Kloosterman sums, modular forms, Mahler measures and sunrise diagrams

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MITP, Mainz, 3 June 2015

The on-shell equal-mass **sunrise** diagram with $N-2$ loops in two space-time dimensions is an integral of a product of N **Bessel** functions. Using **Schwinger** parameters one may try to guess **modular forms** that prevent reduction to polylogarithms for $N>4$. This was achieved, for $N=5,6,8$, and led to evaluations of integrals of N Bessel functions in terms of **L-series** of modular forms with weight $N-2$. The case $N=7$ proved harder, but was conquered last week, thanks to work by Ronald Evans, on **Kloosterman sums**, and an inspired suggestion for the functional equation of the corresponding L-series, from Anton Mellit.

1. **Elliptic** polylogarithms for off-shell sunrise
2. **Modular** forms for Bessel moments on-shell
3. **Kloosterman** sums for $N=7$ Bessel functions
4. **Mahler** measures for $N<7$ Bessel functions

1 Elliptic polylogarithms for off-shell sunrise

Two-loop massive sunrise diagram in $D = 2$ space-time dimensions:

$$I(p^2, m_1, m_2, m_3) \equiv \frac{1}{\pi^2}\left(\prod_{k=1}^{3}\int\frac{\mathrm{d}^2 p_k}{p_k^2 - m_k^2 + \mathrm{i}\epsilon}\right)\delta^{(2)}(p - p_1 - p_2 - p_3)$$

determined by its discontinuity across the cut (Källén, 1955; Sabry, 1962):

$$I(w^2, a, b, c) = 4\int_0^\infty I_0(wt)K_0(at)K_0(bt)K_0(ct)\, t\mathrm{d}t = 8\pi\int_{a+b+c}^\infty \frac{A(x)\, x\, \mathrm{d}x}{x^2 - w^2}$$

$$A(w) = \frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{F(w)\cos^2\theta + 16abcw\sin^2\theta}}$$

$$F(w) = (w + a + b + c)(w + a - b - c)(w - a + b - c)(w - a - b + c).$$

From Gauss (1799), the reciprocals of the complete elliptic integrals

$$A(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(w) - F(-w)}\right)}, B(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(-w)}\right)}$$

are **arithmetic-geometric means**. They define Jacobi's **nome** (1828)

$$q(w) \equiv \exp(-\pi B(w)/A(w)).$$

Now set $a = b = c = 1$ and regard $I(w^2, 1, 1, 1)$ as a function of q.

1.1 Bloch-Vanhove elliptic dilogarithm

Define a character with $\chi(n) = \pm 1$ for $n = \pm 1 \bmod 6$ and $\chi(n) = 0$ otherwise. The ODE of Broadhurst, Fleischer and Tarasov (1993) gives

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I}{24\sqrt{3}A}\right) = \sum_{n>0} \frac{n^2(q^n - q^{5n})}{1 - q^{6n}} = \sum_{n>0}\sum_{k>0} n^2\chi(k)q^{nk}.$$

Integrating twice and using the known imaginary part on the cut, we get

$$\frac{I(w^2,1,1,1)}{4A(w)} = E_2(q) = -\pi\log(-q) - 3\sqrt{3}\sum_{k>0}\frac{\chi(k)}{k^2}\frac{1+q^k}{1-q^k} = -E_2(1/q)$$

with constants of integration that make I finite on-shell, where $q = -1$.

1.2 Bloch-Kerr-Vanhove elliptic trilogarithm

Thanks to Geoffrey Joyce (1971), the three-loop sunrise integral $J(t)$ gives

$$\frac{2J(t)}{(\widetilde{w}B(\widetilde{w}))^2} = E_3(q) = (-2\log(q))^3 + \sum_{k>0}\frac{\psi(k)}{k^3}\frac{1+q^k}{1-q^k} = -E_3(1/q),$$

$$3\log(q) = -2\pi A(\widetilde{w})/B(\widetilde{w}), \quad \widetilde{w} = \sqrt{1-t/4} + \sqrt{4-t/4}$$

with $\psi(k) = -5760,\ -48,\ 720,\ 384,\ -720,\ 48$, for $k = 0,1,2,3,4,5 \bmod 6$.

