AROUND MOTIVIC STRUCTURE

OF QUANTUM COHOMOLOGY

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Plan

- **Part I: INTRODUCTION.**

MOTIVIC REFLEXIVITY OF QUANTUM COHOMOLOGY

- **Part II: GENUS ZERO AND ONE MODULI MOTIVES**

- **Part III: ZETA POLYNOMIALS**

INTRODUCTION AND OVERVIEW

CLASSICAL MOTIVES

- **Choose an adequate intersection theory** A^***:**

(i) For each smooth projective (or proper) variety $X \mapsto A^*(X)$**,**

$A^*(X) =$ **algebraic (or Hodge) cycles on** X **modulo an adequate equivalence relation, over a coefficient ring Z, Q,** $\mathbf{C}, \mathbf{Q}_l \ldots$**.**

(ii) Functorial pullback/pushforward: $f : X \to Y$ **induces**

$$f^* : A^*(Y) \to A^*(X), \quad f_* : A^*(X) \to A^*(Y)$$

(iii) Compatibility with product: canonical morphisms

$$A^*(X) \otimes A^*(Y) \to A^*(X \times Y)$$

(iv) The diagonal map $\Delta_X : X \to X \times X$ **makes of** $A^*(X)$ **a graded commutative ring with multiplication induced by** Δ_X^***. It satisfies the projection formula** $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$**.**

- **Define $\mathbf{A}^*$ – correspondences as graded morphisms** :

If X is of pure dimension d, put

$$Corr^r(X, Y) := A^{d+r}(X \times Y).$$

Composition:

$$Corr^r(X, Y) \otimes Corr^s(Y, Z) \to Corr^{r+s}(X, Z) :$$

$$f \otimes g \mapsto g \circ f := p_{XZ*}(p_{XY}^*(f) \cdot p_{YZ}^*(g))$$

- **Define category of graded correspondences** :

Objects: **varieties (smooth projective manifolds)/k;**

Morphisms: $Corr^*(X \times Y)$**.**

• **Define monoidal category of classical motives** $(Mot_k, \otimes)$**:**

Objects: (X, p, m)**,** X **a variety,**

$$p = p^2 \in Corr^0(X, X), \;\; m \in \mathbf{Z}$$

Morphisms:

$$Hom_{Mot_k}((Y, q, n), (X, p, m)) := q \circ Corr^{n-m}(X, Y) \circ p \subset Corr^*(X, Y).$$

Monoidal (tensor) product (on objects):

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

Another monoidal structure: $\oplus$ **extending** $\coprod$**.**

- **<u>Motives as target category of a cohomology theory</u>** :

$$h : Var_k^{opp} \to Mot_k :$$

$$h(X) := (X, id, 0),$$

$$h(\varphi : X \to Y) := [\Gamma_\varphi] \in Corr^0(X,Y) = Hom_{Mot_k}(h(Y), h(X)).$$

- **<u>Unit and Lefschetz motives</u>** :

$$\mathbf{1} := (Spec\, k, id, 0), \quad \mathbf{L} := (Spec\, k, id, -1).$$

Fact:

$$h(\mathbf{P}^n) \cong \mathbf{1} \oplus \mathbf{L} \oplus \ldots \oplus \mathbf{L}^{\otimes n}.$$

- **<u>The Tate twist</u>** :

$$X(n) := X \otimes \mathbf{L}^{-n}$$

SUMMARY:

Classical motives are obtained from varieties by :

- **"Linearizing morphisms":** $\{f\} \;\; => \{\sum_i a_i f_{i*} f_i^*\}$**.**
- **Adding kernels/cokernels of projectors .**
- **Twisting by** $\mathbf{L}^{\otimes n}$, $n \in \mathbf{Z}$**.**

A QUESTION:

Consider "total (classical) motives": $h(V)$ **where** $V \in Var_k$**.**

Question. *What additional motivic structures are naturally supported by total motives rather than arbitrary ones?*

Example. **Each total motive is in a natural way** *a unital commutative algebra* **in the monoidal category of motives:** Γ_{Δ_X} **induces the multiplication**

$$\cup : h(V) \otimes h(V) \to h(V)$$

This structure is immensely generalized by the following:

BASIC DISCOVERY OF QUANTUM COHOMOLOGY:

Each total motive in a natural way is *an algebra over the cyclic modular operad*

$$\mathcal{HM}(n-1) := \bigoplus_{g} h(\overline{M}_{g,n})$$

in the monoidal category of ind–motives.

