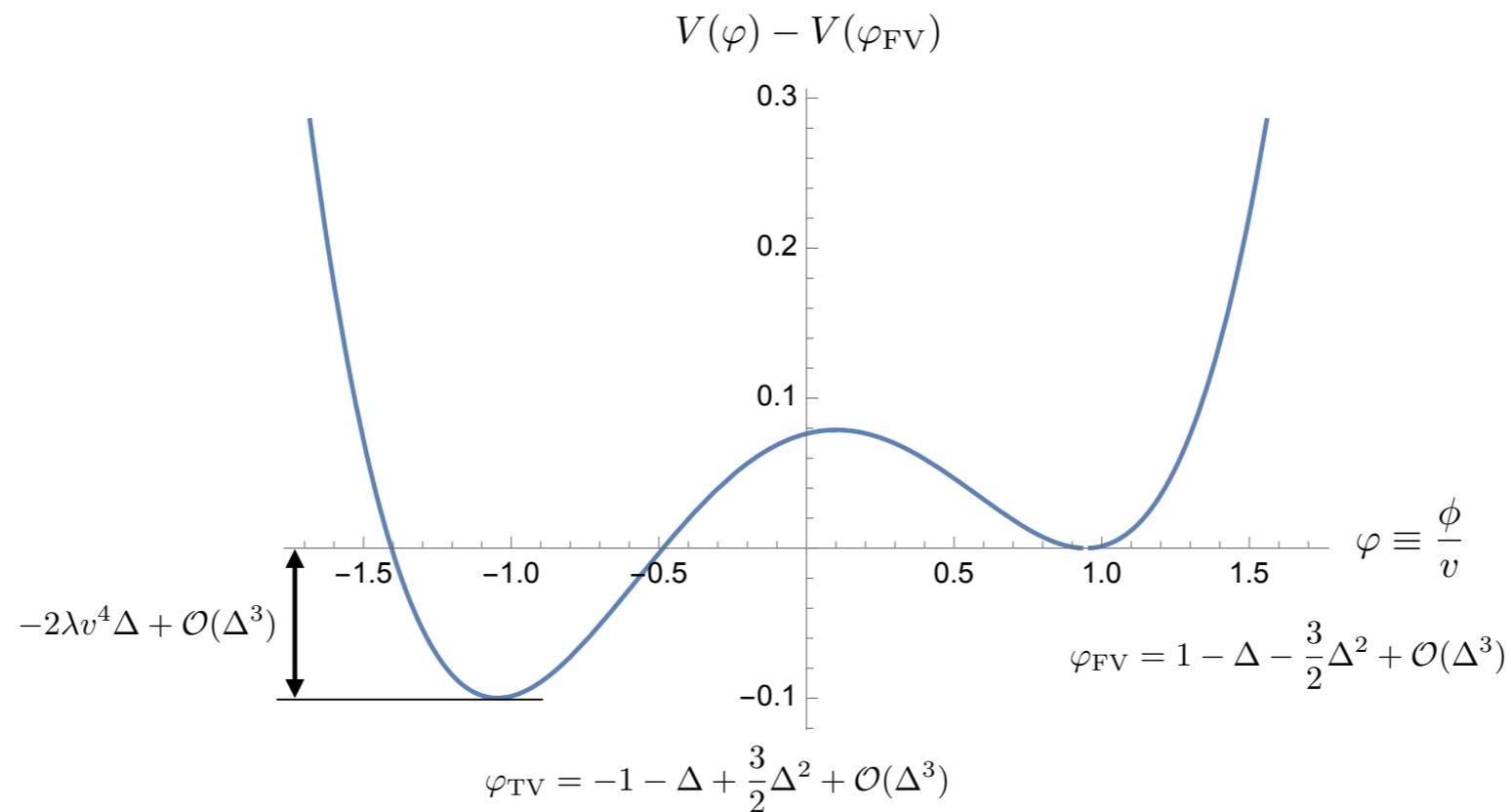
$$[\lambda] = 4 - D$$

Scale of the model

$$[v] = D/2 - 1$$

TW parameter

$$[\Delta] = 0$$



Perturbative

$$0 < \lambda \ll 1, \quad 0 < \Delta \ll 1, \quad \Delta_{\text{max}} = 3^{-3/2}$$

TW = near degenerate

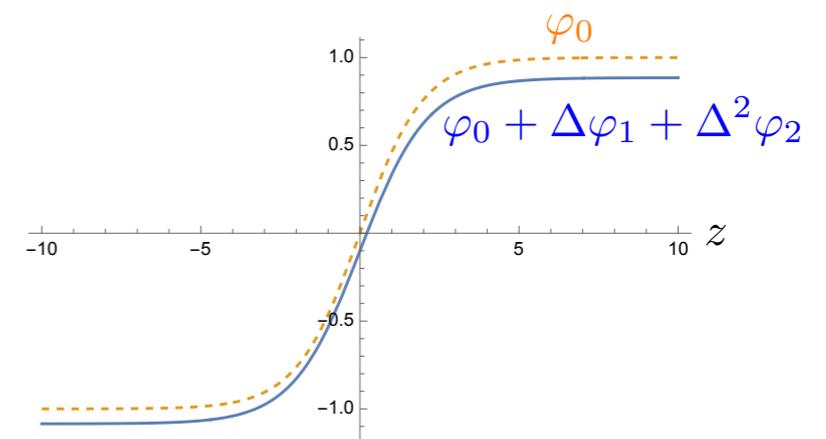
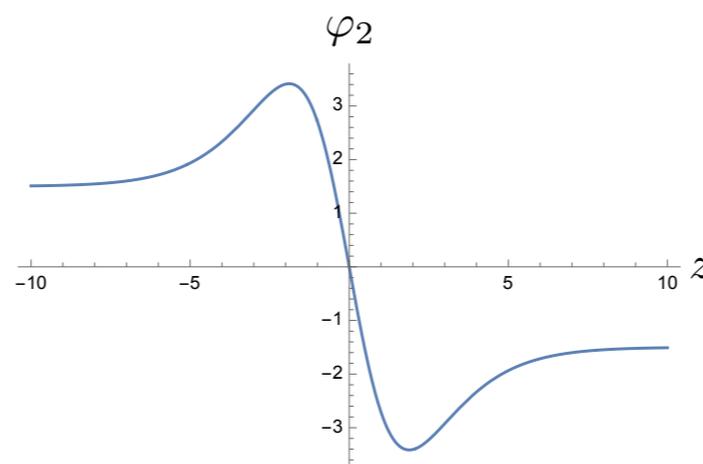
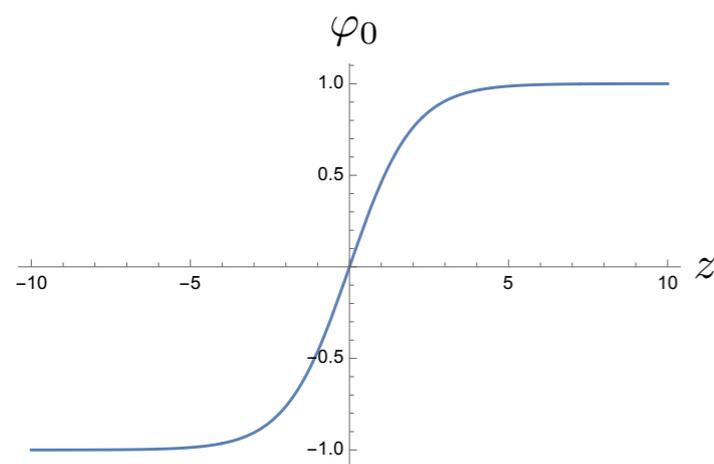
$$V = \frac{\lambda}{8} (\phi^2 - v^2)^2 + \lambda \Delta v^3 (\phi - v)$$

Expand the field around the bounce radius

$$\varphi(z) = \sum \varphi_n(z) \Delta^n, \quad z = \sqrt{\lambda v} \rho - r, \quad r = \frac{1}{\Delta} \sum r_n \Delta^n$$

We solve the bounce equation

$$\begin{aligned} \varphi_0 &= \text{th} \frac{z}{2}, & \varphi_1 &= -1, \\ r_0 &= \frac{D-1}{3}, & r_1 &= 0 \end{aligned}$$



See the paper for φ_2, φ_3 , physical bubble profile

$$r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D-1)}$$

Thin Wall Action

Ivanov, Matteini, MN, Ubaldi '22

λ and v factor out

$$S = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \times \tilde{S}[\Delta^2]$$

Leading order

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \left(\frac{D-1}{3} \right)^{D-1} \frac{2}{3D}$$

Higher orders

$$S_2 = S_0 \left(1 + \Delta^2 \left(\frac{1 + D(25 - 8D - 3\pi^2)}{2(D-1)} \right) + \mathcal{O}(\Delta^4) \right)$$

New & relevant

Perturbativity

$D = 4$: FV decay at $T = 0$

Coleman '77

$D = 3$: FV nucleation at finite T

Affleck '81, Linde '83

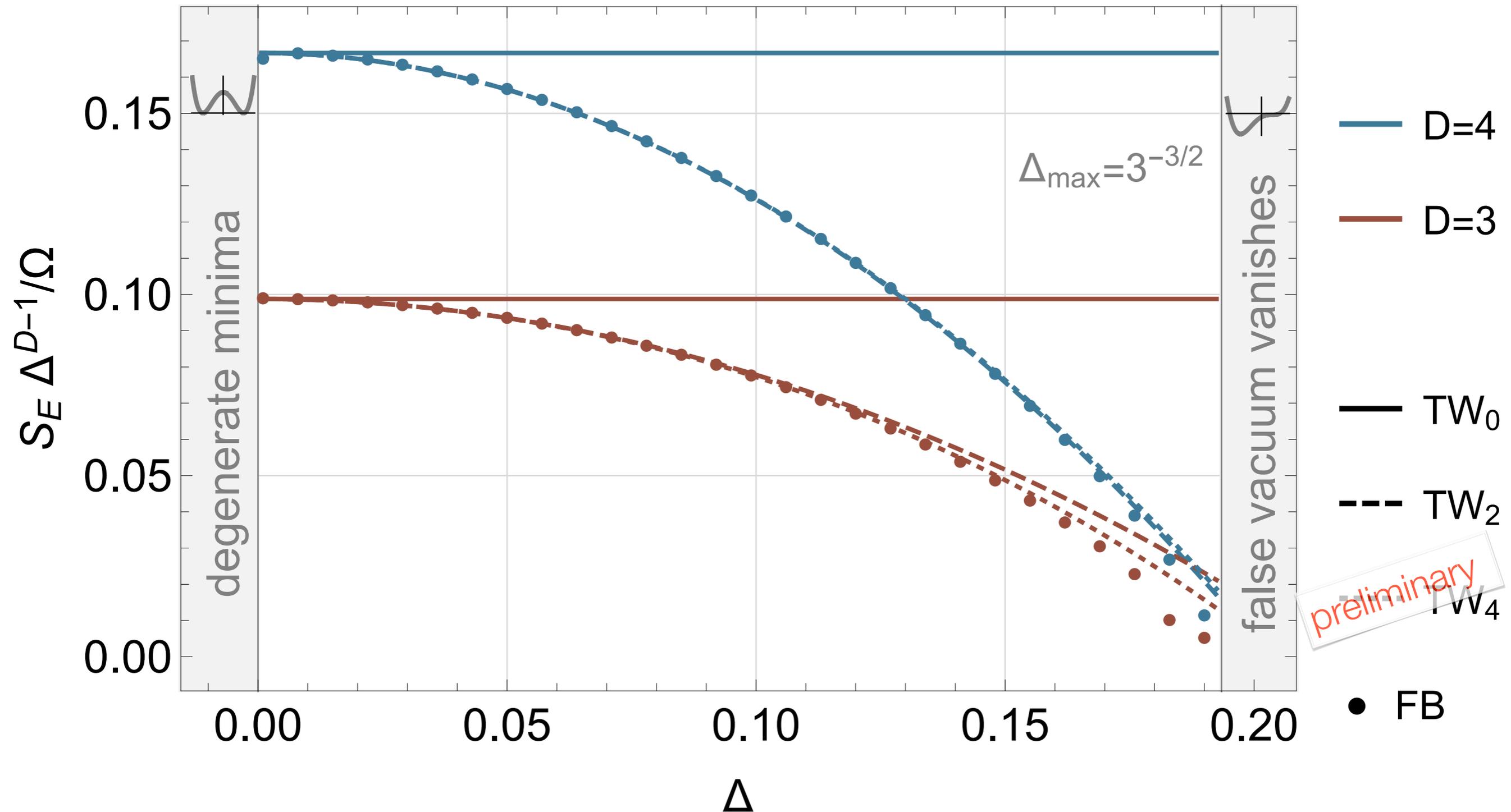
$$S_0 = \frac{1}{\Delta^{D-1}} \begin{cases} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1 \right) \Delta^2 \right), & D = 3, \\ \frac{\pi^2}{3\lambda} \left(1 - \left(2\pi^2 + \frac{9}{2} \right) \Delta^2 \right), & D = 4 \end{cases}$$

Dimensions

Thin Wall Action

Ivanov, Matteini, MN, Ubaldi '22

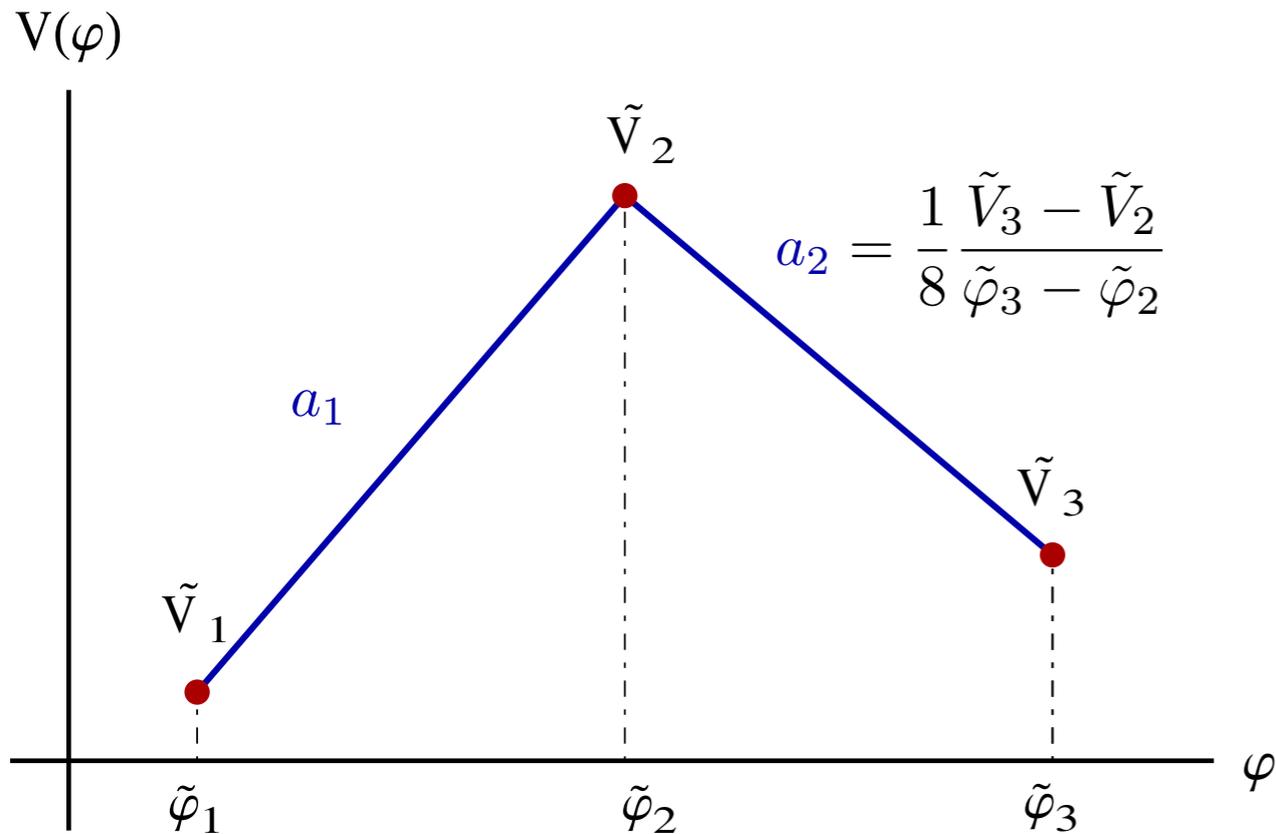
How good/useful is TW? Pretty good all the way to Δ_{\max} !