2 Modular forms for Bessel moments on-shell

The L-loop on-shell sunrise diagram is an integral of $L+2$ Bessel functions. More generally, define an N-Bessel moment at L loops by

$$S_{N,L} \equiv 2^L \int_0^\infty [I_0(t)]^{N-L-1}[K_0(t)]^{L+1} t\mathrm{d}t.$$

Then the on-shell sunrise diagram is the dilog $S_{4,2} = I(1,1,1,1) = \pi^2/4$. At 3 loops, $S_{5,3} = J(1)$. $S_{6,4}$ indicates a challenge that Stefano Laporta encounters for the magnetic moment of the electron at 4 loops, with diagrams with 5-fermion intermediate states in 4 space-time dimensions.

Convergence requires that $L < N < 2L+3$ and $L > 1$ when $N = 2L+2$. Bailey, Borwein, Broadhurst and Glasser, `arXiv:0801.0891`, proved that

$$S_{1,0} = S_{2,1} = 1, \quad S_{3,1} = \frac{2\pi}{3\sqrt{3}}, \quad S_{3,2} = \frac{4\,\mathrm{Cl}_2(\pi/3)}{\sqrt{3}},$$

$$S_{4,2} = \frac{\pi^2}{4}, \quad S_{4,3} = 7\zeta(3), \quad S_{5,2} = \frac{\sqrt{3}}{120\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).$$

We also conjectured and checked to 1000 digits that

$$S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}, \quad S_{6,4} = \frac{4\pi^2}{3} S_{6,2}, \quad S_{8,5} = \frac{18\pi^2}{7} S_{8,3}.$$

2.1 Example of an L-series of a modular form

$$\eta(q) \equiv q^{1/24} \prod_{n>0} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}$$

then for $\Im z > 0$,

$$\eta(\exp(2\pi \mathrm{i} z)) = (\mathrm{i}/z)^{1/2} \eta(\exp(-2\pi \mathrm{i}/z)).$$

If $f(z) = (\sqrt{-N}/z)^w f(-N/z)$, we say that f is a modular form of modular weight w and level N. Here is a well known example with modular weight 12 and level 1:

$$[\eta(q)]^{24} = \sum_{n>0} A(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 \ldots$$

Its Fourier coefficients are multiplicative: $A(mn) = A(m)A(n)$ for $\gcd(m, n) = 1$, and are determined by $A(p)$ at the primes p:

$$L(s) \equiv \sum_{n>0} \frac{A(n)}{n^s} = \prod_p \frac{1}{1 - A(p)p^{-s} + p^{11-2s}}.$$

Moreover, we can analytically continue to values inside the critical strip:

$$\Lambda(s) \equiv \frac{\Gamma(s)L(s)}{(2\pi)^s} = \sum_{n>0} A(n) \int_1^\infty \mathrm{d}x \left(x^{s-1} + x^{11-s}\right) \exp(-2\pi n x) = \Lambda(12 - s).$$

2.2 A modular form of weight 3 for 5 Bessel functions

Let $L_{3,15}(s)$ be the Dirichlet L-function defined by the modular form

$$f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$$

with weight 3 and level 15, where $\eta_n \equiv \eta(q^n)$. I discovered that

$$S_{5,2} = 3L_{3,15}(2), \quad S_{5,3} = \frac{48}{5}\zeta(2)L_{3,15}(1),$$

where $S_{5,3}$ is the the on-shell 3-loop sunrise diagram.

The modular form was identified by counts of zeros of the denominator of

$$S_{5,3} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca}$$

in finite fields, by Francis Brown. We still lack a proof that

$$L_{3,15}(2) = \frac{\sqrt{3}}{360\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).$$

Anton Mellit has an idea of how to prove this, using a j-invariant.