This means that we have for any V canonical correspondences

$$I_{g,n}(V) \in Corr^{*}(\overline{M}_{g,n} \times V^{n})$$

which, when considered as morphisms in Mot_k, satisfy a host of identities: axioms of a modular operad and its representations.

EXPLANATIONS AND WARNINGS

• **Explanation 1. Why ind–motives rather than simply motives? Two reasons:**

(i) For each "arity" n, we have infinitely many genera g.

(ii) Each $I_{g,n}(V)$ is in fact an *infinite* **sum of cycles indexed by** *effective numerical equivalence classes* β **of curves in V:**

$$I_{g,n} = \sum_{\beta} I_{g,n}(V, \beta)$$

• **Warning 2. Moduli spaces of stable curves of genus g with $n+1$ marked points $\overline{M}_{g,n+1}$ generally** *are not smooth varieties*, **they are** *smooth Deligne–Mumford stacks/orbifolds.*

For smooth orbifolds, there are *two different* **Chow ring functors (coinciding upon** Var_k**):**

A^* **(A. Vistoli et al.) and** A^*_χ **(B. Toën):**

A. Vistoli. *Intersection theory on algebraic stacks and their moduli spaces.*
Inv. Math. 97 (1989), 613–669.

B. Toën. *On motives for Deligne–Mumford stacks.* **IMRN, 17 (2000), 909–928.**

Used as correspondences, these constructions give rise to two *a priori different* **categories of classical motives of orbifolds.**

In fact, the categories are *the same*, **but the respective** *motivic cohomologies* **differ.**

Theorem. (B. Toën). *(i) The monoidal categories of classical motives, generated by Var_k, resp. orbifolds, using A^*, resp. A^*_χ, are naturally equivalent.*

(ii) The motivic cohomology functors h and h_χ coincide upon Var_k. However, on orbifolds, h is a generally non–trivial direct factor of h_χ.

• **Warning – Question 3. If one wants to extend the Quantum Cohomology theory to orbifolds (e.g. for defining QuCoho of components of the modular operad $\overline{M}_{*,*}$), what versions of the Chow groups are appropriate?**

Tentative answer (B. Toën, letter of Jan. 2, 2009)

(i) One expects that $h(\overline{M}_{g,n})$ **(cyclically) acts upon** $h_\chi(V)^{\otimes n}$**, and all operadic axioms are satisfied.**

(ii) Probably, this becomes wrong for $h_\chi(\overline{M}_{g,n})$.

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In this talk, I want to draw attention to the reflexivity problem:

HOW THE MODULAR OPERAD ACTS ON ITS OWN COMPONENTS?

GENERA 0, 1, AND TAUTOLOGICAL RINGS

• Chow rings of $\overline{M}_{g,n}$ are notoriously difficult to study, and our knowledge of them and respective motives is very incomplete.

• G. Faber, R. Pandharipande et al. introduced and studied the *tautological subrings* **of Chow groups with coefficients Q: $R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n})$.**

Rougly speaking, they are the smallest subrings closed under push–forwards via all boundary maps and all maps forgetting some marked points.

• For genus zero, the whole Chow ring is tautological.

Motives of $\overline{M}_{0,n}$, $n \geq 3$, are sums of Tate motives, and their structure is in principle very well understood.

Hence we may hope to describe explicitly the Gromov–Witten correspondences

$$I_{0,n}(\overline{M}_{0,m}, \beta) \in A^*(\overline{M}_{0,n} \times (\overline{M}_{0,m})^n).$$

In fact, we did it, jointly with Maxim Smirnov, <u>for the case of boundary classes β.</u>

This will be explained in the next Part of the talk.

• **For genus one, the tautological subring of the Chow ring is isomorphic to** $H^{ev}(\overline{M}_{1,n})$**. (Ezra Getzler, Dan Petersen).**

For $n < 11$**, it is isomorphic to the whole Chow ring, but with appearance of** *cusp forms* **of the respective weight this does not hold anymore.**

Parts of cohomology of moduli spaces in genus one corresponding to the spaces of cusp modular forms provide an exciting challenge: to study their quantum cohomology action.