Polygonal bounces

Triangle

Duncan, Jensen '92



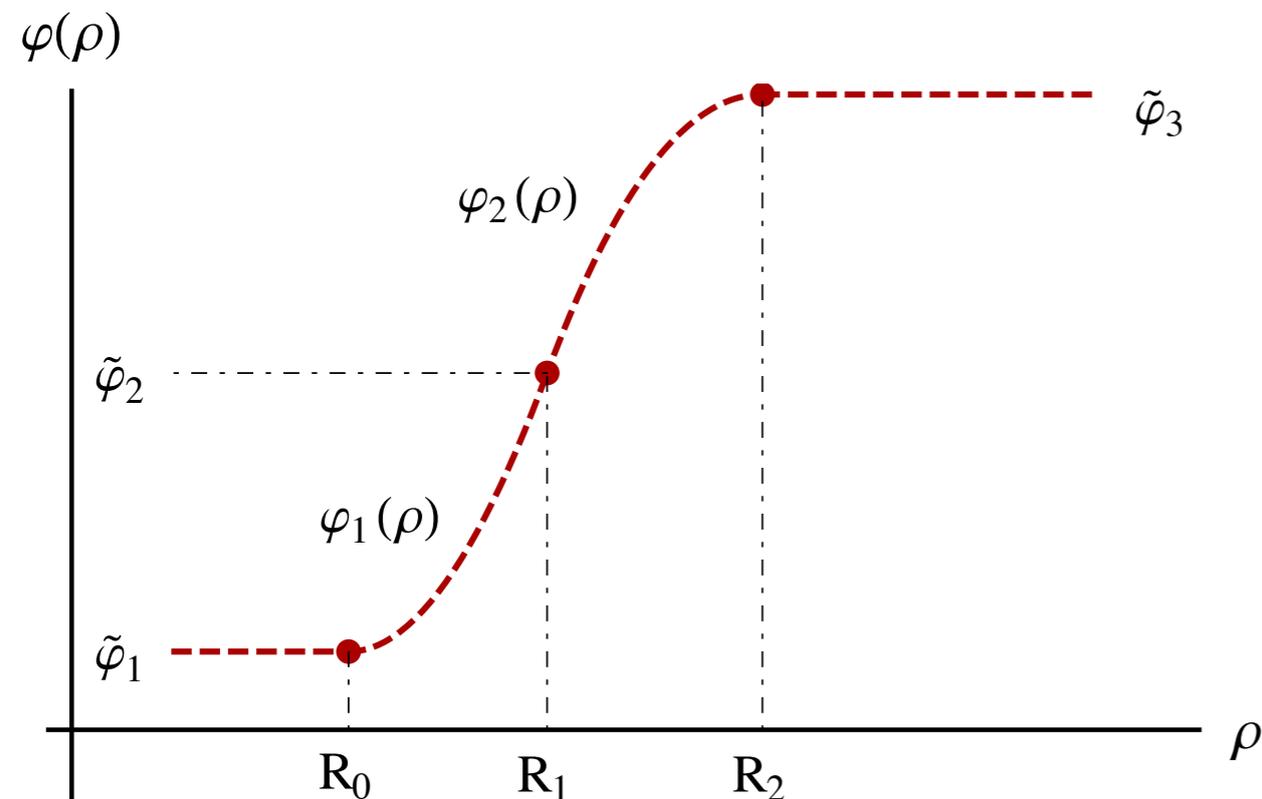
Linear potentials

- triangle and box

Exact solution

$$\ddot{\varphi} + \frac{3}{\rho} \dot{\varphi} = dV = 8a$$

$$\varphi = v + a\rho^2 + \frac{b}{\rho^2}$$



Initial conditions @ R_0

shoot in φ_0 or R_0

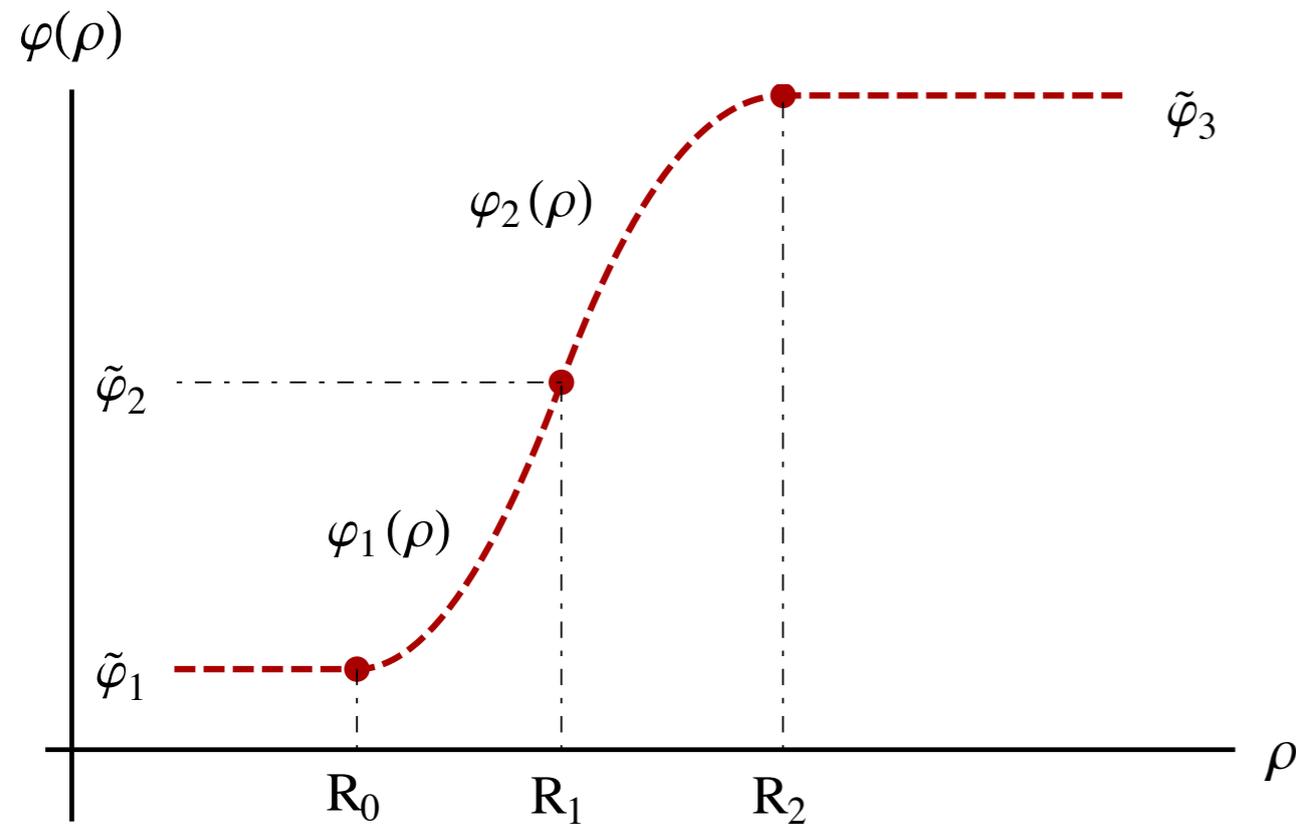
- a) $\varphi_1(0) = \varphi_0, \quad \dot{\varphi}_1(0) = 0$

$$v_1 = \varphi_0, \quad b_1 = 0$$

- b) $\varphi_1(R_0) = \tilde{\varphi}_1, \quad \dot{\varphi}_1(R_0) = 0$

$$v_1 = \tilde{\varphi}_1 - 2a_1 R_0^2, \quad b_1 = a_1 R_0^4$$

Triangle



Matching conditions @ R_1

$$\varphi_1(R_1) = \varphi_2(R_1) = \tilde{\varphi}_2, \quad \dot{\varphi}_1(R_1) = \dot{\varphi}_2(R_1)$$

Final conditions @ R_2

$$\varphi_2(R_2) = \tilde{\varphi}_3, \quad \dot{\varphi}_2(R_2) = 0$$

$$v_2 = \tilde{\varphi}_3 - 2a_2 R_2^2, \quad b_2 = a_2 R_2^4$$

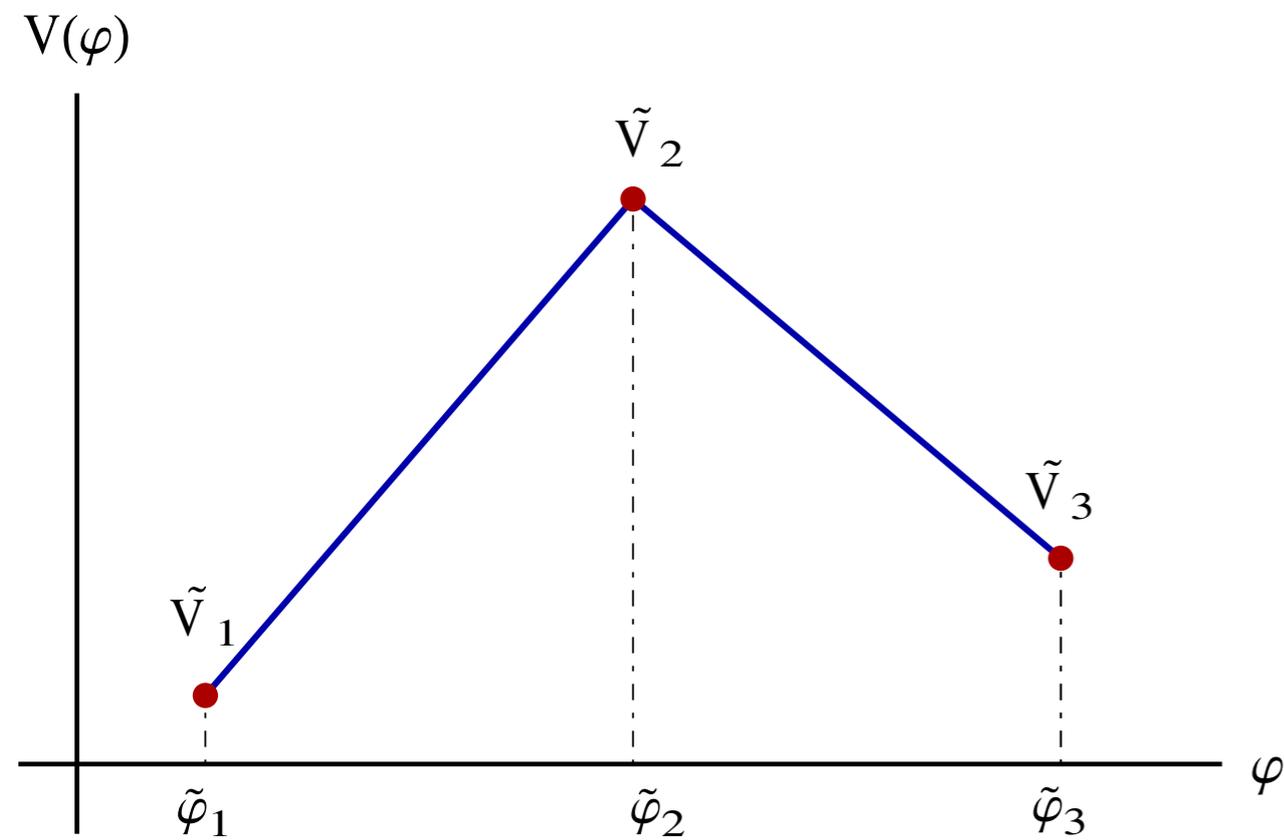
Complete solution

• **a)** $\varphi_0 = \frac{\tilde{\varphi}_3 + c\tilde{\varphi}_2}{1+c}, \quad c = 2\frac{a_2 - a_1}{a_1} \left(1 - \sqrt{\frac{a_2}{a_2 - a_1}}\right) \quad R_1 = \sqrt{\frac{D}{4} \left(\frac{\tilde{\varphi}_2 - \varphi_0}{a_1}\right)}$

• **b)** $R_1 = \frac{1}{2} \frac{\tilde{\varphi}_3 - \tilde{\varphi}_1}{\sqrt{a_1(\tilde{\varphi}_2 - \tilde{\varphi}_1)} - \sqrt{-a_2(\tilde{\varphi}_3 - \tilde{\varphi}_2)}} \quad R_0^2 = R_1 \left(R_1 - \sqrt{\frac{\tilde{\varphi}_2 - \tilde{\varphi}_1}{a_1}} \right)$

$R_2^2 = R_1 \left(R_1 + \sqrt{\frac{\tilde{\varphi}_3 - \tilde{\varphi}_2}{-a_2}} \right)$

Triangle summary



Complete exact analytic solution

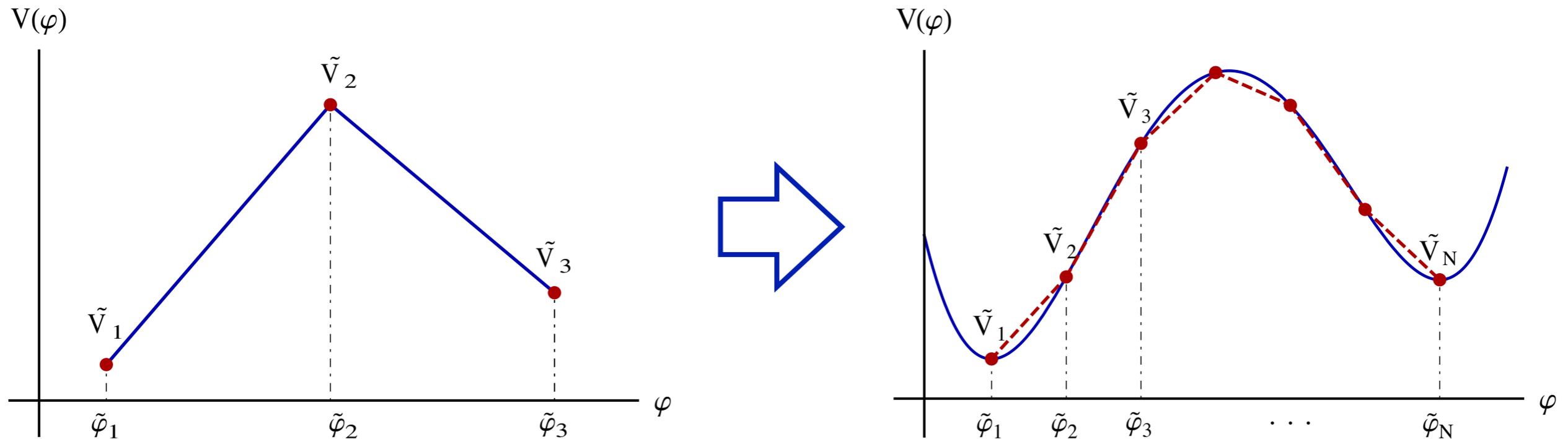
Solved in terms of Euclidean radius

Stable in thin wall, goes over to TW

Limited validity outside the TW

Polygonal bounces

Extend to more segments and D dimensions

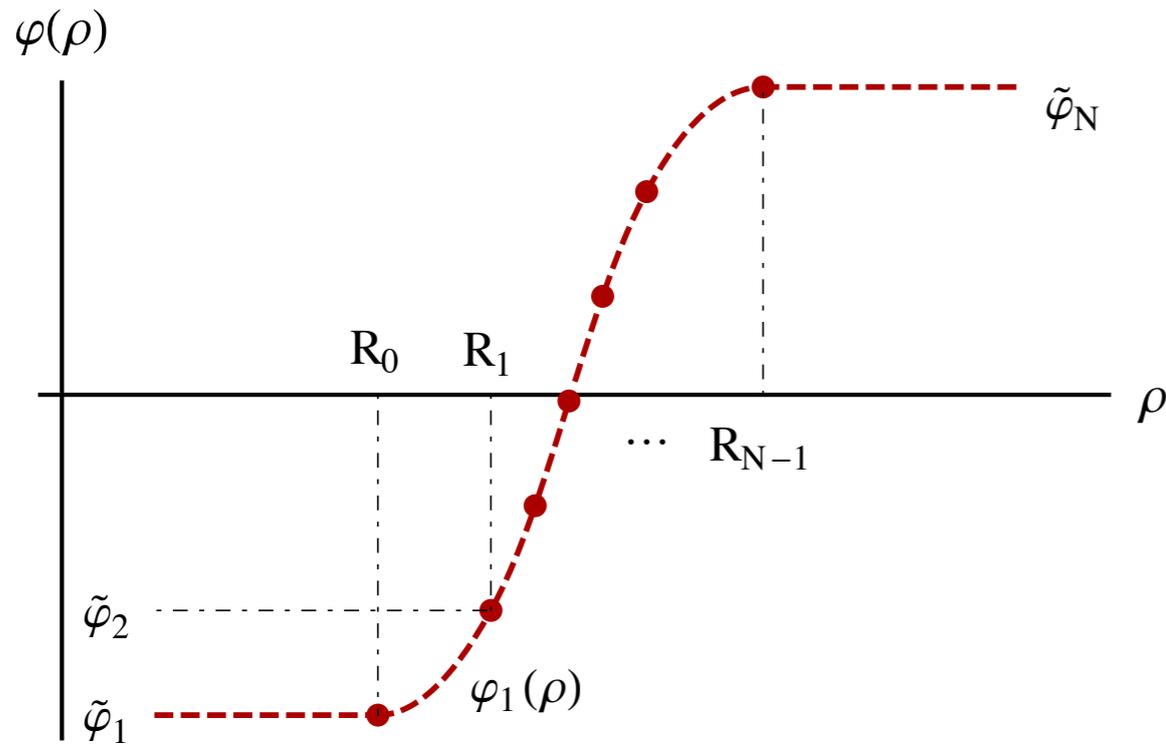


Approximates any V when $N \rightarrow \infty$, controlled precision

Geometric insight of segmentation, cover non-trivial features/unstable Vs

Semi-analytic solution for algebraic manipulation/deformation

Polygonal construction



$$\ddot{\varphi}_i + \frac{D-1}{\rho} \dot{\varphi}_i = dV_i = 8a_i$$

$$\varphi_i = v_i + \frac{4}{D} a_i \rho^2 + \frac{2}{D-2} \frac{b_i}{\rho^{D-2}}$$

Initial/final conditions remain the same

Matching conditions @ R_i 3 parameters and 3 unknowns/segment

$$\varphi_i(R_1) = \varphi_{i+1}(R_i) = \tilde{\varphi}_{i+1}, \quad \dot{\varphi}_i(R_i) = \dot{\varphi}_{i+1}(R_i)$$

The bounce defined recursively

• a) $R_0 = 0$

$$v_n = \varphi_0 - \frac{4}{D-2} \left(a_1 R_0^2 + \sum_{i=1}^{n-1} (a_{i+1} - a_i) R_i^2 \right)$$

• b) $\varphi_0 = \tilde{\varphi}_1$

$$b_n = \frac{4}{D} \left(a_1 R_0^D + \sum_{i=1}^{n-1} (a_{i+1} - a_i) R_i^D \right)$$

Radii computed at each segment from matching the fields

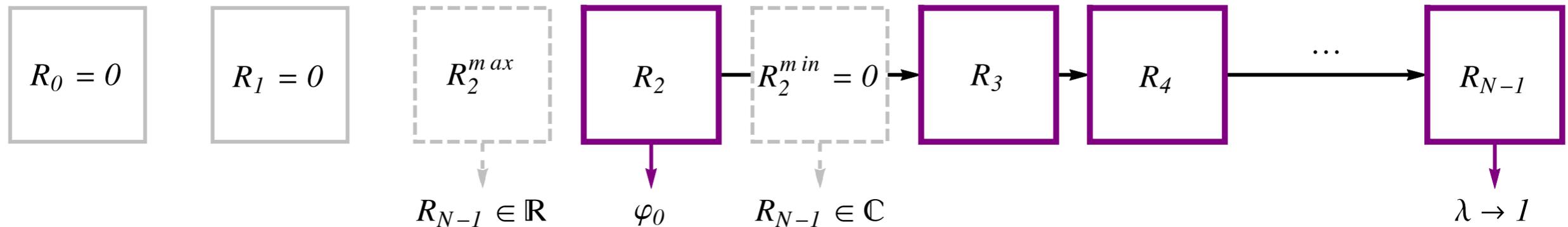
$$\varphi_n(R_n) = \tilde{\varphi}_{n+1}$$

fewnomial

$$R_n^D - \frac{D}{4} \frac{\delta_n}{a_n} R_n^{D-2} + \frac{D}{2(D-2)} \frac{b_n}{a_n} = 0$$

$$\delta_n = \tilde{\varphi}_{n+1} - v_n$$

require real positive roots



radii solutions

$$D = 3 : \quad 2R_n = \frac{1}{\sqrt{a_n}} \left(\frac{\delta_n}{\xi} + \xi \right), \quad \xi^3 = \sqrt{36a_n b_n^2 - \delta_n^3} - 6\sqrt{a_n b_n},$$

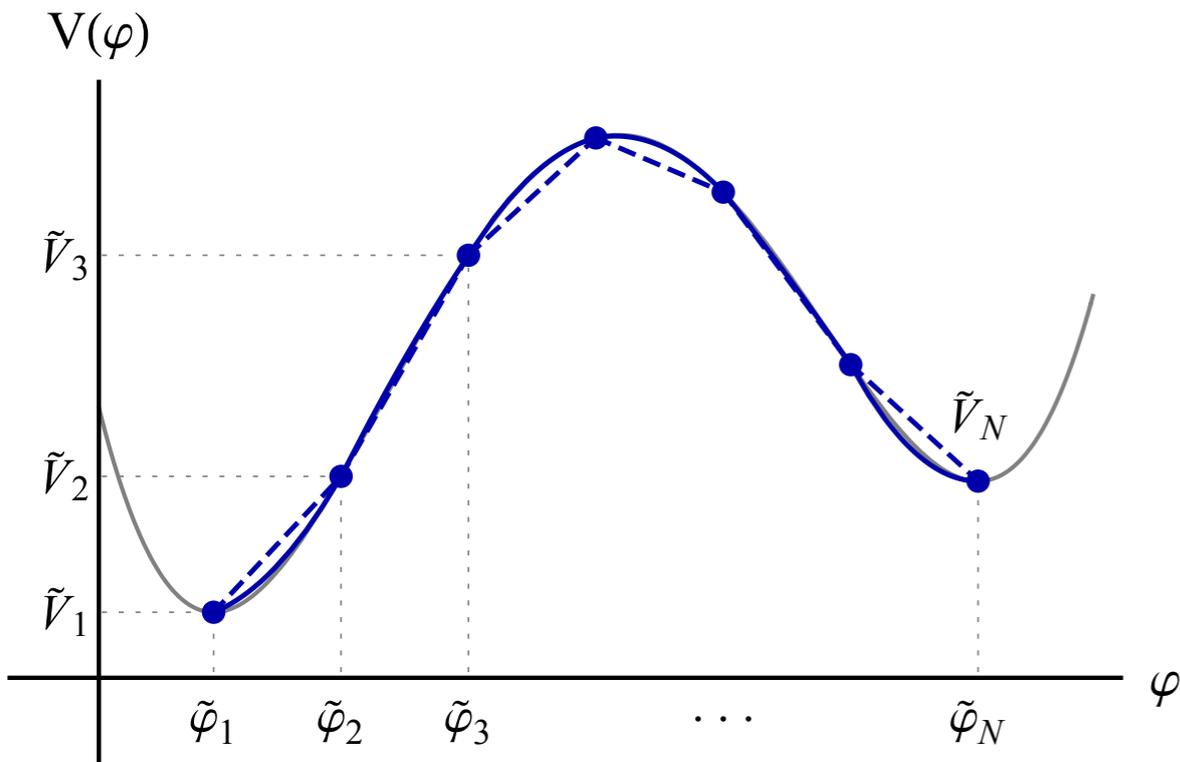
simple cubic

$$D = 4 : \quad 2R_n^2 = \frac{1}{a_n} \left(\delta_n + \sqrt{\delta_n^2 - 4a_n b_n} \right) \quad \text{quadratic}$$