2.3 A modular form of weight 4 for 6 Bessel functions

Let $L_{4,6}(s)$ be the Dirichlet L-function defined by the modular form

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and checked to 1000 digits that

$$S_{6,2} = 6L_{4,6}(2), \quad S_{6,3} = 12L_{4,6}(3), \quad S_{6,4} = 48\zeta(2)L_{4,6}(2),$$

where $S_{6,4}$ is the on-shell 4-loop sunrise diagram.

2.4 A modular form of weight 6 for 8 Bessel functions

Let $L_{6,6}(s)$ be the Dirichlet L-function defined by the modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$S_{8,3} = 8L_{6,6}(3), \quad S_{8,4} = 36L_{4,6}(4), \quad S_{8,5} = 216L_{4,6}(5),$$

but have no result for the on-shell 6-loop sunrise diagram, $S_{8,6}$.

3 Kloosterman sums for $N = 7$ Bessel functions

Extending an analysis by Ronald Evans to a finite field $\mathbf{F}_q$ with characteristic p, I define Kloosterman sums

$$K(a) \equiv \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi\mathrm{i}}{p}\mathrm{Trace}\left(x + \frac{a}{x}\right)\right),$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. Then we obtain integers

$$c(q) \equiv -\frac{1 + T_7}{q^2}$$

$$T_N \equiv \sum_{a \in \mathbf{F}_q^*} \sum_{k=0}^{N} [g(a)]^k [h(a)]^{N-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$.

For these 7th moments, Evans conjectures that, for prime $p > 7$,

$$c(p) \stackrel{?}{=} \left(\frac{p}{105}\right)\left(|b(p)|^2 - p^2\right)$$

where $b(p)$ is the p-th Hecke eigenvalue for a weight-3 newform on $\Gamma_0(525)$ with eigenfield $\mathbf{Q}(\mathrm{i}, \sqrt{6}, \sqrt{14})$, discovered by William Klein, using `Sage`.

3.1 Defining the L-series

I shall suppose that the relevant L-series is of the form

$$L_{5,105}(s) = \sum_{n>0} \frac{A_n}{n^s} = \prod_p \frac{1}{Z_p(p^{-s})}$$

with weight 5, conductor 105, and with $A_p = c(p)$ for prime p. To determine the local factors, I require that $Z_p(T)$ is a polynomial of degree no greater than 3 that reproduces $c(p^n)$ via the Lefschetz formula

$$\log(Z_p(T)) = -\sum_{n>0} \frac{c(p^n)}{n} T^n.$$

Then for prime p that does not divide 105, I infer that

$$Z_p(T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |b(p)|^2) T + p^4 T^2\right).$$

At the bad primes, I obtain quadratic polynomials

$$\begin{aligned} Z_3(T) &= 1 - 10T + (9T)^2, \\ Z_5(T) &= 1 - (25T)^2, \\ Z_7(T) &= 1 + 70T + (49T)^2. \end{aligned}$$

3.2 A weight-5 modular functional equation

As an **aside**, consider the weight-5 level-7 modular form

$$f_{5,7}(q) = \sum_{n>0} A_n q^n = (\eta_1\eta_7)^3 \left(\sum_{m,n\in\mathbf{Z}} q^{m^2+mn+2n^2}\right)^2 .$$

Then convergence of the L-series

$$L_{5,7}(s) = \sum_{n>0} \frac{A_n}{n^s} = \frac{1}{1-7^{2-s}} \prod_{p\neq 7} \frac{1}{1-\frac{A_p}{p^s}+\left(\frac{p}{7}\right)\frac{p^4}{p^{2s}}}$$

may be accelerated using the functional equation for

$$\Lambda_{5,7}(s) = \frac{\Gamma(s)\,L_{5,7}(s)}{(2\pi/\sqrt{7})^s} = \sum_{n>0} A_n \int_1^\infty \frac{\mathrm{d}x}{x}\left(x^s+x^{5-s}\right)\exp\left(-\frac{2\pi n x}{\sqrt{7}}\right)$$