They can be included into the general picture with the help of the foliowing new definition:

The following definition is motivated by the fact that the notion of modular operad with components of all genera is essentially equivalent to the notion of a *functor on the category of stable modular graphs with values in DM–stacks***.**

- **Definition.** **The q–tautological rings**

$$R_q^*(\overline{M}_{g,\tau}) \subset A^*(\overline{M}_{g,\tau})$$

are defined simultaneously for all stable modular graphs τ as the minimal set of subrings containing all fundamental classes of $A^*(\overline{M}_{0,\tau})$ and constituting a modular suboperad.

GENUS ZERO QUANTUM COHOMOLOGY

• **Notation and preliminaries (arbitrary genus). Fix a finite set Σ, a genus $g \geq 0$, a smooth projective manifold W, and an effective class $\beta \in A_1(W)$.**

Then one can define a (proper DM)–stack $\overline{M}_{g,\Sigma}(W,\beta)$.

For a k–scheme T, one object of the groupoid $\overline{M}_{g,\Sigma}(W,\beta)(T)$ of T–points of this stack consists of a diagram of schemes of the following structure:

$$\begin{array}{ccc} \mathcal{C}_T & \xrightarrow{f_T} & W \\ \big\downarrow{\scriptstyle h_T} & & \\ T & & \end{array}$$

and a family of sections $x_{j,T} : T \to \mathcal{C}_T$, $j \in \Sigma$, $h_T \circ x_{j,T} = id_T$.

They must satisfy the following conditions:

(a) $\mathcal{C}_T \to T$ and $(x_{j,T})$ constitute a flat *prestable* T–family of curves of genus g.

(b) $f_T : (\mathcal{C}_T; (x_{j,T})) \to W$, is a *stable map* of class β.

Given such a diagram with sections, we call (W, β) its *target*, T its *base*, and the whole setup *a T–family of stable maps.* Isomorphisms of families, lifting id_T, must be identical also on W. Base changes are defined in an evident way.

The stack $\overline{M}_{g,\Sigma}(W, \beta)$ is defined as the base of the universal family of this type with given target (W, β):

$$\begin{array}{ccc} \overline{C}_{g,\Sigma}(W,\beta) & \xrightarrow{\ f\ } & W \\ \Big\downarrow h & & \\ \overline{M}_{g,\Sigma}(W,\beta) & & \end{array}$$

It is endowed with sections $x_j : \overline{M}_{g,\Sigma}(W,\beta) \to \overline{C}_{g,\Sigma}(W,\beta)$**,** $j \in \Sigma$**. Naturally,** $\overline{C}_{g,\Sigma}(W,\beta)$ **is a stack as well.**

If W **is a point,** $\beta = 0$**, we routinely omit the target and write simply** $\overline{M}_{g,\Sigma}$**,** $\overline{C}_{g,\Sigma}$ **etc.**

The defining stack diagram produces the *evaluation/stabilization* **diagram**

$$\begin{array}{ccc} \overline{M}_{g,\Sigma}(W,\beta) & \xrightarrow{st} & \overline{M}_{g,\Sigma} \; . \\ \Big\downarrow{\scriptstyle ev} & & \\ W^{\Sigma} & & \end{array}$$

Here

$$ev = (ev_j = f \circ x_j \,|\, j \in \Sigma): \quad \overline{M}_{g,\Sigma}(W,\beta) \to W^{\Sigma}.$$

In the case $2g+|\Sigma| \geq 3$, the absolute stabilization morphism st discards the map f and stabilizes the remaining prestable family of curves

$$st : \overline{M}_{g,\Sigma}(W,\beta) \to \overline{M}_{g,\Sigma}.$$

The *virtual fundamental class*, **or the J–class $[\overline{M}_{g,\Sigma}(W,\beta)]^{virt}$, is a canonical element in the Chow ring $A_*(\overline{M}_{g,\Sigma}(W,\beta))$:**

$$J_{g,\Sigma}(W,\beta) \in A_D(\overline{M}_{g,\Sigma}(W,\beta))\,,$$

where D is the virtual dimension (Chow grading degree)

$$(-K_W,\beta) + |\Sigma| + (\dim W - 3)(1-g).$$

The respective *Gromov–Witten correspondence*, **defined for** $2g + |\Sigma| \geq 3$, **is the proper pushforward**

$$I_{g,\Sigma}(W,\beta) := (ev, st)_*(J_{g,\Sigma}(W,\beta)) \in A_D(W^\Sigma \times \overline{M}_{g,\Sigma})$$

Understanding these correspondences is the content of *motivic* **quantum cohomology.**

- **<u>Example :</u>** $g = 0$, $\beta = 0$. **In this case the universal family is**

$$\begin{array}{ccc} W \times \overline{C}_{0,\Sigma} & \xrightarrow{pr_1} & W \\ \downarrow {\scriptstyle id_W \times h} & & \\ W \times \overline{M}_{0,\Sigma} & & \end{array}$$

with structure sections $id_W \times x_j$.