$D = 2, 6, 8$ in the paper, other D s possible numerically

Amariti '20

Higher orders



Expand to higher orders

- improves convergence
- important @ extrema

$$\begin{aligned}
 \text{---} & V_i \simeq \tilde{V}_i - \tilde{V}_N + \partial \tilde{V}_i (\varphi_i - \tilde{\varphi}_i) \\
 \text{—} & + \frac{\partial^2 \tilde{V}_i}{2} (\varphi_i - \tilde{\varphi}_i)^2
 \end{aligned}$$

Perturbative expansion $\varphi = \varphi_{PB} + \xi$

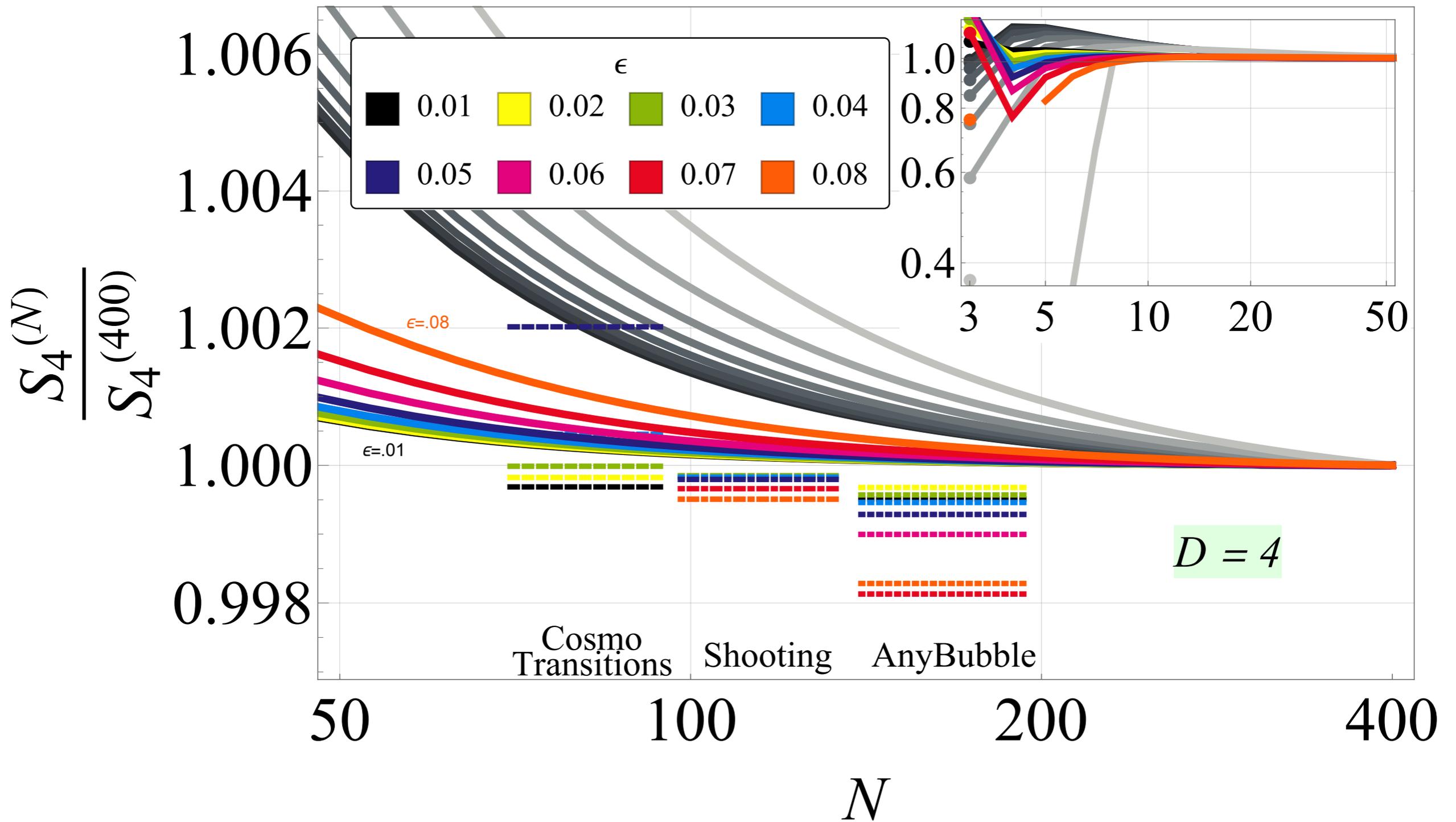
$$\ddot{\varphi} + \frac{D-1}{\rho} \dot{\varphi} = 8(a + \alpha) + \delta dV(\varphi_{PB}(\rho))$$

$$\ddot{\xi} + \frac{D-1}{\rho} \dot{\xi} = 8\alpha + \delta dV(\rho)$$

$$\xi = \nu + \frac{2}{D-2} \frac{\beta}{\rho^{D-2}} + \frac{4}{D} \alpha \rho^2 + \mathcal{I}(\rho)$$

$$\mathcal{I}_s^{D=4} = \partial^2 \tilde{V}_s \left(\frac{v_s - \tilde{\varphi}_s}{8} \rho^2 + \frac{b_s}{2} \ln \rho + \frac{a_s}{24} \rho^4 \right)$$

$$V(\varphi) = \frac{\lambda}{8} (\varphi^2 - v^2)^2 + \varepsilon \left(\frac{\varphi - v}{2v} \right)$$



Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho} \dot{\varphi}_i = \frac{dV}{d\varphi_i}$$

$$\varphi_i(0) = \varphi_{0i}$$

- **CosmoTransitions** **Wainwright '11**

bounce and path deformation separate,
oscillations, Runge-Kutta PDE solver

- **AnyBubble** **Masoumi, Olum, Shlaer '16**

multiple shooting, damping approximations

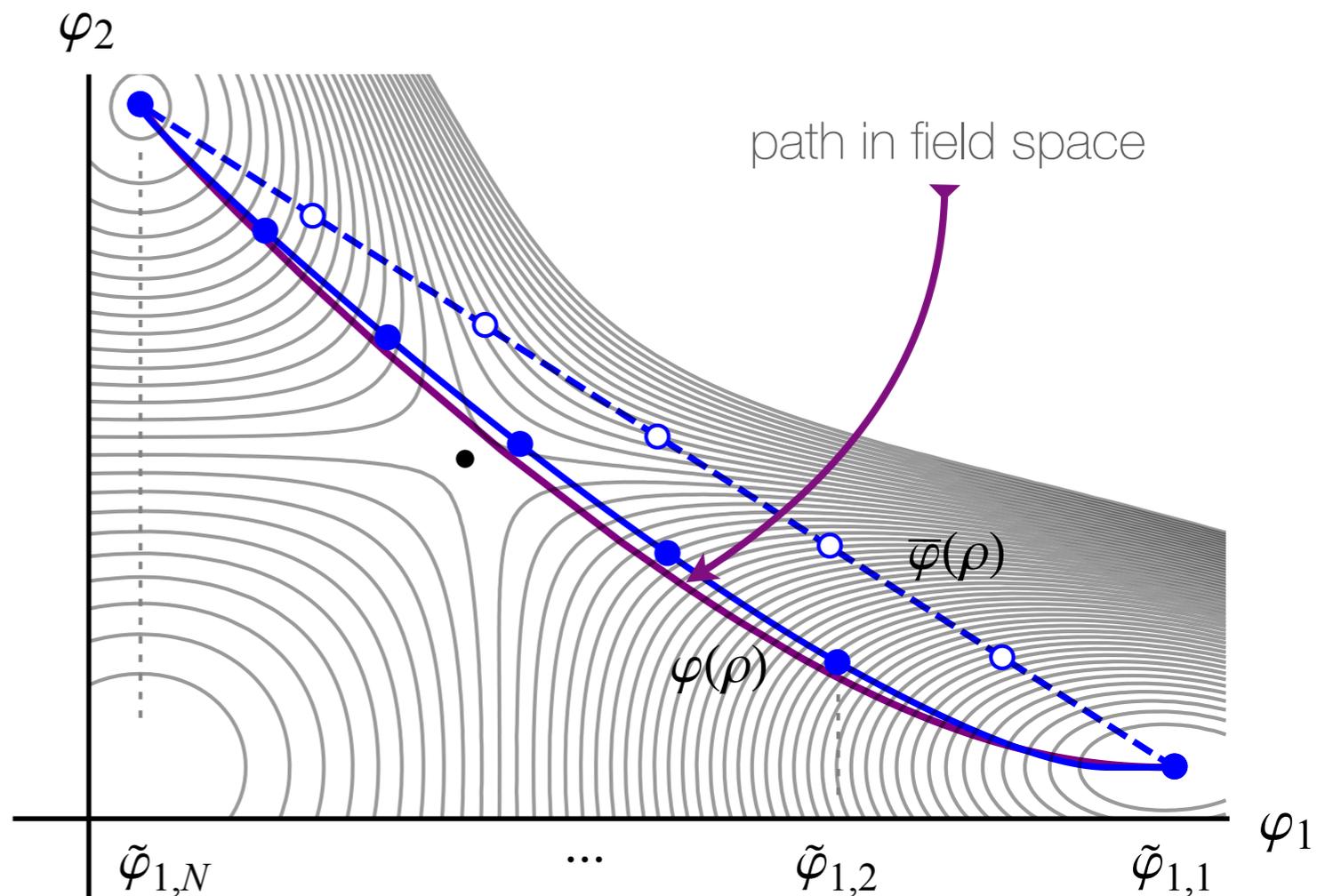
- **Other recent approaches**

tunnelling potential

Espinosa, Konstandin '18

machine learning

Piscopo, Spannowsky, Waite '19

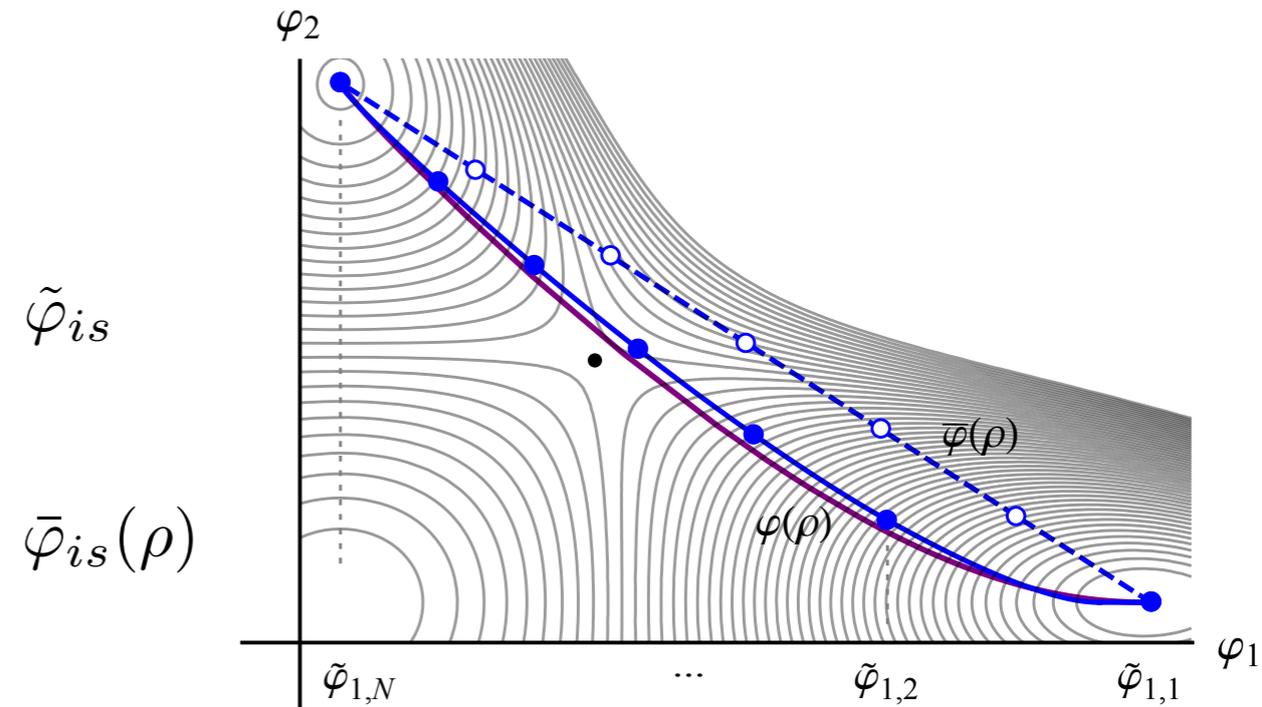


gradient flow

Sato '19

Polygonal approach with many fields

- **Initial ansatz** straight line, via saddle, custom segmentation
- **Initial solution** longitudinal single field PB



Crucial idea #1

- perturbation up to linear term in V , keeps the PB

$$\underbrace{\ddot{\varphi}_{is} + \frac{D-1}{\rho} \dot{\varphi}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho} \dot{\zeta}_{is}}_{8a_{is}} = \frac{dV}{d\varphi_i} (\bar{\varphi} + \zeta)$$

$$\zeta_{is} = v_{is} + \frac{2}{D-2} \frac{b_{is}}{\rho^{D-2}} + \frac{4}{D} a_{is} \rho^2$$

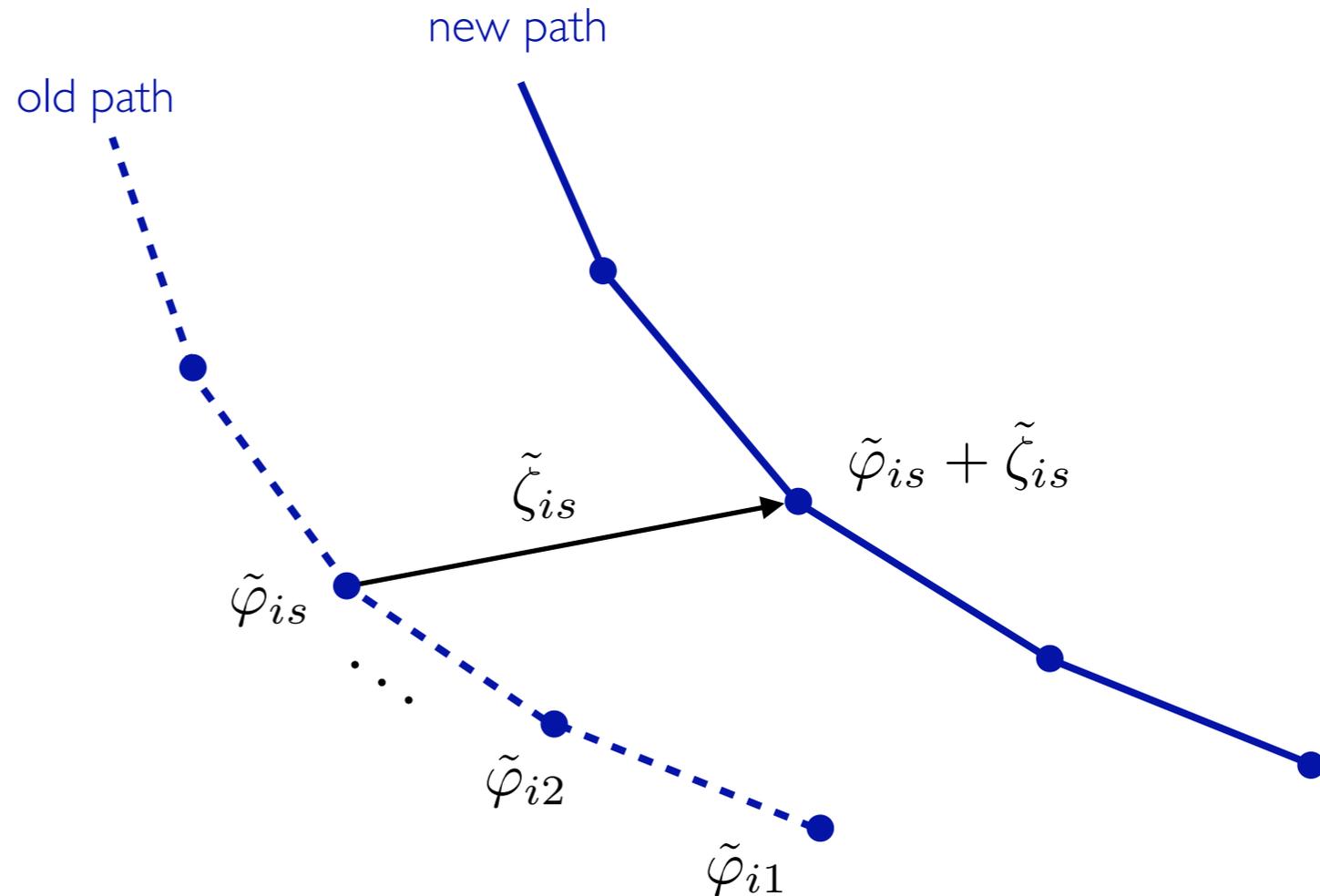
$$\underbrace{\ddot{\varphi}_{is} + \frac{D-1}{\rho} \dot{\varphi}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho} \dot{\zeta}_{is}}_{8a_{is}} = \frac{dV}{d\varphi_i} (\bar{\varphi} + \zeta)$$

Crucial idea #2

$$8a_{is} \simeq \frac{dV}{d\varphi_i} (\tilde{\varphi}_{is} + \tilde{\zeta}_{is}) - 8\bar{a}_{is}$$