with the reflection symmetry $\Lambda_{5,7}(s) = \Lambda_{5,7}(5-s)$. In this modular case we may easily compute

$$\begin{aligned} L_{5,7}(3) &= \sum_{n>0} \frac{A_n}{n^3}\left(1+\frac{3\pi n}{\sqrt{7}}+\frac{4\pi^2 n^2}{7}\right)\exp\left(-\frac{2\pi n}{\sqrt{7}}\right) \\ L_{5,7}(4) &= \sum_{n>0} \frac{A_n}{n^4}\left(1+\frac{2\pi n}{\sqrt{7}}+\frac{2\pi^2 n^2}{7}+\frac{8\pi^3 n^3}{21\sqrt{7}}\right)\exp\left(-\frac{2\pi n}{\sqrt{7}}\right) \end{aligned}$$

to high precision and quickly discover that

$$\begin{aligned} L_{5,7}(4) &= \frac{\sqrt{7}\pi}{8} L_{5,7}(3) = \frac{\pi^4}{96}\left(\sum_{n=-\infty}^{\infty} \exp\left(-\sqrt{7}\pi n^2\right)\right)^8 \\ &= \frac{1}{42}\left(\frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}{4\pi}\right)^4. \end{aligned}$$

But the functional equation for the 7-Bessel problem is **not** of this simple modular form. Thanks to this workshop, it has been **found**.

3.3 The functional equation for 7 Bessel moments

Using the Fourier series, Anton Mellit inferred the functional equation

$$\Lambda_{5,105}(s) \equiv \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L_{5,105}(s) = \Lambda_{5,105}(5-s)$$

and then we were able to use Tim Dokchitser's `computel` to discover that

$$S_{7,4} \equiv 2^4 \int_0^\infty [I_0(t)]^2 [K_0(t)]^5 t\mathrm{d}t = 20\zeta(2) L_{5,105}(2)$$

which I have checked to 1000 digits, using 50,00 Fourier coefficients.

4 Mahler measures for $N < 7$ Bessel functions

A Laurent polynomial $P(x_1, \ldots, x_n)$ has a logarithmic Mahler measure

$$m(P) \equiv \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_n \log\left(\left|P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right)\right|\right).$$

Christopher Deninger conjectured that

$$m\left(1 + x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2}\right) = L'_{2,15}(0) = \frac{15}{4\pi^2} L_{2,15}(2)$$

where the L-series comes from the weight-2 and level-15 modular form

$$f_{2,15} = \eta_1 \eta_3 \eta_5 \eta_{15}.$$

This was proven by Mathew Rogers and Wadim Zudilin `arXiv:1102.1153`.

David Boyd conjectured and Anton Mellit proved, `arXiv:1207.4722`, that

$$m\left(1 + x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_1 x_2 + \frac{1}{x_1 x_2}\right) = L'_{2,14}(0) = \frac{7}{2\pi^2} L_{2,14}(2)$$

where the L-series comes from the weight-2 and level-14 modular form

$$f_{2,14} = \eta_1 \eta_2 \eta_7 \eta_{14}.$$

It is instructive to note that

$$
\begin{aligned}
m(1+x_1+x_2) &= \frac{\sqrt{3}}{4\pi}S_{3,2} = \frac{\mathrm{Cl}_2(\pi/3)}{\pi} \\
m(1+x_1+x_2+x_3) &= \frac{1}{2\pi^2}S_{4,3} = \frac{7\zeta(3)}{2\pi^2}
\end{aligned}
$$

are simply related to related to 2 and 3-loop **vacuum** diagrams, with 3 and 4 Bessel functions. One might expect the 5 and 6-Bessel modular forms,

$$f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15}), \quad f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2,$$

to determine Mahler measures, since I have proven that

$$m(1+x_1+\ldots+x_{N-1}) = -\log(2) - \gamma - \int_0^\infty \mathrm{d}t \log(t)\frac{\mathrm{d}}{\mathrm{d}t}[J_0(t)]^N.$$

This illuminates the conjectures by Fernando Rodriguez Villegas,

$$
\begin{aligned}
m(1+x_1+x_2+x_3+x_4) &= -L'_{3,15}(-1) = 6\left(\frac{\sqrt{15}}{2\pi}\right)^5 L_{3,15}(4), \\
m(1+x_1+x_2+x_3+x_4+x_5) &= -8L'_{4,6}(-1) = 3\left(\frac{\sqrt{6}}{\pi}\right)^6 L_{4,6}(5).
\end{aligned}
$$

which David Bailey has checked to 1000 digits, using the N-Bessel formula.