The stabilization morphism is simply the projection

$$st = pr_2 : \ W \times \overline{M}_{0,\Sigma} \to \overline{M}_{0,\Sigma}.$$

The evaluation morphism is the projection followed by the diagonal embedding Δ_Σ:

$$ev : \ W \times \overline{M}_{0,\Sigma} \to W \to W^\Sigma.$$

We have:

$$J_{0,\Sigma}(W,0) = [\overline{M}_{0,\Sigma}(W,0)] = [W] \otimes [\overline{M}_{0,\Sigma}].$$

The virtual dimension is

$$|\Sigma| + \dim\, W - 3 = \dim\,(W \times \overline{M}_{0,\Sigma}).$$

Thus, finally, the Gromov–Witten correspondence is the class

$$I_{0,\Sigma}(W,0) = [\Delta_\Sigma(W)] \otimes [\overline{M}_{0,\Sigma}] \in A_*(W^\Sigma \times \overline{M}_{0,\Sigma}).$$

• **Genus zero moduli spaces acting upon genus zero moduli spaces.**

We will now study the Gromov–Witten correspondences *of genus zero* **for** $W = \overline{M}_{0,S}$**,** $\beta =$ *class of a boundary curve* **in** $\overline{M}_{0,S}$**.**

In particular , we need to understand the relevant J**–classes and the diagrams**

$$ev : \overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta) \to \overline{M}_{0,S}^{\Sigma}, \quad st : \overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta) \to \overline{M}_{0,\Sigma}.$$

We also want to be able to trace various functorialities, in particular, in *both* S **and** Σ**.**

In the remaining parts of this section we describe a more general situation. Afterwards we will show that our main problem is contained in it.

• **Setup, part I.**

(a) $b: E \to W$ **:= a morphism of smooth irreducible projective manifolds.**

(b) β_E **:= an effective genus zero curve class on** E**, and** $\beta := b_*(\beta_E)$ **its pushforward to** W**.**

Any stable map $\mathcal{C}_T/T \to E, (x_j : T \to \mathcal{C}_T \,|\, j \in \Sigma)$**, of class** β_E**, induces, after composition with** b **and stabilization, a stable map with target** (W, β)**.**

Thus, we get a map

$$\widetilde{b}: \ \overline{M}_{0,\Sigma}(E, \beta_E) \to \overline{M}_{0,\Sigma}(W, \beta)$$

that clearly fits into the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,\Sigma}(E, \beta_E) & \xrightarrow{\widetilde{b}} & \overline{M}_{0,\Sigma}(W, \beta) \\ \Big\downarrow{\scriptstyle (ev_E, st_E)} & & \Big\downarrow{\scriptstyle (ev_W, st_W)} \\ E^{\Sigma} \times \overline{M}_{0,\Sigma} & \xrightarrow{b^{\Sigma} \times id} & W^{\Sigma} \times \overline{M}_{0,\Sigma} \end{array}$$

• **Proposition**. *(i) Assume that*

$$J_{0,\Sigma}(W,\beta) = \widetilde{b}_*(J_{0,\Sigma}(E,\beta_E)).$$

Then

$$I_{0,\Sigma}(W,\beta) = (b^\Sigma \times id)_*(I_{0,\Sigma}(E,\beta_E)).$$

(ii) Let $\gamma_j \in H^*(W)$, $j \in \Sigma$, be a finite family of cohomology classes marked by Σ. Then from (i) we have

$$pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(W,\beta) =$$

$$= (b^\Sigma \times id)_*[pr_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)].$$

Here $pr_W : W^\Sigma \times \overline{M}_{0,\Sigma} \to W^\Sigma$ and $pr_E : E^\Sigma \times \overline{M}_{0,\Sigma} \to E^\Sigma$ are the respective projection morphisms, and H^* can be any standard cohomology theory.

• **<u>Remark</u>.**

In our applications to the case $W = \overline{M}_{0,S}$, E will be a boundary stratum containing the boundary curve representing β, and the virtual fundamental classes $J_{0,\Sigma}$ will coincide with the usual fundamental classes since the relevant deformation problem will be *unobstructed.*

Moreover, E will have a very special additional structure.