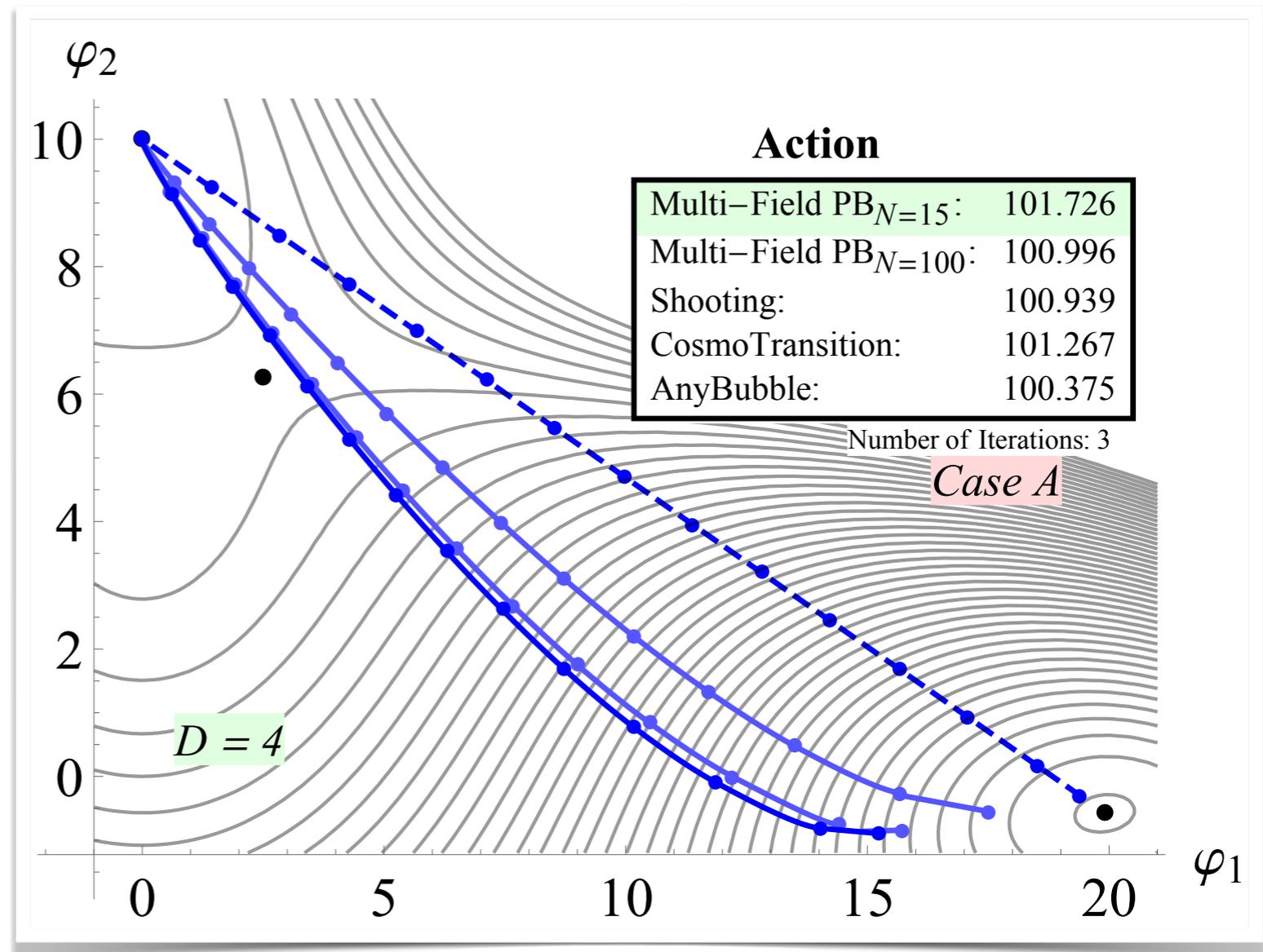
$$\frac{dV}{d\varphi_i} \simeq \frac{1}{2} \left(d_i \tilde{V}_s + d_i \tilde{V}_{s+1} + d_{ij}^2 \tilde{V}_s \tilde{\zeta}_{js} + d_{ij}^2 \tilde{V}_{s+1} \tilde{\zeta}_{js+1} \right)$$

- simultaneous solution for the bounce and path deformation
- linear system for r_{i0} (as in the single field expansion) and $\tilde{\zeta}_{is}$
- iterate until $\tilde{\zeta}_{is} < \varepsilon_{\Delta\varphi}$



$$V(\varphi_i) = \sum_{i=1}^2 (-\mu_i^2 \varphi_i^2 + \lambda_i^2 \varphi_i^4) + \lambda_{12} \varphi_1^2 \varphi_2^2 + \tilde{\mu}^3 \varphi_2$$

- no oscillations
- converges in a few iterations
- works for thin wall
- works for $D=3$ and 4
- tested for up to 20 fields

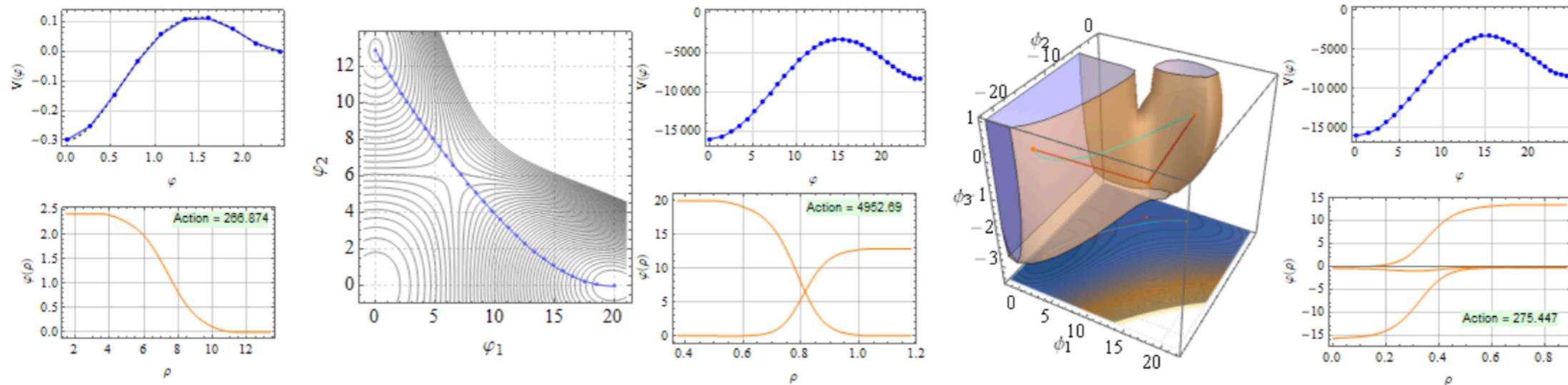


FindBounce

FindBounce <https://github.com/vguada/FindBounce/releases>

FindBounce is a [Mathematica](#) package that computes the bounce configuration needed to compute the false vacuum decay rate with multiple scalar fields.

We kindly ask the users of the package to cite the two papers that describe the working of the *FindBounce* package: the paper with the original proposal by [Guada, Maiezza and Nemevšek \(2019\)](#) and the software release manual by [Guada, Nemevšek and Pintar \(2020\)](#).



Installation

To use the *FindBounce* package you need Mathematica version 10.0 or later. The package is released in the `.paclet` file format that contains the code, documentation and other necessary resources. Download the latest `.paclet` file from the repository ["releases"](#) page to your computer and install it by evaluating the following command in the Mathematica:

```
(* Path to .paclet file downloaded from repository "releases" page. *)
PacletInstall["full/path/to/FindBounce-X.Y.Z.paclet"]
```

Load the package as usual

```
In[1]:= Needs["FindBounce`"]
```

Define a metastable potential

```
In[2]:= V[x_] := 0.5 x^2 + 0.5 x^3 + 0.12 x^4;
```

```
In[3]:= extrema = x/.Sort@Solve[D[V[x],x]==0];
```

Compute the bounce - obtain bf = the bounce function

```
In[4]:= bf = FindBounce[V[x],x,{extrema[[1]],extrema[[3]]}]
```

```
Out[4]= BounceFunction[  ]
```

```
In[5]:= bf["Action"]
```

```
Out[5]= 73496.
```

```
In[6]:= bf["Dimension"]
```

Retrieve the
bounce properties

```
Out[6]= 4
```

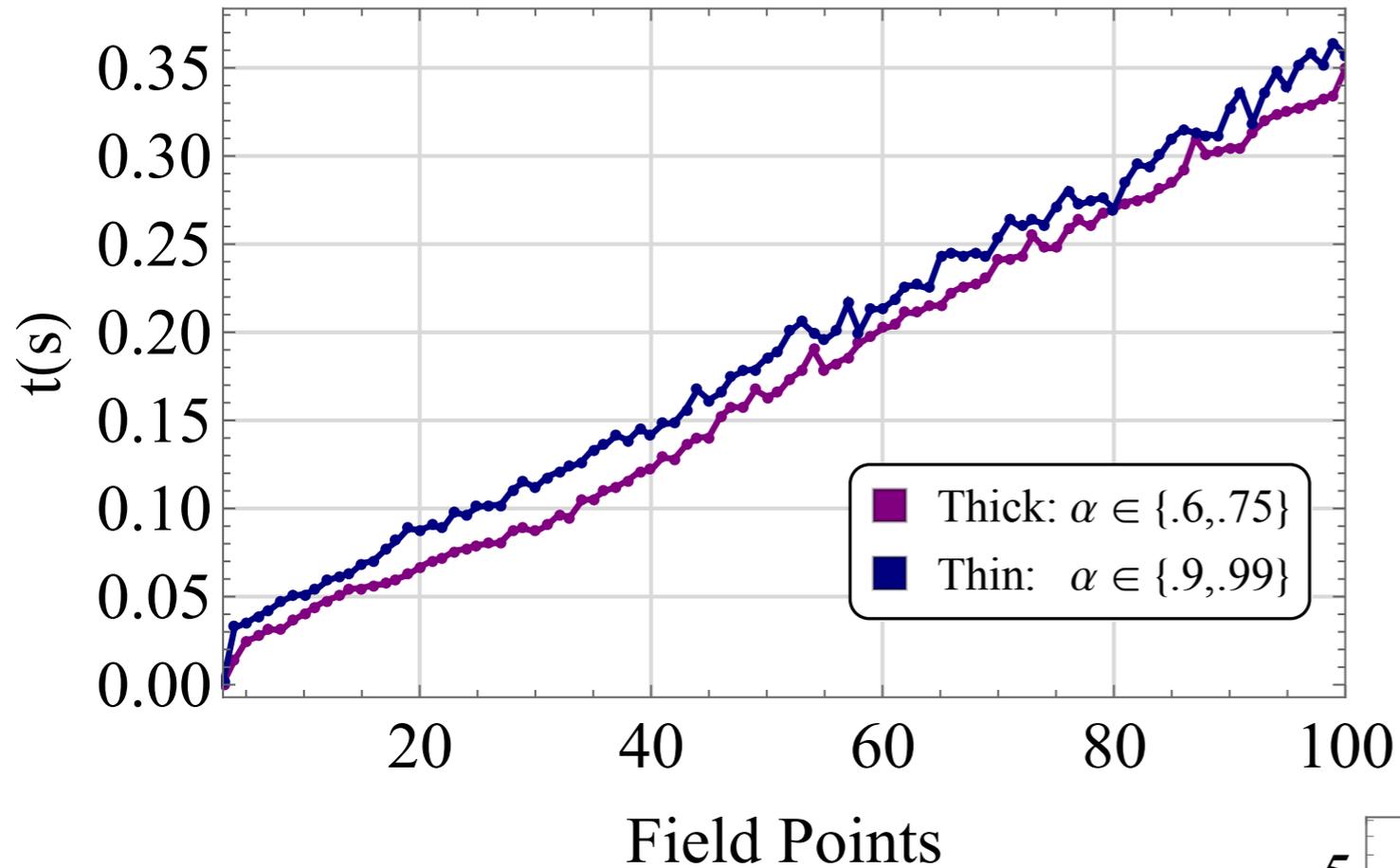
Run on single points

```
FindBounce[{{x1,V1},{x2,V2},...}]
```

And multifiends

```
FindBounce[V[x,y,...],{x,y,...},{m1,m2}]
```

Time demand



Scales linearly by construction

Works in thin and thick regimes

Tested up to 20 fields

CT - CosmoTransitions

Wainwright '11

BP - BubbleProfiler

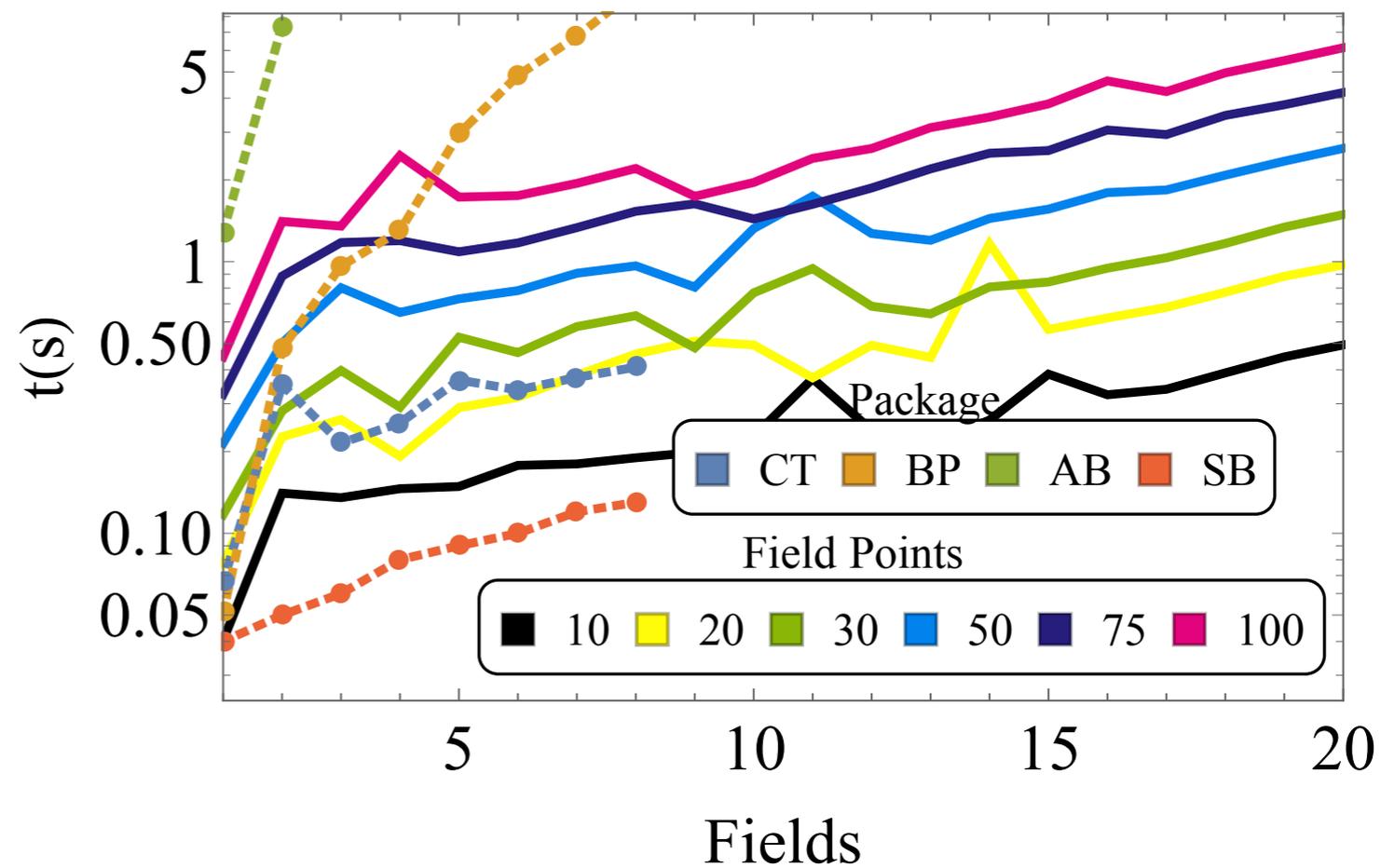
Masoumi et al. '16

AB - AnyBubble

Athron et al. '19

SB - SimpleBounce

Sato '20





$$\int \mathcal{D}\varphi \rightarrow \prod \lambda_i$$

Prefactor

* Thin wall

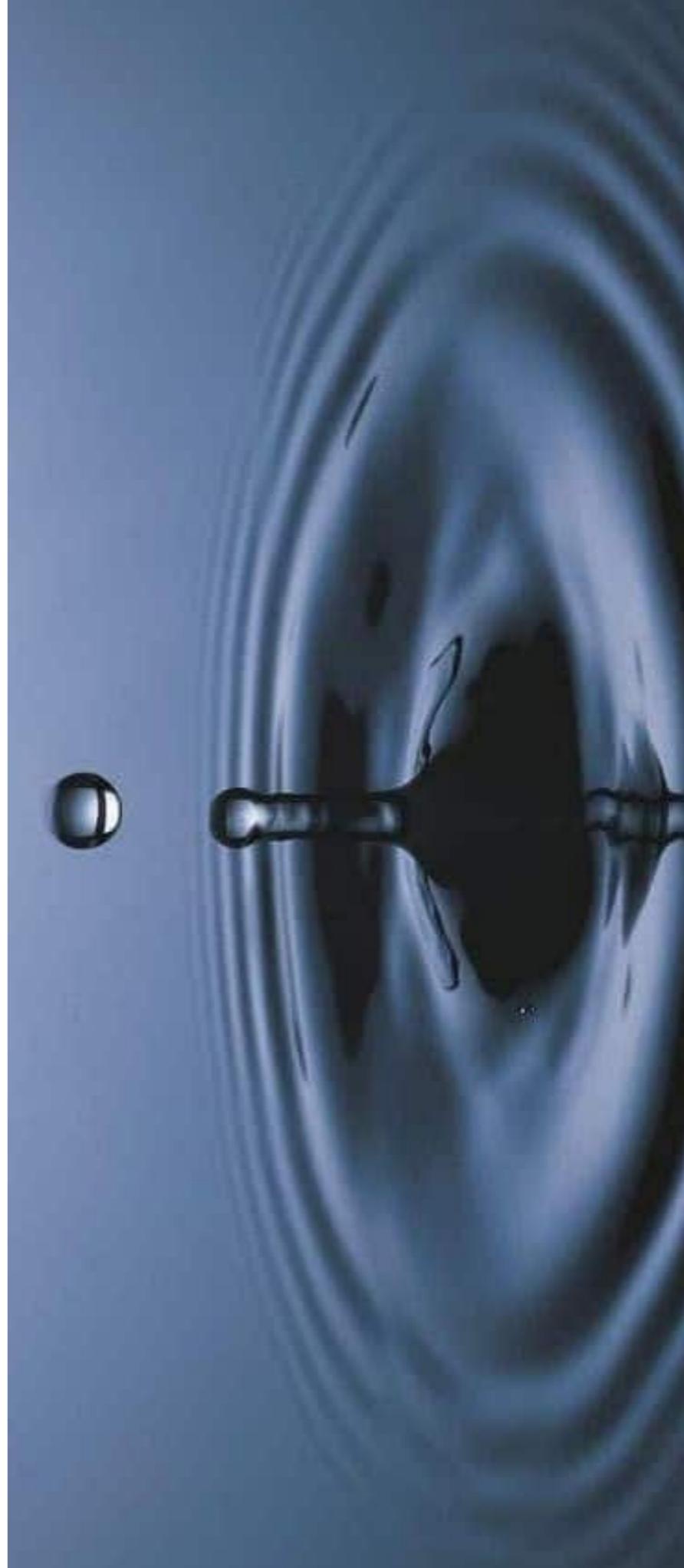
Ivanov, Matteini, MN, Ubaldi '22

* Polygonal bounce

Guada, Maiezza, MN '18

* Quartic-quartic

Guada, MN '20



TW rate at one loop - introduction

Coleman '88

$$\frac{\Gamma}{\mathcal{V}} = \left(\frac{S_R}{2\pi\hbar} \right)^{\frac{D}{2}} \left| \frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}} e^{-\frac{S_R}{\hbar} - S_{\text{ct}}} (1 + \mathcal{O}(\hbar))$$

