Elliptic multiple zeta values and superstring one-loop amplitudes

Part II

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based on joint work with Carlos Mafra, Nils Matthes and Oliver Schlotterer
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Amplitudes, Motives and Beyond
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Starting point

previous talk by Oliver Schlotterer:

- multiple elliptic zeta values (eMZVs) defined: $\omega(n_1, \ldots, n_r; \tau)$
- Fay and shuffle identities lead to relations & indecomposable eMZVs:

w_ω \ ℓ_ω	2	3	4
1	$\omega(0,1)$		$\omega(0,0,1,0)$
3	$\omega(0,3)$		$\omega(0,0,0,3)$
5	$\omega(0,5)$		$\omega(0,0,0,5)$ $\omega(0,0,2,3)$
7	$\omega(0,7)$		$\omega(0,0,0,7)$ $\omega(0,0,2,5)$ $\omega(0,0,4,3)$

w_ω \ ℓ_ω	2	3	4	5
2		$\omega(0,0,2)$		$\omega(0,0,0,0,2)$
4		$\omega(0,0,4)$		$\omega(0,0,0,0,4)$ $\omega(0,0,0,1,3)$
6		$\omega(0,0,6)$		$\omega(0,0,0,0,6)$ $\omega(0,0,0,1,5)$ $\omega(0,0,0,2,4)$ $\omega(0,0,2,2,2)$

Goals in this talk:

- Fay and shuffle algebraically involved:
  $\rightarrow$ different and more efficient way to identify relations?
- is there a formalism or pattern explaining / leading to the above table?

q-expansion

$$\omega(n_1,\ldots,n_r;\tau) = \omega_0(n_1,\ldots,n_r) + \sum_{k=1}^{\infty} c_k q^k \qquad\qquad q = e^{2\pi i\,\tau}$$

- constant term ω_0 can be evaluated $\to$ combination of usual MZVs [Enriquez]
- q-expansion from differential equation (τ suppressed): [Enriquez]

$$\begin{aligned}
2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau}\,\omega(n_1,\ldots,n_r) &= -4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q}\,\omega(n_1,\ldots,n_r) \\
&= n_1\,\mathrm{G}_{n_1+1}\,\omega(n_2,\ldots,n_r) - n_r\,\mathrm{G}_{n_r+1}\,\omega(n_1,\ldots,n_{r-1}) \\
&+ \sum_{i=2}^{r}\Big\{(-1)^{n_i}(n_{i-1}+n_i)\,\mathrm{G}_{n_{i-1}+n_i+1}\,\omega(n_1,\ldots,n_{i-2},0,n_{i+1},\ldots,n_r) \\
&- \sum_{k=0}^{n_{i-1}+1}(n_{i-1}-k)\binom{n_i+k-1}{k}\mathrm{G}_{n_{i-1}-k+1}\,\omega(n_1,\ldots,n_{i-2},k+n_i,n_{i+1},\ldots,n_r) \\
&+ \sum_{k=0}^{n_i+1}(n_i-k)\binom{n_{i-1}+k-1}{k}\mathrm{G}_{n_i-k+1}\,\omega(n_1,\ldots,n_{i-2},k+n_{i-1},n_{i+1},\ldots,n_r)\Big\}
\end{aligned}$$

- non-modular G_2 shows up for divergent eMZVs $(n_1 = 1,\ n_r = 1)$ exclusively.

q-expansion

$$\omega(n_1,\ldots,n_r;\tau) = \omega_0(n_1,\ldots,n_r) + \sum_{k=1}^{\infty} c_k q^k \qquad q = e^{2\pi i\tau}$$

- constant term ω_0 can be evaluated $\rightarrow$ combination of usual MZVs [Enriquez]
- q-expansion from differential equation: [Enriquez]

$$q\frac{\mathrm{d}}{\mathrm{d}q}\underbrace{\omega(n_1,\ldots,n_r)}_{w_\omega,\ \ell_\omega} \rightarrow \sum \mathrm{G}_{\tilde{n}}\ \underbrace{\omega(\tilde{n}_1,\ldots,\tilde{n}_{r-1})}_{\text{shorter}}$$

- apply recursively, use known q-expansions of Eisenstein series:

$$\mathrm{G}_k(\tau) = 2\,\zeta_k + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!}\sum_{m,n=1}^{\infty} m^{k-1}q^{mn} \qquad k \text{ even},\ k>0\,.$$

- Typical result:

$$\omega(0,0,0,0,6) \rightarrow -\zeta_6/60 + \zeta_8\left(6720q + 216720q^2 + \frac{4901120q^3}{3} + \ldots\right).$$