Below, we axiomatize the relevant geometry.

- **Setup, part II.**

Additional assumptions:

(c) E is explicitly represented as $E = B \times C$ *where C is isomorphic to* $\mathbf{P}^1$*.*

This identification, including the projections $p = pr_B : E \to B$ *and* $pr_C : E \to C$*, will constitute a part of structure.*

(d) β_E *is the (numerical) class of any fiber of p.*

(f) The deformation problem for any fiber C_0 *of p embedded via* b_0 *in W is trivially unobstructed in the sense of Behrend:*

$$H^1(C_0, b_0^*(\mathcal{T}_W)) = 0\,.$$

(f) The map $\widetilde{b}$ *is an isomorphism.*

These assumptions are quite strong, and with them we can complete the explicit computation of $I_{0,\Sigma}(W, \beta)$

• **<u>End of computations</u>. (A) First of all, we have**

$$pr_{B*}(\beta_E) = 0, \quad pr_{C*}(\beta_E) = \mathbf{1}$$

where 1 is the fundamental class $[C]$ in the Chow ring of C.

Thus, the two projections induce the map

$$(\widetilde{pr}_B, \widetilde{pr}_C) : \overline{M}_{0,\Sigma}(E, \beta_E) \to \overline{M}_{0,\Sigma}(B, 0) \times \overline{M}_{0,\Sigma}(C, \mathbf{1})\,.$$

Stabilization maps embed this morphism into the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,\Sigma}(E, \beta_E) & \longrightarrow & \overline{M}_{0,\Sigma}(B, 0) \times \overline{M}_{0,\Sigma}(C, \mathbf{1}) \\ {\scriptstyle st_E}\downarrow & & \downarrow{\scriptstyle st_B \times st_C} \\ \overline{M}_{0,\Sigma} & \xrightarrow{\Delta_{\overline{M}_{0,\Sigma}}} & \overline{M}_{0,\Sigma} \times \overline{M}_{0,\Sigma} \end{array}$$

where the lower line is the diagonal embedding.

(B) Similarly, evaluation maps produce the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,\Sigma}(E,\beta_E) & \longrightarrow & \overline{M}_{0,\Sigma}(B,0)\times\overline{M}_{0,\Sigma}(C,\mathbf{1}) \\ {\scriptstyle ev_E}\downarrow & & \downarrow{\scriptstyle ev_B\times ev_C} \\ E^\Sigma & \xrightarrow{\quad s \quad} & B^\Sigma\times C^\Sigma \end{array}$$

where the lower line is now the evident permutation isomorphism induced by $E=B\times C$.

(C) Combining these two diagrams, we get

$$\begin{array}{ccc} \overline{M}_{0,\Sigma}(E,\beta_E) & \longrightarrow & \overline{M}_{0,\Sigma}(B,0)\times\overline{M}_{0,\Sigma}(C,\mathbf{1}) \\ {\scriptstyle (ev_E,st_E)}\downarrow & & \downarrow{\scriptstyle (ev_B,st_B)\times(ev_C,st_C)} \\ E^\Sigma\times\overline{M}_{0,\Sigma} & \xrightarrow{\widetilde{\Delta}} & B^\Sigma\times\overline{M}_{0,\Sigma}\times C^\Sigma\times\overline{M}_{0,\Sigma} \end{array}$$

Here the lower line is an obvious composition of permutations and the diagonal embedding of $\overline{M}_{0,\Sigma}$.

(D) It follows that

$$I_{0,\Sigma}(E,\beta_E) = \widetilde{\Delta}^!(I_{0,\Sigma}(B,0) \otimes I_{0,\Sigma}(C,\mathbf{1}))\,.$$

Here for $x \in A_*(X), y \in A_*(Y)$ we denote by $x \otimes y \in A_*(X \times Y)$ the image of $x \otimes y \in A_*(X) \otimes A_*(Y)$ wrt the canonical map $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$.

Furthermore,

$$I_{0,\Sigma}(B,0) = [\Delta_\Sigma(B) \times \overline{M}_{0,\Sigma}] \in A_*(B^\Sigma \times \overline{M}_{0,\Sigma}).$$

Finally, describe the space $\overline{M}_{0,\Sigma}(C,\mathbf{1})$ and the class $I_{0,\Sigma}(C,\mathbf{1})$.