$\mathcal{O} = -\partial_\mu \partial^\mu + V^{(2)}$ fluctuations around the bounce

Bubble deformation, zeroes for symmetries, single negative

a) Renormalized bounce action and counter-terms

b) Functional determinant

Pedagogical notes on functional determinants

Dunne '07

c) Zero removal

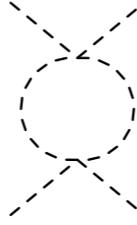
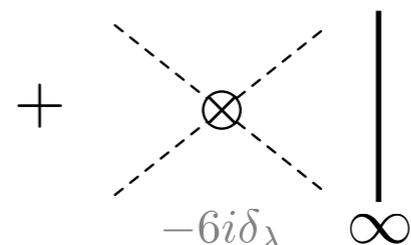
$\det' \mathcal{O}$

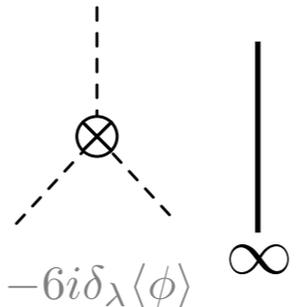
a) Renormalized bounce action

At one loop, the action needs counter-terms

Peskin & Schröder

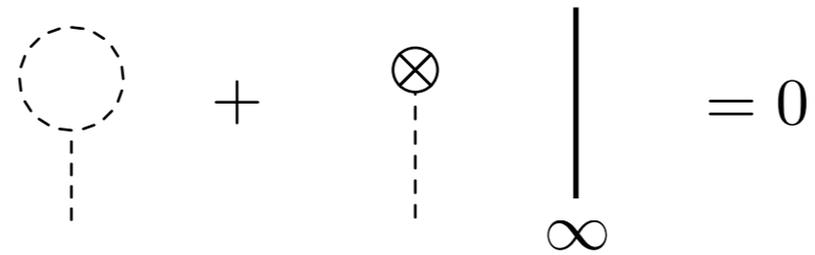
$$V_{\text{ct}} = \frac{\delta_{m^2}}{2} \phi^2 + \frac{\delta_\lambda}{4} \phi^4, \quad \langle \phi \rangle = \frac{\mu}{\sqrt{\lambda}}, \quad V^{(n)} \equiv \frac{d^n V}{d\phi^n} (\langle \phi \rangle)$$

$\delta_\lambda = \frac{1}{32\pi^2 \epsilon} V^{(4)2}$ set by the 4-p function  +  = 0 $\epsilon = 4 - D$

cancels the 3-p  +  = 0 if $V^{(3)} = V^{(4)} \langle \phi \rangle$

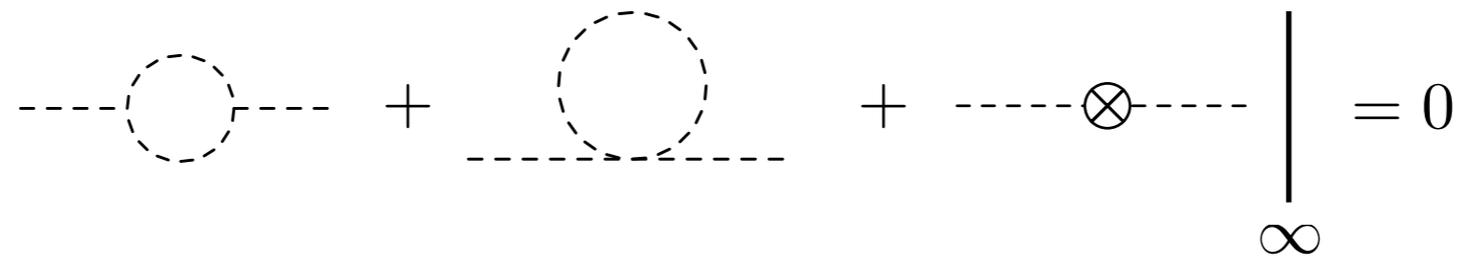
automatic if 3-p interactions comes from a single quartic

$$V^{(4)} = 4! \lambda, \quad \underline{V^{(3)} = 4 \times 3! \lambda \langle \phi \rangle = V^{(4)} \langle \phi \rangle}$$

Remove the tadpoles...  = 0

$$\delta_{m^2} = \frac{1}{(4\pi)^2 \varepsilon} V^{(4)} \left(V^{(2)} - \frac{1}{2} V^{(4)} \langle \phi \rangle^2 \right)$$

...and the 2-p mass divergence cancels out

 = 0

$$\langle \phi \rangle \simeq v (1 - \Delta + \dots)$$

the vev shifts

$$\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2 2\varepsilon},$$

$$\delta_{m^2} = -\frac{3\lambda^2 v^2}{(4\pi)^2 2\varepsilon}$$

independent of Δ

Δ counter-term is zero, and does not run (2 loops?)

a) Renormalized bounce action and counter-terms

Counter-term for the Euclidean action

$$S_{\text{ct}} = \int_D (V_{\text{ct}} - V_{\text{ctFV}}) = \frac{3\lambda^2}{8(4\pi)^2 \varepsilon} \int_D (3(\phi^4 - \phi_{\text{FV}}^4) - 2v^2(\phi^2 - \phi_{\text{FV}}^2)) \simeq -\frac{3}{16\varepsilon\Delta^3}.$$

Running of the Euclidean action

$$S_R + S_{\text{ct}} = S \left(1 - \frac{9\lambda_0}{(4\pi)^2} \left(\frac{1}{\varepsilon} + \ln \frac{\mu}{\mu_0} \right) \right)$$

Non-trivial check: the $\frac{1}{\varepsilon}$ pole and $\ln \mu$ cancel with the determinant

b) Functional determinant

We wish to get the eigenvalues of O and multiply them

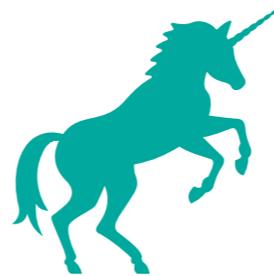
The bounce and fluctuations are spherically symmetric, orbital decomposition

$$\left| \frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}} = \left| \prod_{l=0}^{\infty} \frac{\det' \mathcal{O}_l}{\det \mathcal{O}_{l\text{FV}}} \right|^{-\frac{1}{2}}$$

$$\mathcal{O}_l = -\frac{d^2}{d\rho^2} - \frac{D-1}{\rho} \frac{d}{d\rho} + \frac{l(l+D-2)}{\rho^2} + V^{(2)}$$

Gel'fand-Yaglom theorem

$$\begin{aligned} \mathcal{O}_l \psi_l &= 0, & \psi_l(0) &\sim \rho^l \\ \mathcal{O}_{l\text{FV}} \psi_{l\text{FV}} &= 0, & \psi_{l\text{FV}}(0) &\sim \rho^l \end{aligned}$$



$$\frac{\det \mathcal{O}_l}{\det \mathcal{O}_{l\text{FV}}} = \left(\frac{\psi_l}{\psi_{l\text{FV}} \Big|_{\infty}} \right)^{d_l}$$

$$d_l = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!} \quad \text{eigenvalue degeneracy}$$

Fluctuations

Define the ratio $R_l \equiv \frac{\psi_l}{\psi_{l\text{FV}}}$ and solve the stable equation

$$\ddot{R}_l + 2 \left(\frac{\dot{\psi}_{l\text{FV}}}{\psi_{l\text{FV}}} \right) \dot{R}_l = \left(V^{(2)} - V_{\text{FV}}^{(2)} \right) R_l$$

$$\psi_{l\text{FV}}(0) \sim \rho^l, \quad R_l(0) = 1, \quad \dot{R}_l(0) = 0$$

IR $l \sim 1$

l

UV $l \gg 1$

Low multipoles $l < \frac{1}{\Delta}$

Multiplicative TW expansion $R_l = \prod_{n \geq 0} R_{ln}^{\Delta^n} \quad x = e^z$

$$R_{l0} = \frac{1}{(1+x)^2},$$

$$\ln R_{l1} = 3(r + \ln x),$$

$$\ln R_{l2}(x \rightarrow \infty) = \frac{3}{4} \frac{(l-1)(l+D-1)}{(D-1)^2} x^2$$

$$R_l(\infty) = \Delta^2 e^{D-1} \frac{3}{4} \frac{(l-1)(l+D-1)}{(D-1)^2}$$

Negative and zero modes ok

High multipoles $l \gg \frac{1}{\Delta}$

Shift multipoles $\nu = l + \frac{D}{2} - 1$

FV part $\psi_{\nu\text{FV}} \simeq e^{k_\nu z}, \quad k_\nu^2 = 1 + \frac{\Delta^2 \nu^2}{r_0^2}$

Leading order $R_{\nu 0}(\infty) = \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)}$

$$\ln R_\nu(\infty) = \ln \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} + \underline{3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1} \right)}$$

‘All-order’ corrections

Correct UV behaviour $\lim_{\nu \rightarrow \infty} R_\nu(\infty) = 1$ not low- l , though, as expected

Renormalized determinant

Sum over multipoles

$$\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \sum_{\nu=D/2-1}^{\infty} d_{\nu} \ln R_{\nu}, \quad d_{\nu} \simeq \frac{2}{(D-2)!} \nu^{D-2}$$

Diverges in the UV as expected from QFT

$$\sum_{\nu \gg 1} d_{\nu} \ln R_{\nu \gg 1} \sim -\frac{3r_0(2-r_0)}{(D-2)!\Delta} \sum_{\nu \gg 1} \nu^{D-2} \left(\frac{1}{\nu} - \frac{1}{\nu^3} \left(\frac{r_0}{2\Delta} \right)^2 \right)$$

quadratic and log in $D=4$

Finite sum

$$\Sigma_D = \sum_{\nu=\nu_0}^{\infty} \sigma_D = \sum_{\nu=\nu_0}^{\infty} d_{\nu} (\ln R_{\nu} - \ln R_{\nu}^a)$$

asymptotic subtraction

Renormalized determinants (subtractions and logs for D s)

Dunne, Min '05
Dunne, Kirsten '06
Hur, Min '08

Renormalized determinant $D=4$

$$\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \sum_{\nu} \nu^2 \left(\ln R_{\nu} - \frac{1}{2\nu} I_1 + \frac{1}{8\nu^3} I_2 \right) - \frac{1}{8} \tilde{I}_2$$

Asymptotic subtractions remove first two divergencies in any D

$$I_1 = \int_0^{\infty} d\rho \rho \left(V^{(2)} - V_{\text{FV}}^{(2)} \right) \simeq -3(2 - r_0) \left(\frac{r_0}{\Delta} \right),$$

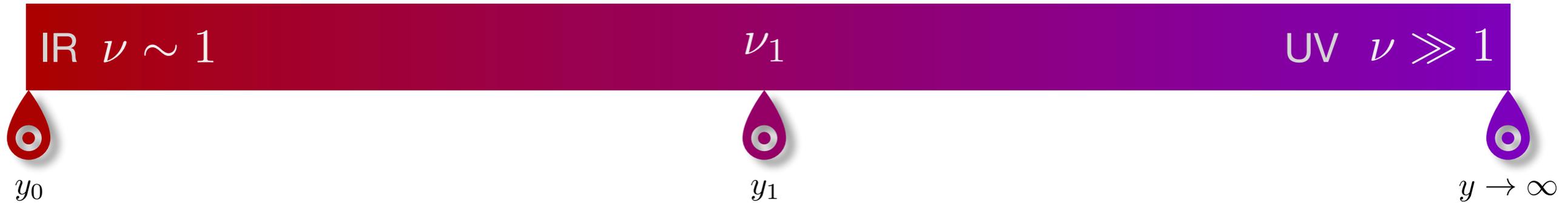
$$I_2 = \int_0^{\infty} d\rho \rho^3 \left(V^{(2)2} - V_{\text{FV}}^{(2)2} \right) \simeq -3(2 - r_0) \left(\frac{r_0}{\Delta} \right)^3$$

The renormalized piece gives the pole and the log scale part (and a finite piece, the 1)

$$\begin{aligned} \tilde{I}_2 &= \int_0^{\infty} d\rho \rho^3 \left(V^{(2)2} - V_{\text{FV}}^{(2)2} \right) \left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln \left(\frac{\mu\rho}{2} \right) \right) \\ &\simeq I_2 \left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln \left(\frac{\mu r_0}{2\sqrt{\lambda}v\Delta} \right) \right) \end{aligned}$$

Final sum done by Euler-Maclaurin, dominated by *large* multipoles, use $y = \frac{\Delta\nu}{r_0}$

$$\Sigma_4^f \simeq \frac{1}{\Delta^3} \int_{y_0}^{\infty} dy y^2 \left(\ln R_\nu + \frac{3}{2y} - \frac{3}{8y^3} \right) = \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{36} + \ln 2y_0 \right)$$



Separate low and high at arbitrary intermediate multipole ν_1 (artefact anyway)

$$\Sigma_D = \Sigma_D^{\text{low}} + \Sigma_D^{\text{high}} = \sum_{\nu=\nu_0}^{\nu_1} \sigma_D + \sum_{\nu=\nu_1+1}^{\infty} \sigma_D$$

Combining low and high, we get the finite sum

$$\Sigma_4 = \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{36} - \gamma_E + \ln 2\Delta \right) \rightarrow \ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \Sigma_4 - \frac{\tilde{I}_2}{8}$$

The γ_E and $\ln \Delta$ cancel with parts in $\tilde{I}_2 \simeq I_2 \left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln \left(\frac{\mu r_0}{2\sqrt{\lambda\nu\Delta}} \right) \right)$

Explicit closed form renormalized rate at one loop

$$\frac{\Gamma}{\mathcal{V}} \simeq \left(\left(\frac{S}{2\pi} \right) \frac{12}{e^{D-1}} \lambda v^2 \right)^{D/2} \exp \left[-S - \frac{1}{\Delta^{D-1}} \begin{cases} \frac{20+9 \ln 3}{54}, & D = 3, \\ \frac{45-4\pi\sqrt{3}}{192}, & D = 4, \end{cases} \right]$$

zero removal
determinant part

Renormalized action

$$S = \frac{1}{\Delta^{D-1}} \begin{cases} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1 \right) \Delta^2 \right), & D = 3, \\ \frac{\pi^2}{3\lambda} \left(1 - \left(2\pi^2 + \frac{9}{2} \right) \Delta^2 \right), & D = 4. \end{cases}$$

General procedure for and D even and odd

Bounces and prefactors

are fascinating

Appendix

Zero removal

$$\Gamma \propto \frac{1}{\sqrt{\det \mathcal{O}'}} = \prod_n \sqrt{\frac{\lambda_{n\text{FV}}}{\lambda'_n}} \propto (v^2)^{D/2}$$

Eigenvalues have $d=2$, drop D of them, sqrt overall

However, we're working with GY, so all the $l=1$ eigenvalues are multiplied together

The trick: modified GY

$$(\mathcal{O}_{l=1} + \mu_\varepsilon^2) \psi_{l=1}^\varepsilon = 0$$

$$R_{l=1}^\varepsilon(\infty) = \frac{\psi_{l=1}^\varepsilon(\infty)}{\psi_{l=1}^{\text{FV}}(\infty)} \simeq \frac{(\mu_\varepsilon^2 + \gamma_1) \prod_{n=2}^\infty \gamma_n}{\prod_{n=1}^\infty \gamma_n^{\text{FV}}} = \mu_\varepsilon^2 R'_{l=1}(\infty)$$

We already have the needed fluctuation functions, easy to off-set

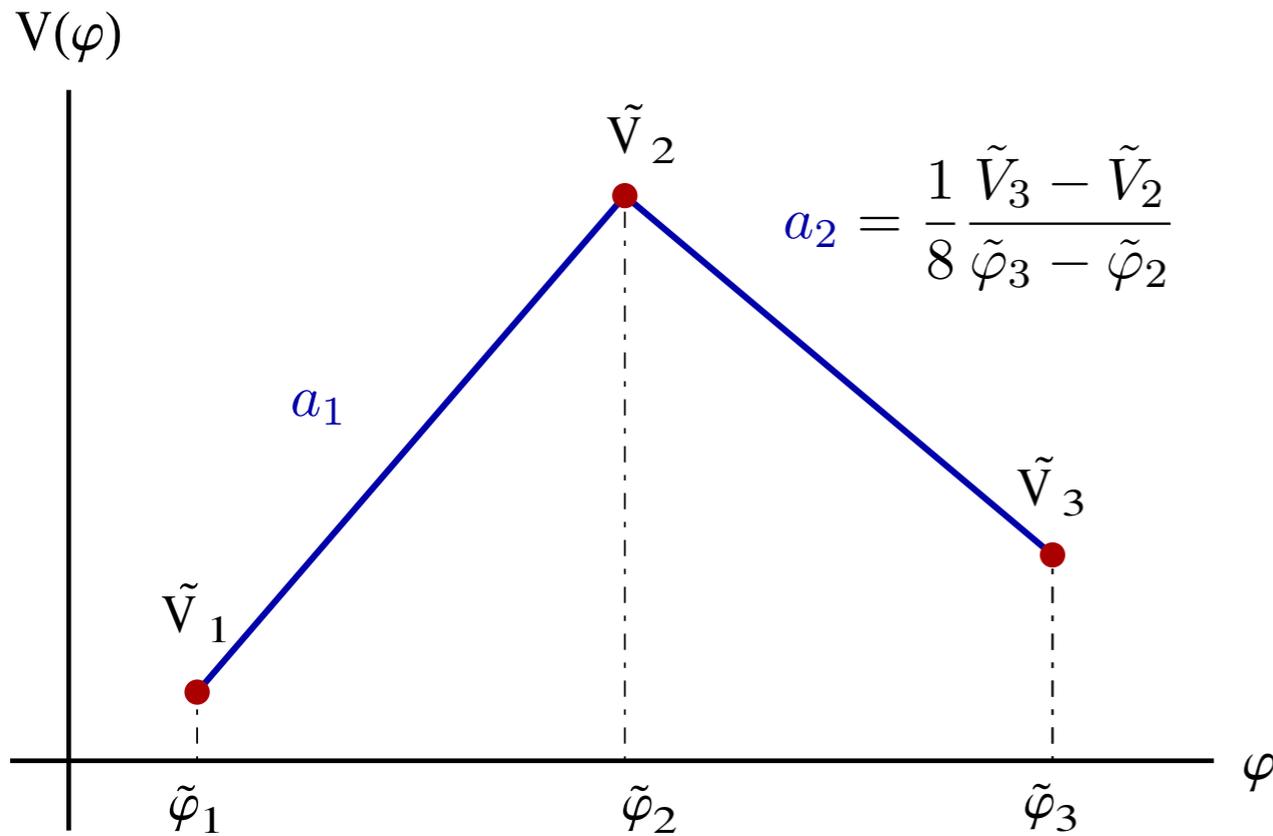
$$R'_{l=1}(\infty) = \lim_{\mu_\varepsilon^2 \rightarrow 0} \frac{1}{\mu_\varepsilon^2} R_{l=1}^\varepsilon(\infty) = \frac{e^{D-1}}{12} \frac{1}{\lambda v^2}$$