Conclusions

1. **Sunrise** diagrams give **elliptic** polylogarithms at 2 and 3 loops.
2. On-shell, they are evaluated by L-series, for $N < 7$ **Bessel** functions.
3. For each $N \in \{3, 4, 5, 6, 8\}$, at least two **moments** $S_{N,L} \equiv 2^L \int_0^\infty [I_0(t)]^{N-L-1}[K_0(t)]^{L+1} t\mathrm{d}t$ are given by a L-series.
4. At $N = 7$, I determined a **L-series with weight 5**.
5. Thanks to Anton Mellit's **functional equation**, we obtained $S_{7,4} = 20\zeta(2)L_{5,105}(2)$, which parallels the sunrise results, $S_{6,4} = 48\zeta(2)L_{4,6}(2)$ and $S_{5,3} = \frac{48}{5}\zeta(2)L_{3,15}(1)$.
6. **Kloosterman** sums yield integers that inform the L-series for $N < 9$.
7. **Mahler measures** are evaluated by sunrise L-series for $N < 7$.
8. We still **lack** results for **vacuum** diagrams with $N > 4$ and for **sunrise** diagrams with $N > 6$ Bessel functions.
9. A solution to one of these **outstanding** problems will give **deep** information about the nature of **Feynman periods**.

Q1: Let $I_D(w^2, m_1, m_2, m_3)$ be the two-loop sunrise integral in D space-time dimensions. What is the minimum number of rational combinations of $\{w, m_1, m_2, m_3\}$ required to evaluate $w\partial I_3/\partial w$?

A: 1 **B:** 2 **C:** 3 **D:** 4

Q2: In Watson's cubic number field $(1-\gamma^2)(1-\gamma) = \gamma$, there are 14 units, u_i, such that $u_i \in [0, 1]$ and $1 - u_i$ is also a unit. Experiment shows that 5 independent $\mathbf{Z}$-linear combinations of $\mathcal{L}_4(u_i)$ with π^4 vanish for the choice

$$\mathcal{L}_4(x) \equiv \mathrm{Li}_4(x) - \frac{\log(x)}{2}\mathrm{Li}_3(x) + \frac{\log^3(x)}{24}\mathrm{Li}_1(x).$$

Which authors encountered this weight-4 combination of polylogarithms?

A: Bloch **B:** Davydychev **C:** Gangl **D:** Kummer
E: Rogers **F:** Ussyukina **G:** Volovich **H:** Wigner
I: Zagier **J:** none of A to I

Q3: Where does Bas Tausk's evaluation of the on-shell massless four-point non-planar scalar double-box diagram give alternating sums?

A: $s = 0$ **B:** $s = t$ **C:** $s = u$ **D:** $s = -t$ **E:** $s = -u$

Answer 1: **A**. This dimensionless number is determined solely by w/m, where $m \equiv m_1 + m_2 + m_3$. In 3 dimensions the volume of phase space is a multiple of $(1 - m/w)$, for $w > m$. Then Källén's method gives

$$w\frac{\partial I_3}{\partial w} \propto w \int_m^\infty \frac{\partial}{\partial w}\left(\frac{x-m}{x^2-w^2}\right) \mathrm{d}x = \frac{m}{w}\mathrm{arctanh}\left(\frac{w}{m}\right) - 1.$$

Answer 2: **J**, as far as I am aware. Please correct me if you know better.

Answer 3: **B** and **C**. `arXiv:hep-ph/9909506` shows the $t \leftrightarrow u$ symmetry.