(E) Recall the Fulton–MacPherson construction.

Let V **be a smooth complete algebraic manifold. For a finite set** Σ**, let** V^{Σ} **be the direct product of a family of** V**'s labeled by elements of** Σ**.**

Denote by $\widetilde{V}^{\Sigma}$ **the blow up of the (small) diagonal in** V^{Σ}**. Finally, define** $V^{\Sigma,0}$ **as the complement to all partial diagonals in** V^{Σ}**.**

The Fulton–MacPherson's *configuration space* $V\langle\Sigma\rangle$ **is the closure of** $V^{\Sigma,0}$ **naturally embedded into the product**

$$V^{\Sigma} \times \prod_{\Sigma' \subset \Sigma, |\Sigma'| \geq 2} \widetilde{V}^{\Sigma'}.$$

It turns out that $\overline{M}_{0,\Sigma}(C, \mathbf{1})$ **can be identified with** $C\langle\Sigma\rangle$ **in such a way that the birational morphism** ev_C **becomes the tautological open embedding when restricted to** $C^{\Sigma,0}$**.**

Therefore, denoting by $D_\Sigma \subset C^\Sigma \times \overline{M}_{0,\Sigma}$ the closure of the graph of the canonical surjective map $C^{\Sigma,0} \to M_{0,\Sigma}$, we get

$$I_{0,\Sigma}(C, \mathbf{1}) = [D_\Sigma]\,.$$

Combining all the above we get:

- **Proposition**. *We have*

$$I_{0,\Sigma}(E, \beta_E) = \widetilde{\Delta}^!([\Delta_\Sigma(B) \times \overline{M}_{0,\Sigma} \times D_\Sigma])$$

and

$$I_{0,\Sigma}(W, \beta) = (b^\Sigma \times id)_* \circ \widetilde{\Delta}^!([\Delta_\Sigma(B) \times \overline{M}_{0,\Sigma} \times D_\Sigma])\,.$$

APPLICATIONS

• **Combinatorics of boundary strata in** $\overline{M}_{0,S}$.

The basic combinatorial invariant of an S**–pointed stable curve** C **of genus zero is its** *dual graph* $\tau = \tau_C$**.**

Its set of *vertices* V_τ **is (bijective to) the set of irreducible components of** C**.**

Each vertex v **is a boundary point of the set of** *flags* $f \in F_\tau(v)$ **which is (bijective to) the set consisting of singular points and** S**–labeled points on this component.**

Put $F_\tau = \cup_{v \in V_\tau} F_\tau(v)$.

If two components of C **intersect, the respective two vertices carry two flags that are grafted to form an** *edge* e **connecting the respective vertices.**

The set of edges is denoted E_τ. The flags that are not pairwise grafted are called *tails.*

Tails form a set T_τ which is naturally bijective to the set of S–labeled points and therefore itself is labeled by S.

Stable curves of genus zero correspond to *stable trees* τ**: each vertex carries at least three flags.**

Finally, the total space $\overline{M}_{0,S}$ is a disjoint union of *locally closed strata* M_τ **indexed by stable S–labeled trees.**

Generally, a stratum M_τ lies in the closure $\overline{M}_\sigma$ of M_σ, iff σ can be obtained from τ by contracting a subset of edges.

Closed strata $\overline{M}_\sigma$ corresponding to trees with nonempty set of edges are called *boundary* **ones. The number of edges is the codimension of the stratum.**

• **Boundary divisors. The classes of boundary divisors** D_σ **bijectively correspond to stable unordered 2–partitions** σ : $S = S_1 \cup S_2, \mathrm{card}\, S_i \geq 2.$

Here and below *an unordered m–partition* **of a set** S **is synonymous to** *an equivalence relation* **on** S **with** m **equivalence classes.**

• **Boundary curves : combinatorics.**

Consider an unordered 4–partition Π **of** S**. Denote by the** $S(\Pi)$ **the set of irs components, that is, the quotient of** S **wrt the respective equivalence relation.**

4–partitions are in a natural bijection with isomorphism classes of *distinguished stable S–labeled trees* π**.**

By definition, such a tree is endowed with one *distinguished vertex* v_0**, with the set of flags** $S(\Pi)$ **at this vertex** $F_\pi(v_0)$**.**

The flags labeled by one–element components $\{s\}$ of Π are tails, carrying the respective labels $s \in S$. The remaining flags are halves of edges.