* two other ways of seeing the same thing give the same answer

This answers the question of dimensional analysis estimate

Triangular

Duncan, Jensen '92



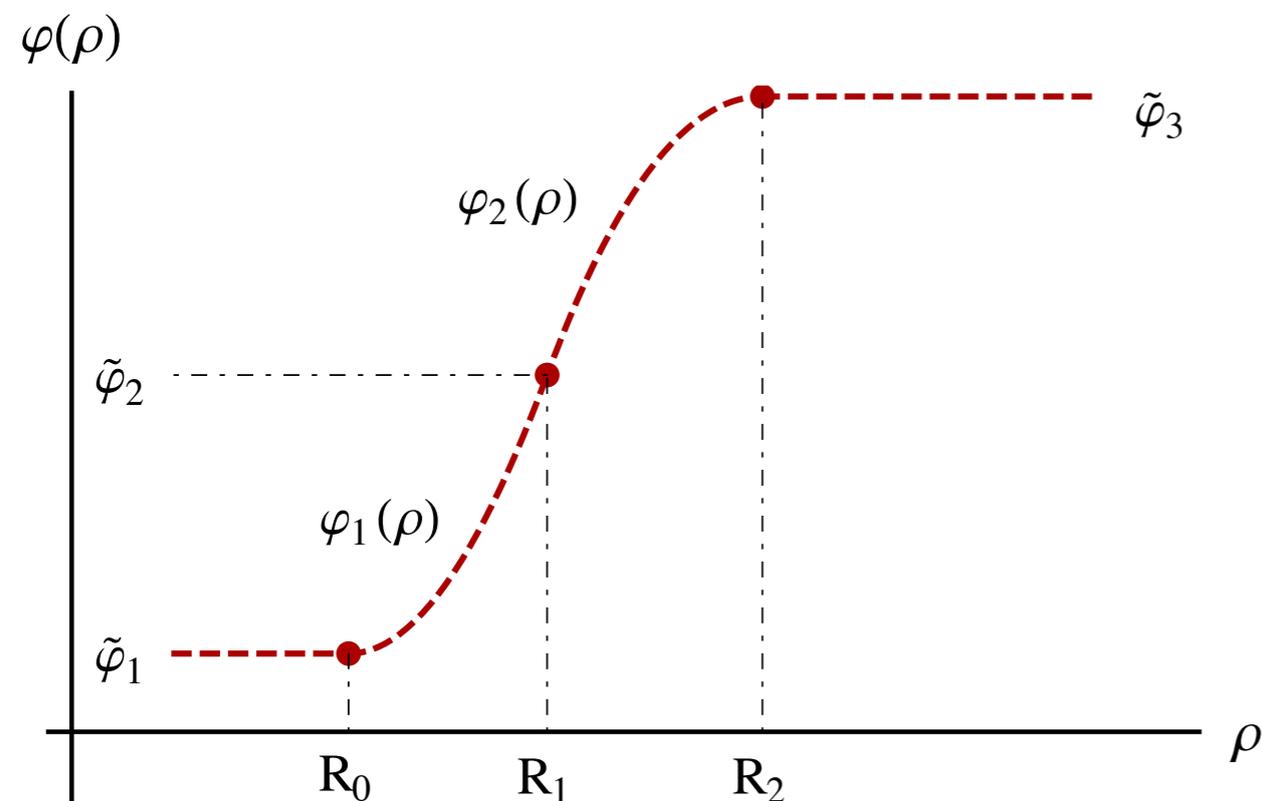
Linear potentials

- triangle and box

Exact solution

$$\ddot{\varphi} + \frac{3}{\rho} \dot{\varphi} = dV = 8a$$

$$\varphi = v + a\rho^2 + \frac{b}{\rho^2}$$



Initial conditions @ R_0

- a) $\varphi_1(0) = \varphi_0, \quad \dot{\varphi}_1(0) = 0$

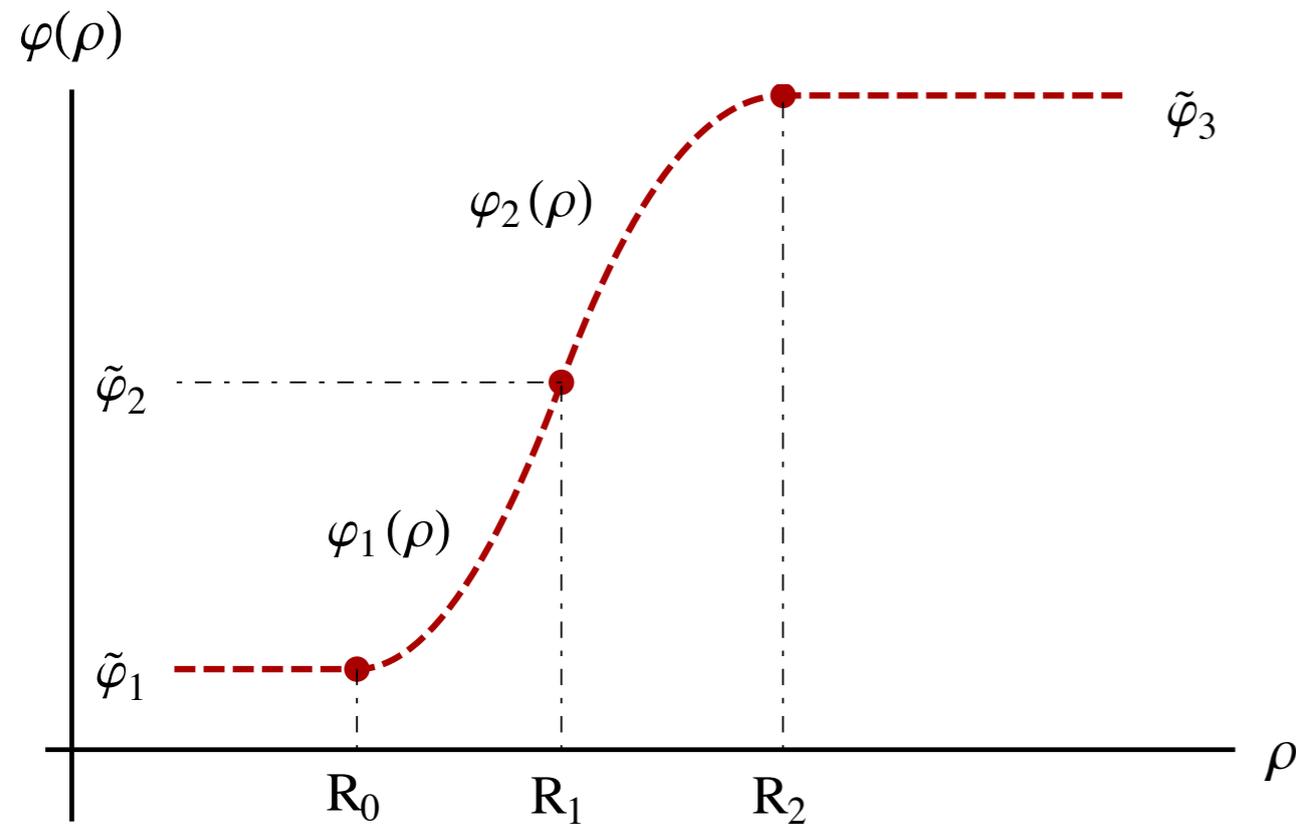
shoot in φ_0

$$v_1 = \varphi_0, \quad b_1 = 0$$

- b) $\varphi_1(R_0) = \tilde{\varphi}_1, \quad \dot{\varphi}_1(R_0) = 0$ or R_0

$$v_1 = \tilde{\varphi}_1 - 2a_1 R_0^2, \quad b_1 = a_1 R_0^4$$

Triangular



Matching conditions @ R_1

$$\varphi_1(R_1) = \varphi_2(R_1) = \tilde{\varphi}_2, \quad \dot{\varphi}_1(R_1) = \dot{\varphi}_2(R_1)$$

Final conditions @ R_2

$$\varphi_2(R_2) = \tilde{\varphi}_3, \quad \dot{\varphi}_2(R_2) = 0$$

$$v_2 = \tilde{\varphi}_3 - 2a_2 R_2^2, \quad b_2 = a_2 R_2^4$$

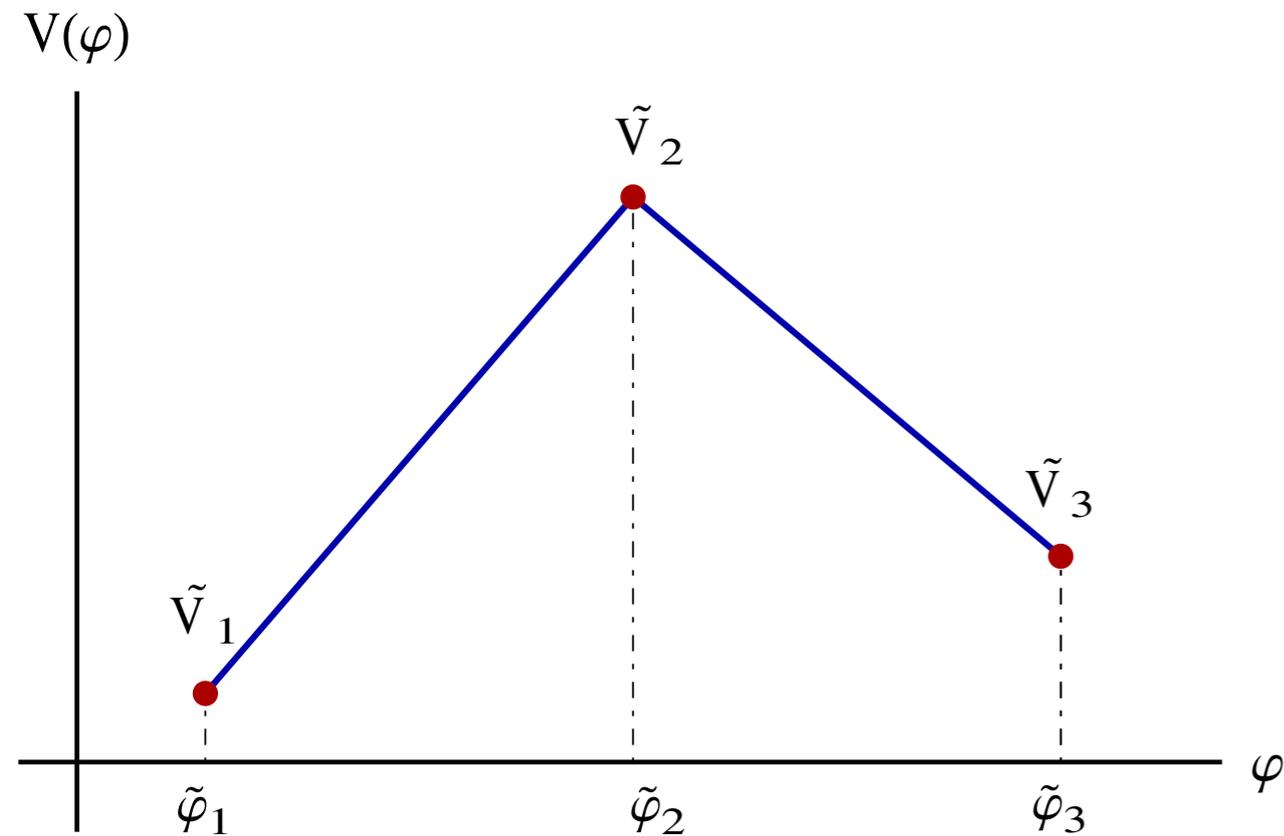
Complete solution - a) works in D-dimensions

$$\bullet \text{ a) } \varphi_0 = \frac{\tilde{\varphi}_3 + c\tilde{\varphi}_2}{1+c}, \quad c = 2\frac{a_2 - a_1}{a_1} \left(1 - \sqrt{\frac{a_2}{a_2 - a_1}}\right) \quad R_1 = \sqrt{\frac{D}{4} \left(\frac{\tilde{\varphi}_2 - \varphi_0}{a_1}\right)}$$

$$\bullet \text{ b) } R_1 = \frac{1}{2} \frac{\tilde{\varphi}_3 - \tilde{\varphi}_1}{\sqrt{a_1(\tilde{\varphi}_2 - \tilde{\varphi}_1)} - \sqrt{-a_2(\tilde{\varphi}_3 - \tilde{\varphi}_2)}} \quad R_0^2 = R_1 \left(R_1 - \sqrt{\frac{\tilde{\varphi}_2 - \tilde{\varphi}_1}{a_1}} \right)$$

$$R_2^2 = R_1 \left(R_1 + \sqrt{\frac{\tilde{\varphi}_3 - \tilde{\varphi}_2}{-a_2}} \right)$$

Summary



Complete exact analytic solution

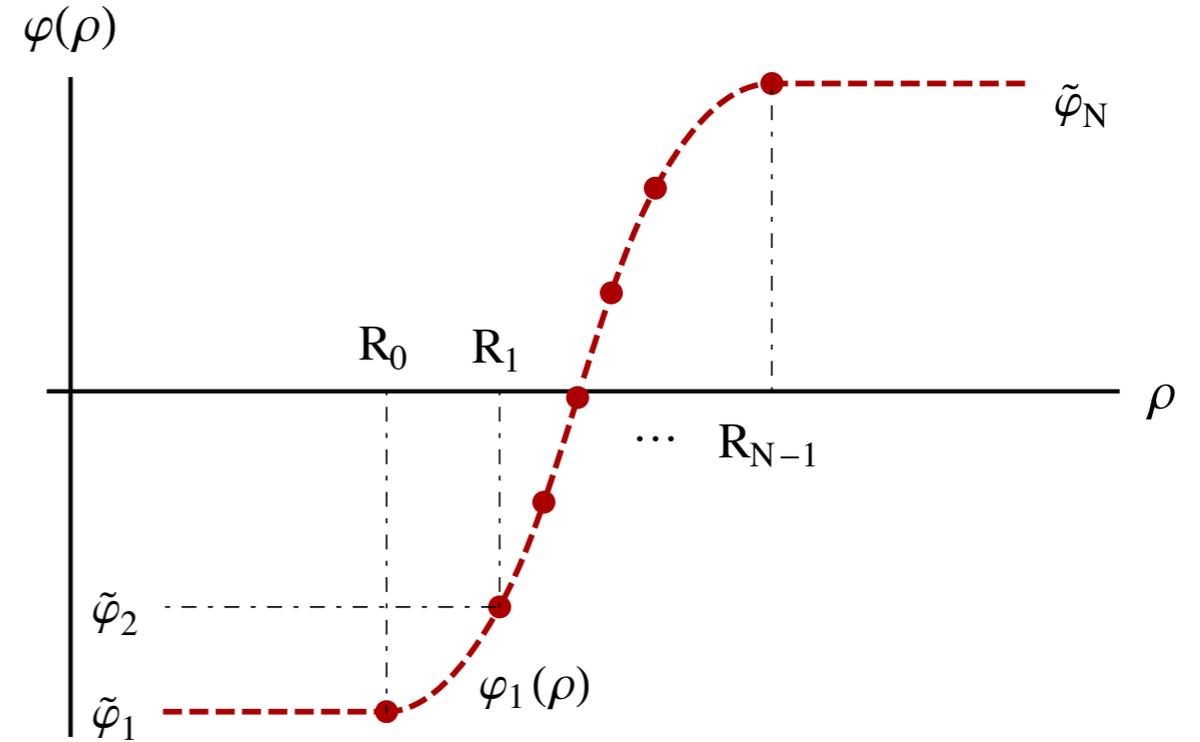
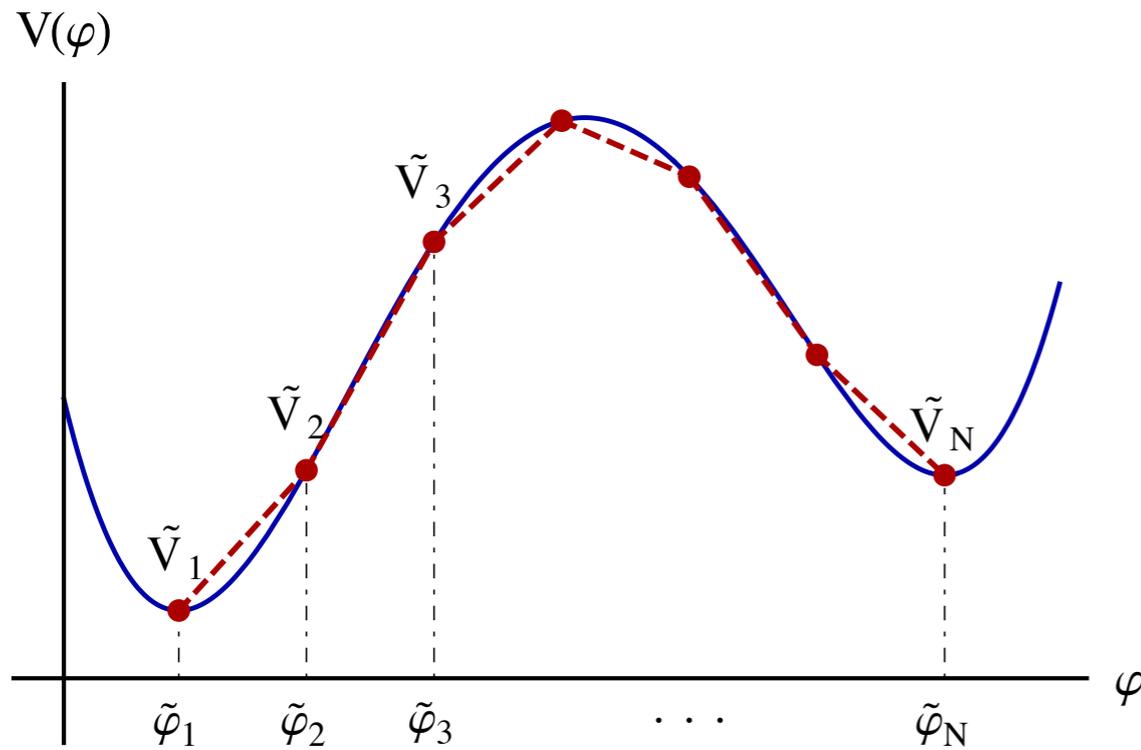
Solved in terms of Euclidean radius

Stable in thin wall, goes over to TWA
with limited validity

Dutta, Hector, Konstandin,
Vaudrevange, Westphal '12

Analytic continuation in Minkowski space
describes the bubble evolution [Pastras '11](#)

Polygonal bounces



$$V_i(\varphi) = \underbrace{\left(\frac{\tilde{V}_{i+1} - \tilde{V}_i}{\tilde{\varphi}_{i+1} - \tilde{\varphi}_i} \right)}_{8 a_i} (\varphi - \tilde{\varphi}_i) + \tilde{V}_i - \tilde{V}_N, \quad dV_i = 8 a_i.$$