The second vertex of an edge, whose one half is labeled by a component S_i carries tails labeled by elements of S_i.

We will routinely identify $F_\pi(v_0)$ with $S(\Pi)$.

Definition. *(i) Given a 4–partition* Π*, denote by* $P = P(\Pi)$ *the set of those stable 2–partitions* σ *of* S*, each component of which is a union of two different components of* Π*. For* $|S| \geq 4$ *we have* $|P(\Pi)| = 3$.

(ii) Denote by $N = N(\Pi)$ *the set of those stable 2–partitions of* S *whose one component coincides with one component of* Π*. We have for* $|S| \geq 5$*:* $1 \leq |N(\Pi)| \leq 4$.

Fact. Π *can be uniquely reconstructed from* $P(\Pi)$*; hence* $P(\Pi)$ *uniquely determines* $N(\Pi)$ *as well.*

Proof. In fact, if $\Pi = (S_1, S_2, S_3, S_4)$ **(numeration arbitrary), then by definition**
$P(\Pi)$ **must consist of partitions**

$$\sigma_1 = (S_1 \cup S_2, S_3 \cup S_4),\ \sigma_2 = (S_1 \cup S_3, S_2 \cup S_4),\ \sigma_3 = (S_1 \cup S_4, S_2 \cup S_3)$$

Hence conversely, knowing $P(\Pi)$**, we can reconstruct** Π**: its components are exactly non–empty pairwise intersections of components of different** $\sigma_i \in P(\Pi)$.

• **Boundary curves : geometry. Each 4–partition Π of S determines the following boundary stratum of $\overline{M}_{0,S}$:**

$$b_\Pi : \quad \overline{M}_\Pi := \cap_{\sigma\in N(\Pi)} D_\sigma \hookrightarrow \overline{M}_{0,S}\,.$$

Equivalently, $\overline{M}_\Pi$ is the stratum, corresponding to the special tree π associated to Π.

In other words, now all components of Π are indexed by the flags $f\in F_\pi(v_0)$ at the special vertex v_0, whereas components of cardinality ≥ 2 are also naturally indexed by the remaining vertices of π:

$$\overline{M}_\Pi = \overline{M}_{0,F_\pi(v_0)} \times \prod_{v\neq v_0} \overline{M}_{0,F_\pi(v)}.$$

Codimension of $\overline{M}_\Pi$ is $|N(P)|$, and $1 \leq |N(\Pi)| \leq 4$.

Since $|F_\pi(v_0)| = 4$, the moduli space $\overline{M}_{0,F_\pi(v_0)}$ is $\mathbf{P}^1$ with three points naturally labeled by the set of stable partitions of $F_\pi(v_0)$ which in turn is canonically bijective to $P(\Pi)$.

Hence we may and will define the projection map

$$p = p_\Pi : \overline{M}_\Pi \to B_\Pi := \prod_{v \neq v_0} \overline{M}_{0,F_\pi(v)}$$

having three canonical disjoint sections canonically labeled by $P(\Pi)$.

Clearly, all fibers of p_Π are rationally equivalent so that they define a class

$$\beta = \beta(\Pi) \in A_1(\overline{M}_{0,S}).$$

Final Lemma. *(i) For* $n := |S| \geq 4$, *each boundary curve (one–dimensional boundary stratum)* C_τ *is a fiber of one of the projections* p_Π.

(ii) $[C_{\tau_1}] = [C_{\tau_2}] \in A_1(\overline{M}_{0,S})$ *iff these curves are fibers of one and the same projection* p_Π.

• **Gromov – Witten correspondences for genus zero moduli spaces.**

Here I will show that one can apply the technique of Setups I, II in order to calculate

$$I_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi)) \in A_*((\overline{M}_{0,S})^\Sigma \times \overline{M}_{0,\Sigma}).$$

I will restrict myself by showing how general data of Setups I,II specialise to this case.

- **Setup, part I.**

(a) $b: E \to W$ **specialises to** $b_\Pi : \overline{M}_\Pi \to \overline{M}_{0,S}$.

(b), (d) β_E **is the class of any fiber of** $p = p_\Pi$**.**

- **Setup, part II.**

(c) Explicit isomorphism $E = B \times C$ **is given by**

$$\overline{M}_\Pi = \prod_{v \neq v_0} \overline{M}_{0,F_\pi(v)} \times \overline{M}_{0,F_\pi(v_0)}.$$

References

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THANK YOU FOR YOUR ATTENTION!