No free parameters, one segment three unknowns v_i , b_i , R_i

Generalize case b), solve R_0 or R_i a), retrieve φ_0

Radii computed at each segment from matching the fields

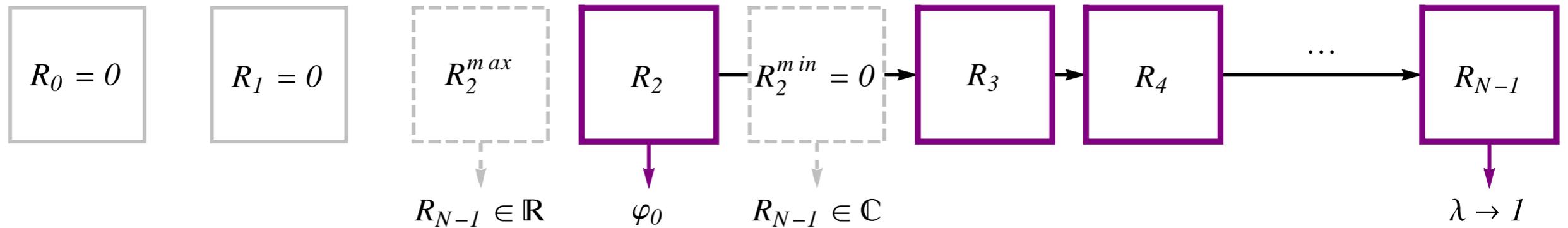
$$\varphi_n(R_n) = \tilde{\varphi}_{n+1}$$

fewnomial

$$R_n^D - \frac{D}{4} \frac{\delta_n}{a_n} R_n^{D-2} + \frac{D}{2(D-2)} \frac{b_n}{a_n} = 0$$

$$\delta_n = \tilde{\varphi}_{n+1} - v_n$$

require real positive roots



Bounce action

Euclidean action

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \rho^{D-1} d\rho \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right)$$

PB action

$$S_{>2} = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \frac{R_0^D}{D} (\tilde{V}_1 - \tilde{V}_N) + \sum_{i=1}^{N-1} \left[\rho^2 \left(\frac{32a_i^2(D+1)\rho^D}{D^2(D+2)} + \frac{16a_i b_i}{D(D-2)} \right. \right. \right. \\ \left. \left. \left. - \frac{2b_i^2}{\rho^D(D-2)} \right) + \frac{\rho^D}{D} \left(8a_i(v_i - \tilde{\varphi}_i) + \tilde{V}_i - \tilde{V}_N \right) \right]_{R_{i-1}}^{R_i} \right\}$$

Total

$$S = \mathcal{T} + \mathcal{V}$$

$$\mathcal{T} \propto \int_0^\infty \rho^{D-1} d\rho \dot{\varphi}^2,$$

kinetic

$$\mathcal{V} \propto \int_0^\infty \rho^{D-1} d\rho V(\varphi)$$

potential

Derrick's theorem

Non-existence of non-trivial static solutions of KG equation, no solitonic scalar 'particles'

Derrick '64

Unstable under re-scaling

$$\varphi(\rho) \rightarrow \varphi(\rho/\lambda)$$

$$\begin{aligned}\lambda \times 0 &= 0, \\ \lambda \times \infty &= \infty\end{aligned}$$

$$S_D^{(\lambda)} = \lambda^{D-2}\mathcal{T} + \lambda^D\mathcal{V}$$

change of variables...remain the same

action is extremized at non-scaled values for true solutions

$$\left. \frac{dS_D^{(\lambda)}}{d\lambda} \right|_{\lambda=1} = 0$$

$$(D-2)\mathcal{T} + D\mathcal{V} = 0$$

relation between kinetic and potential

$$\left. \frac{d^2 S_D^{(\lambda)}}{d\lambda^2} \right|_{\lambda=1} < 0$$

Caveat for PB

$$R \rightarrow \lambda R$$

Works for $N \gg 1$

Benchmarks

Back to thin wall

Coleman '77

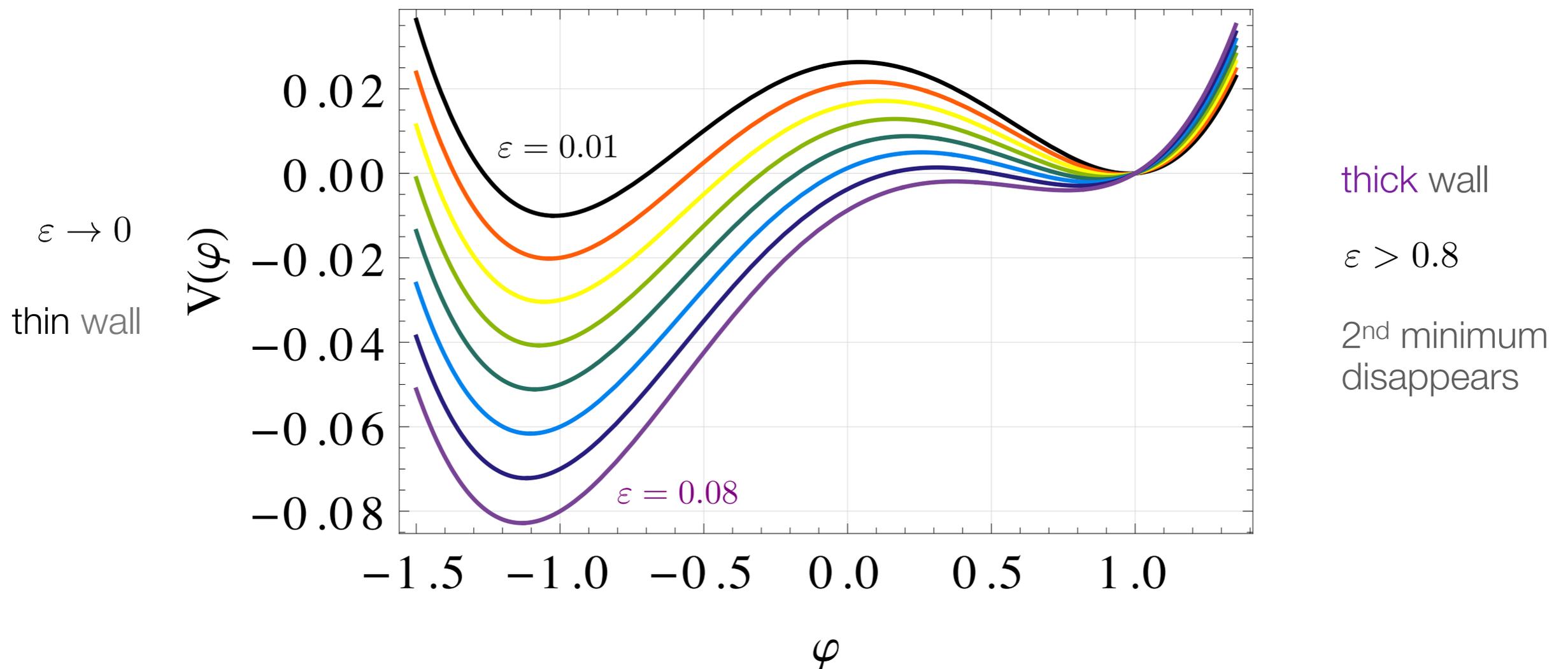
$$V(\varphi) = \frac{\lambda}{8} (\varphi^2 - v^2)^2 + \varepsilon \left(\frac{\varphi - v}{2v} \right)$$

Benchmark for testing

$\lambda = 0.25, \quad v = 1$

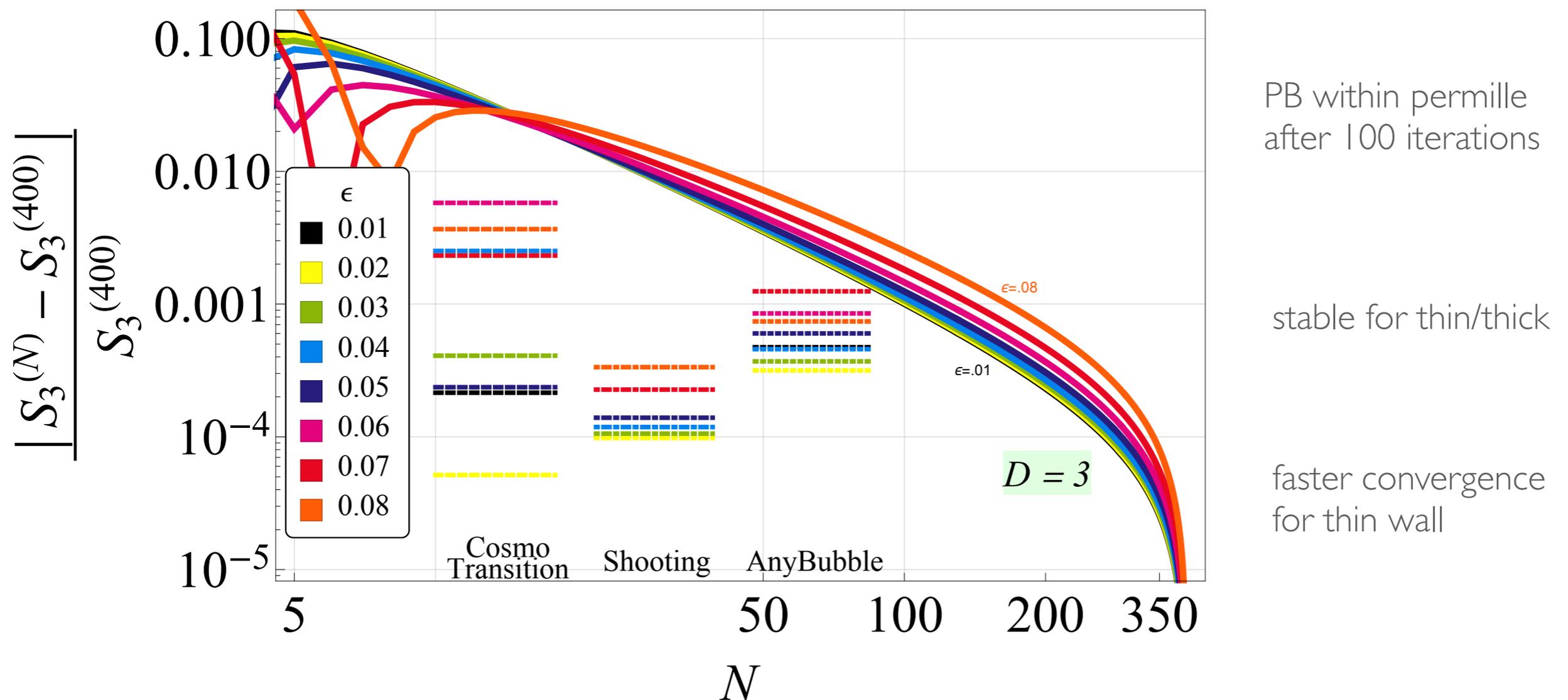
rescaling

Sarid '98



Euclidean action, comparisons

- **CosmoTransitions** Runge-Kutta PDE solver, initial value approximations **Wainwright '11**
discontinued
- **AnyBubble** multiple shooting, damping approximations **Masoumi, Olum, Shlaer '16**
- **Shooting** Mathematica, precise setting of initial values, issues with 0, infinity

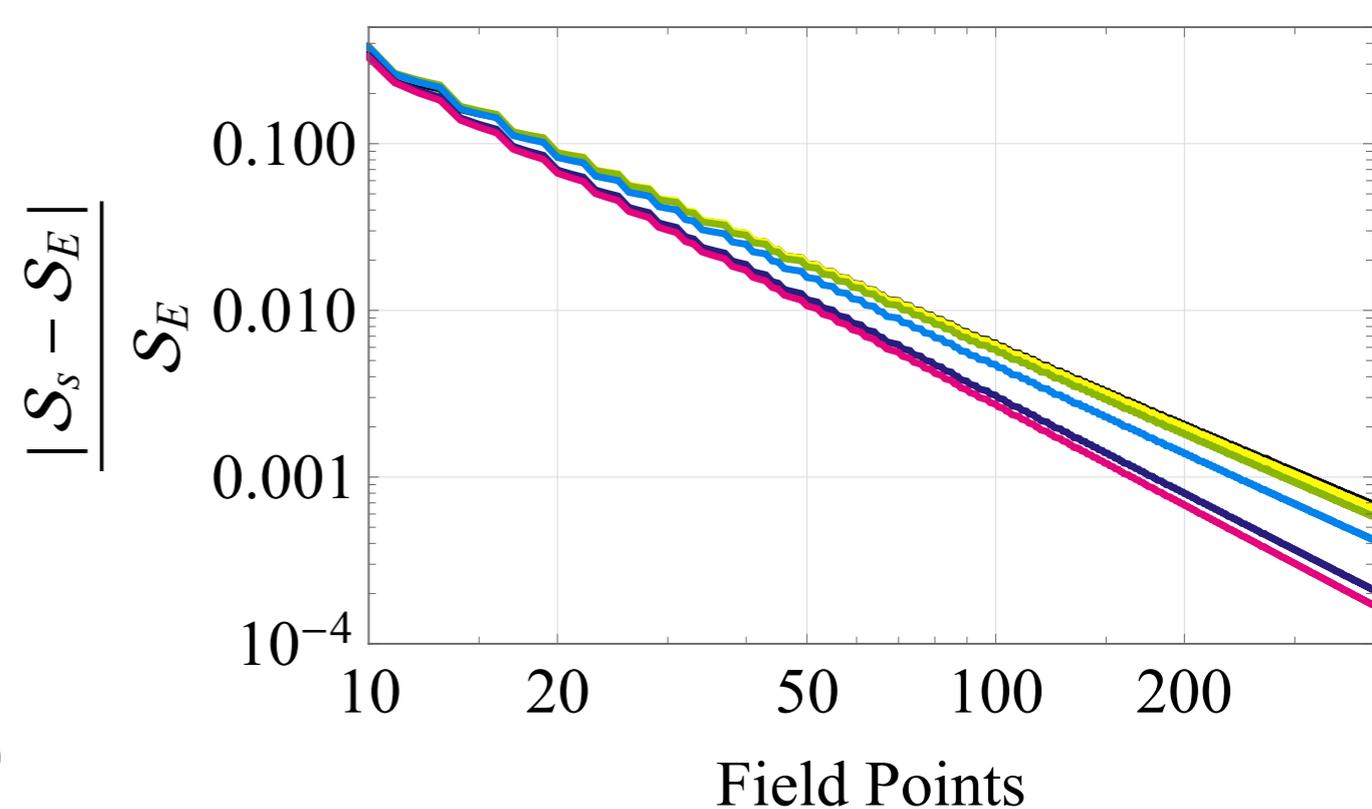
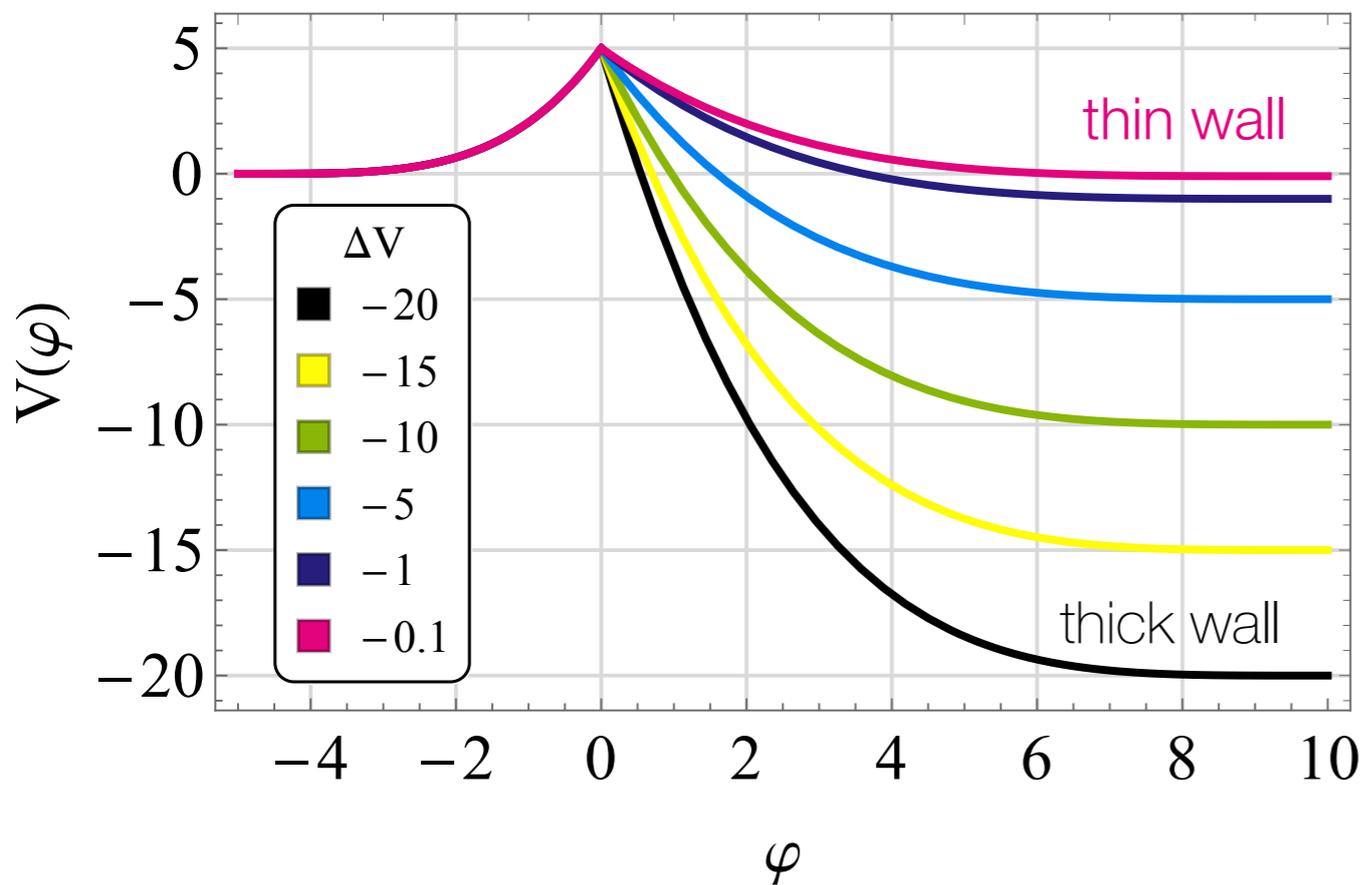


Bi-quartic

Other exact $N=3$ potentials, quartic-linear, quartic-quartic

Dutta, Hector, Vaudrevange, Westphal '11

known exact solution, 'fair' comparison and test for the PB method



● CosmoTransitions

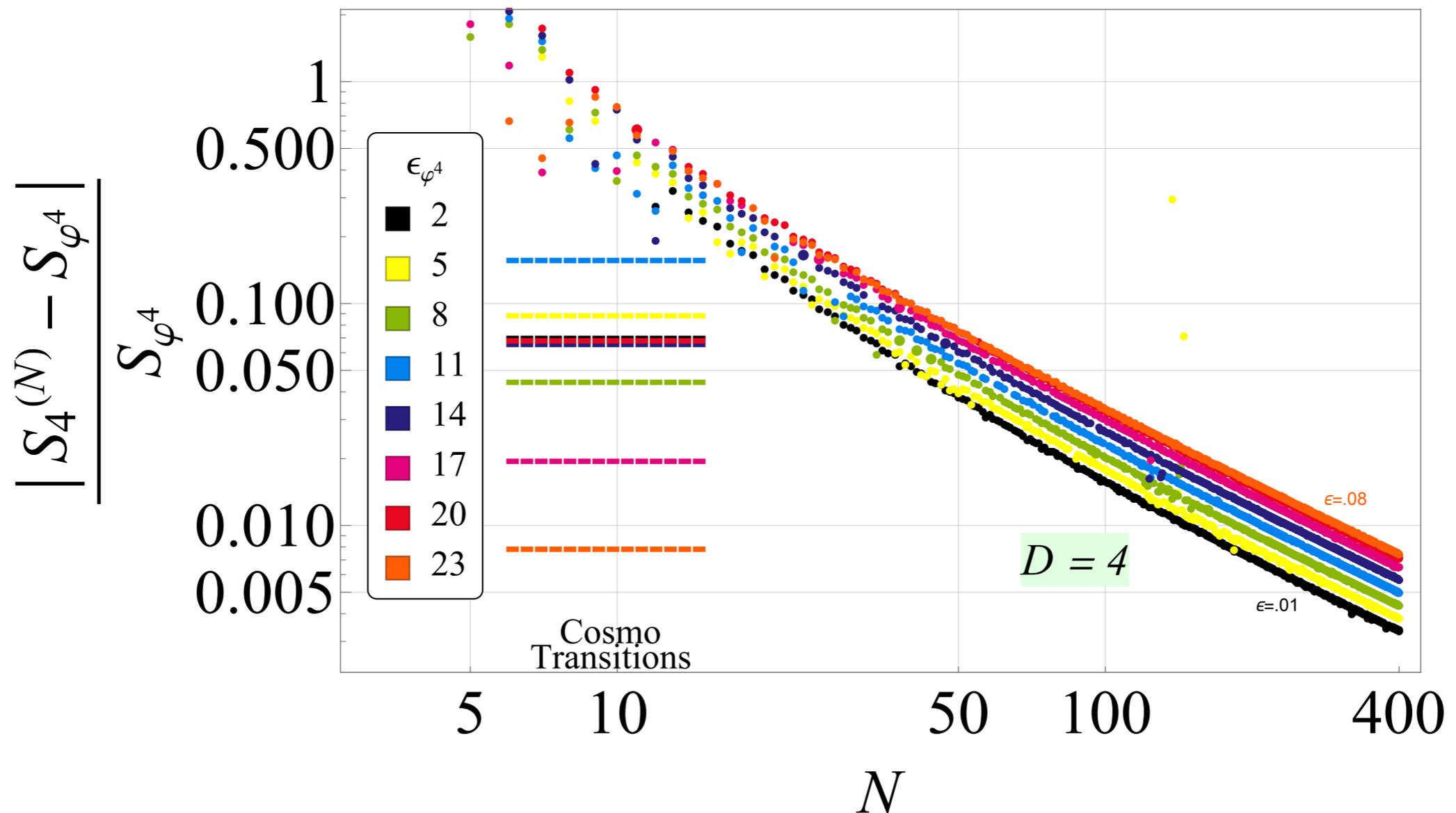
fails with the action, possible to repair by hand,
precise from 20% to 0.5%

● AnyBubble

fails to compute

● Polygonal bounce

works smoothly with a bi-homogeneous
segmentation



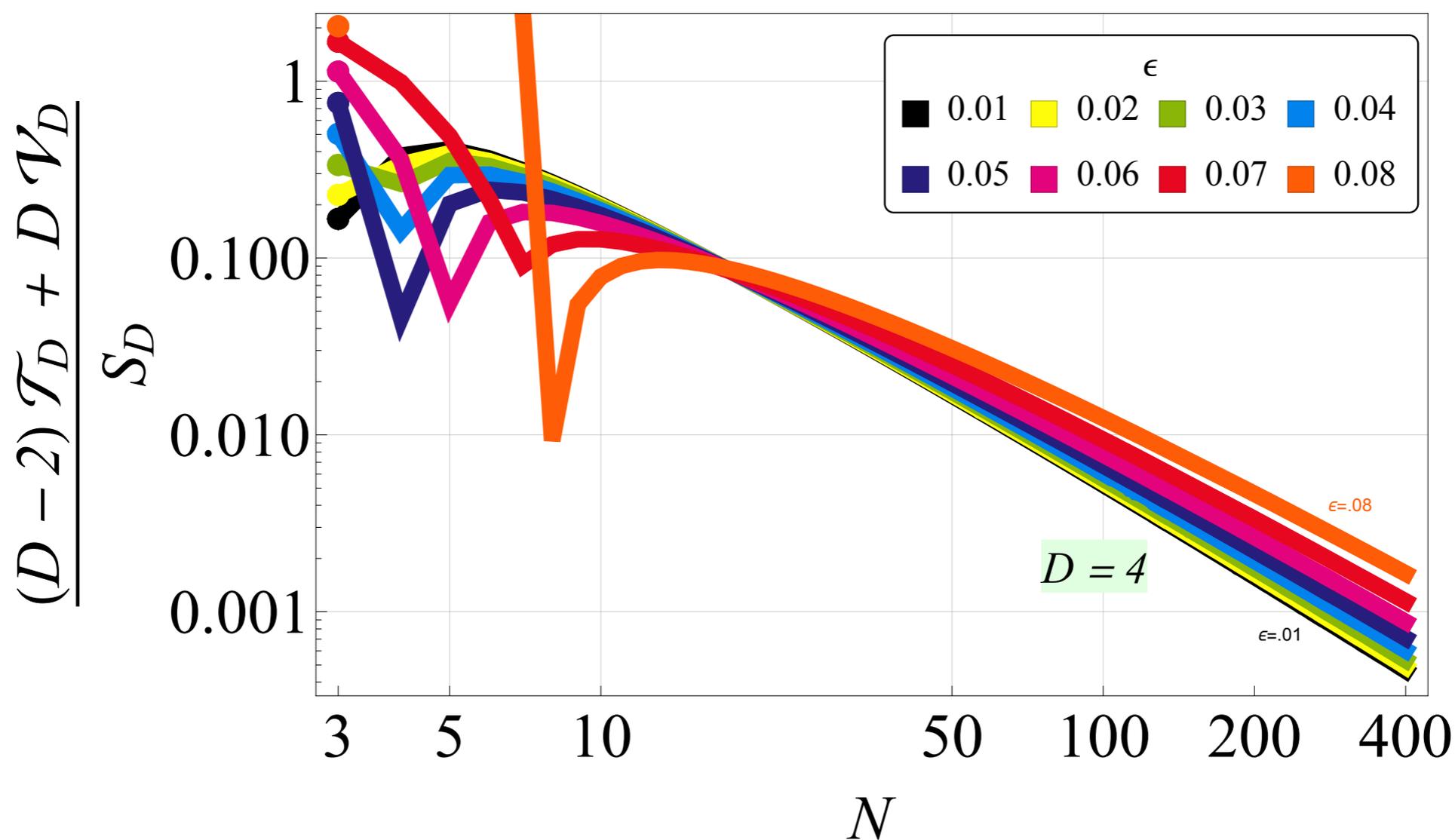
Derrick's theorem

$$(D - 2)\mathcal{T} + D\mathcal{V} \rightarrow 0$$

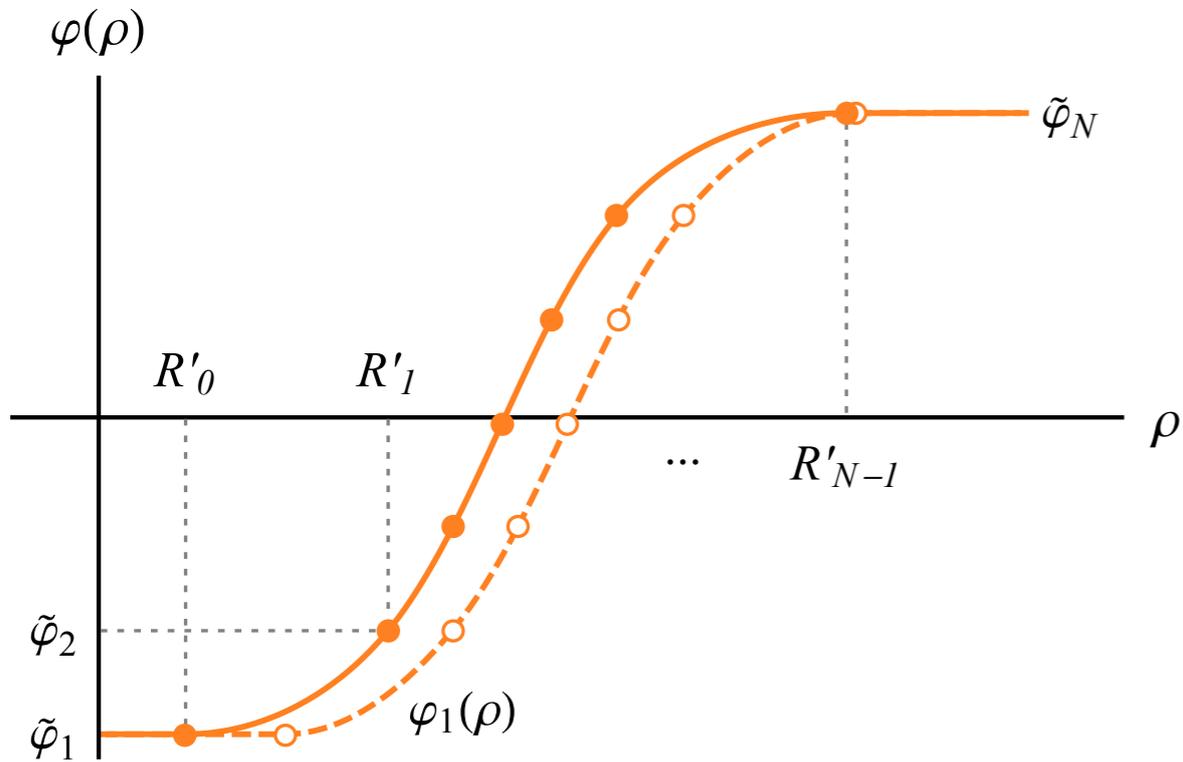
finite part corrections up to $N \simeq 10$

independent measure of goodness of approximation

above relation 'exact' for the PB potential



Higher orders



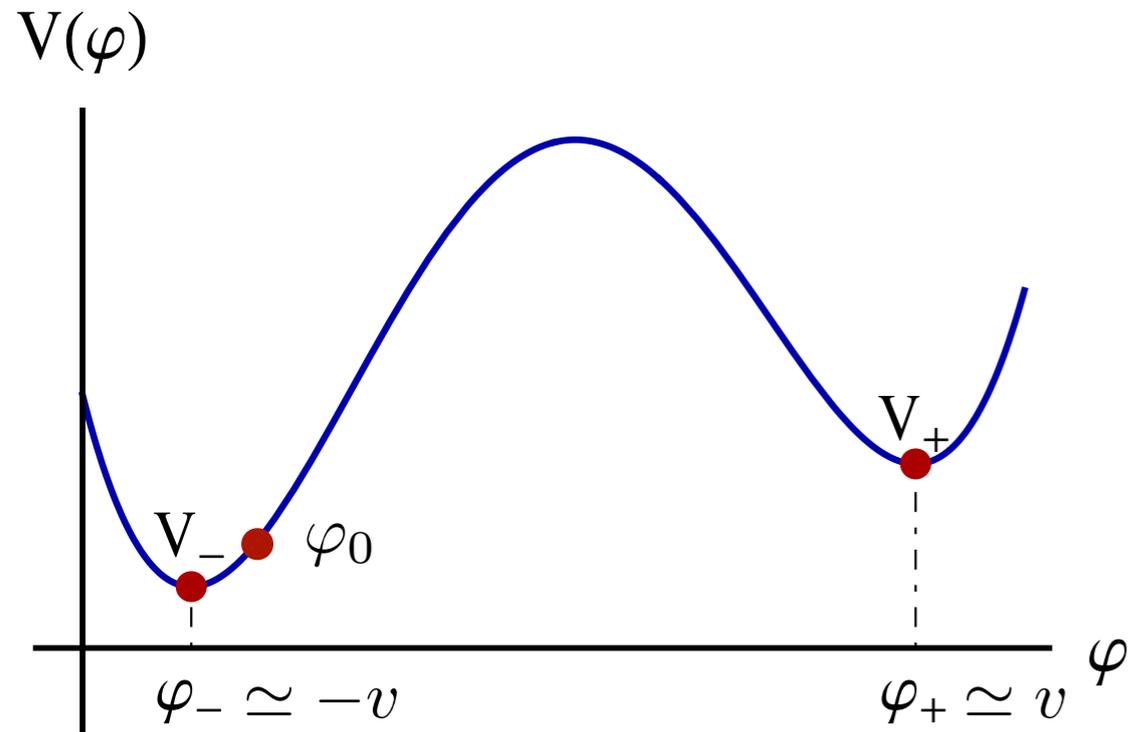
Match at perturbed radii

$$R_s \rightarrow R_s (1 + r_s), \quad r_s \ll 1$$

Rederive the matching conditions

A single linear equation = very fast

$$r_s = \frac{\beta_s + \frac{D-2}{2} \left(\nu_s + \mathcal{I}_s + \frac{4}{D} \alpha_s R_s^2 \right) R_s^{D-2}}{(D-2) \left(b_s - \frac{4}{D} a_s R_s^D \right)}$$



Thin wall approximation

Coleman '77

$$V(\varphi) = \frac{\lambda}{8} (\varphi^2 - v^2)^2 + \varepsilon \left(\frac{\varphi - v}{2v} \right), \quad S_1 = \frac{v^3 \sqrt{\lambda}}{3}$$

small ε limit $\varphi_0 \simeq \varphi_-$ until $\rho = R$

field solution

$$\varphi(\rho) = \begin{cases} -v, & \rho \ll R \\ \varphi_1(\rho - R), & \rho \approx R \\ v, & \rho \gg R \end{cases} \quad \varphi_1(\rho) = v \tanh\left(\frac{\sqrt{\lambda}v}{2}\rho\right)$$

extremize the action

bounce action

$$S_E = 2\pi^2 \int_0^\infty \rho^3 d\rho \left(\frac{1}{2} \dot{\varphi}^2 + V \right)$$

$$= -\frac{1}{2} \pi^2 R^4 \varepsilon + \pi^2 R^3 S_1$$

volume surface

$$\frac{dS_E}{dR} = 0 \quad \Rightarrow \quad R = \frac{3S_1}{\varepsilon}$$

$$S_E = \frac{27\pi^2}{2} \frac{S_1^4}{\varepsilon^3}$$

runaway

$$\frac{d^2 S_E}{dR^2} < 0$$

Coleman '77
Bödeker, Moore '09, '17

Custom options control the input

`"Dimension"`

sets the Euclidean spacetime dimension, 3 or 4

`"FieldPoints"`

number of field points defines the segmentation

`"Gradient"`

one can pre-calculate the gradient, or make it numerical

`"Hessian"`

similar to the gradient, needed for multi-fields

`"MaxPath"`

limits the number of path iterations, typically small

`"MidFieldPoint"`

one can define a starting fixed point (e.g. the saddle)

`"PathTolerance"` &
`"ActionTolerance"`

set a goal for the precision of the path variation and the
Euclidean action

Output is a bundled container that can be easily accessed

"Action"

sets the Eucludian spacetime dimension, 3 or 4

"Bounce"

number of field points defines the segmentation

"Coefficients"

one can pre-calculate the gradient, or make it numerical

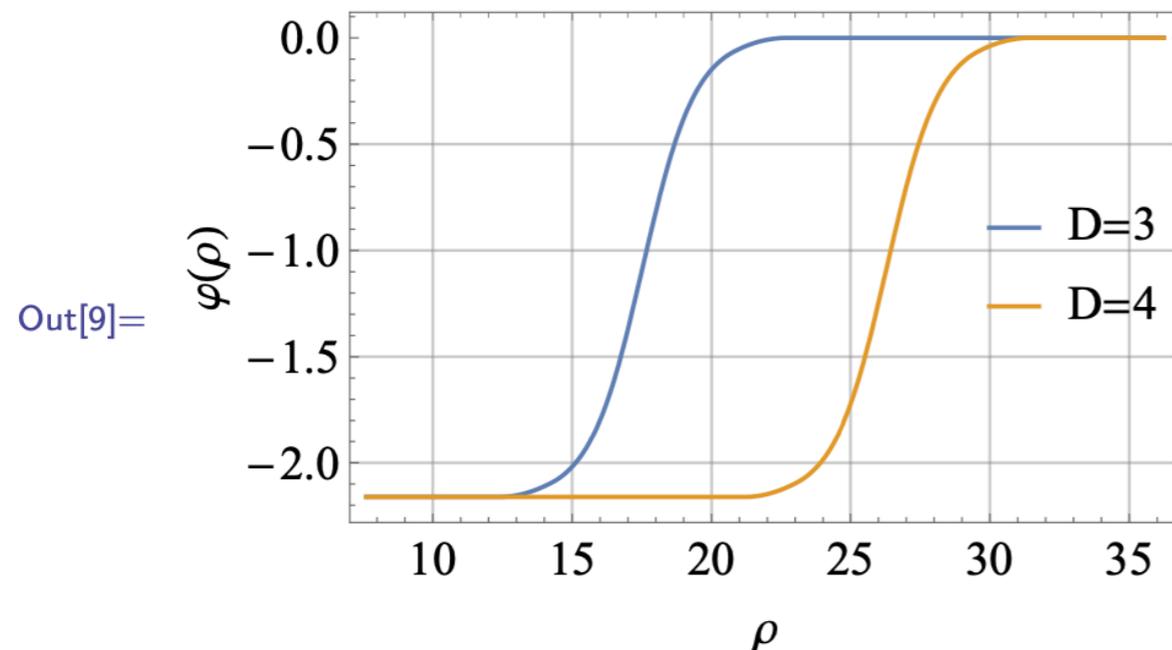
"Path"

similar to the gradient, needed for multi-fields

"Radii"

limits the number of path iterations, typically small

```
In[9]:= BouncePlot[{bf3,bf}, PlotLegends-> Placed[{"D=3","D=4"}, {Right,Center}]]
```



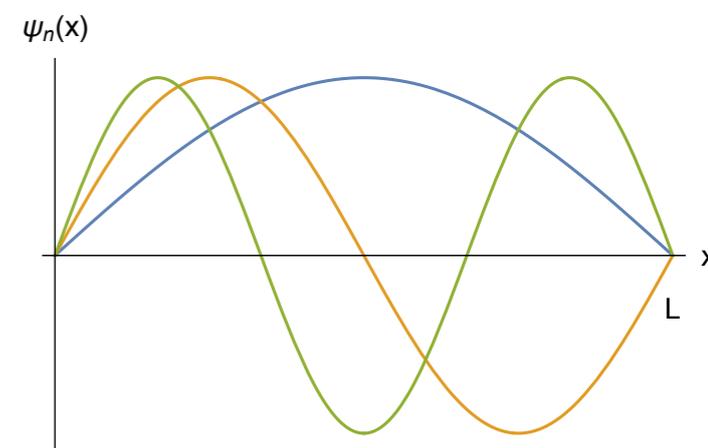
custom function for convenient plotting

Gel'fand-Yaglom aside

simplest instance of GY formalism 'magic'

$$\mathcal{O} = -\frac{d^2}{dx^2} + m^2 \qquad \mathcal{O}_{\text{FV}} = -\frac{d^2}{dx^2}$$

QM well, classical ID string



a) impose Dirichlet (fixed) boundary condition at L

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 + m^2, \quad \lambda_{n\text{FV}} = \left(\frac{n\pi}{L}\right)^2, \quad \frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} = \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_{n\text{FV}}} = \frac{\sinh(mL)}{mL}$$

b) solve the Cauchy (open) boundary condition

$$\psi'' - m^2\psi = 0, \quad \psi(0) = 0, \quad \psi'(0) = 1, \quad \psi = \frac{\sinh(mx)}{m}, \quad \psi_{\text{FV}} = x$$

$$\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} = \frac{\psi}{\psi_{\text{FV}}}(L) = \frac{\sinh(mL)}{mL}$$

Classical physical
significance?