Relations: $\sum_i a_i\, \omega(n_{i_1}, \ldots, n_{i_r}) + \sum_j b_j\, \omega_{\text{ind}} \sqcup \omega_{\text{ind}} + \sum_k c_k\, \zeta_k\, \omega_{\text{ind}} \sqcup \ldots + \cdots \overset{!}{=} 0$

$\rightarrow$ *compare coefficients for each power of* q

w_ω \ ℓ_ω	2	3	4
1	$\omega(0,1)$		$\omega(0,0,1,0)$
3	$\omega(0,3)$		$\omega(0,0,0,3)$
5	$\omega(0,5)$		$\omega(0,0,0,5)$ $\omega(0,0,2,3)$
7	$\omega(0,7)$		$\omega(0,0,0,7)$ $\omega(0,0,2,5)$ $\omega(0,0,4,3)$
9	$\omega(0,9)$		$\omega(0,0,0,9)$ $\omega(0,0,2,7)$ $\omega(0,0,4,5)$ $\omega(0,1,3,5)$
11	$\omega(0,11)$		$\omega(0,0,0,11)$ $\omega(0,0,2,9)$ $\omega(0,0,4,7)$ $\omega(0,1,3,7)$ $\omega(0,3,3,5)$
13	$\omega(0,13)$		$\omega(0,0,0,13)$ $\omega(0,0,2,11)$ $\omega(0,0,4,9)$ $\omega(0,1,3,9)$ $\omega(0,1,5,7)$ $\omega(0,3,3,7)$ $\omega(0,3,5,5)$

w_ω \ ℓ_ω	2	3	4	5
2		$\omega(0,0,2)$		$\omega(0,0,0,0,2)$
4		$\omega(0,0,4)$		$\omega(0,0,0,0,4)$ $\omega(0,0,0,1,3)$
6		$\omega(0,0,6)$		$\omega(0,0,0,0,6)$ $\omega(0,0,0,1,5)$ $\omega(0,0,0,2,4)$ $\omega(0,0,2,2,2)$
8		$\omega(0,0,8)$ $\omega(0,3,5)$		$\omega(0,0,0,0,8)$ $\omega(0,0,0,1,7)$ $\omega(0,0,0,2,6)$ $\omega(0,0,1,2,5)$ $\omega(0,0,2,2,4)$
10		$\omega(0,0,10)$ $\omega(0,3,7)$		$\omega(0,0,0,0,10)$ $\omega(0,0,0,1,9)$ and 7 more
12		$\omega(0,0,12)$ $\omega(0,3,9)$		$\omega(0,0,0,0,12)$ $\omega(0,0,0,1,11)$ $\omega(0,0,0,2,12)$ and 11 more
14		$\omega(0,0,14)$ $\omega(0,3,11)$ $\omega(0,5,9)$		$\omega(0,0,0,0,14)$ $\omega(0,0,0,1,13)$ $\omega(0,0,0,2,12)$ and many more

- method reproduces all known relations
- results available up to very high weight and length
- comparing q-expansions limited: maximal power in q (here: up to q^{160})
- *need a different and more powerful language!*

Iterated Eisenstein integrals

- q-derivative:

$$q\frac{\mathrm{d}}{\mathrm{d}q}\underbrace{\omega(n_1,\ldots,n_r)}_{w_\omega,\,\ell_\omega}\to\sum \mathrm{G}_{\tilde{n}}\,\underbrace{\omega(\tilde{n}_1,\ldots,\tilde{n}_{r-1})}_{\text{shorter}}$$

- why not capture everything in terms of Eisenstein series?

$$\gamma(k_1,k_2,\ldots,k_n;q)\equiv\frac{1}{4\pi^2}\int\limits_{0\leq q'\leq q}\mathrm{d}\log q'\;\gamma(k_1,\ldots,k_{n-1};q')\,\mathrm{G}^0_{k_n}(q')\qquad k_1\neq 0$$

where

$$\mathrm{G}^0_0\equiv -1,\qquad \mathrm{G}^0_k\equiv \mathrm{G}_k-2\,\zeta_k\;.$$

- *shuffle relations*: obvious

$$\gamma(n_1,\ldots,n_r;q)\,\gamma(k_1,\ldots,k_s;q)=\gamma\left((n_1,\ldots,n_r)\sqcup\!\sqcup\,(k_1,\ldots,k_s);q\right),$$

other relations: tested to very high power – negative result.

Iterated Eisenstein integrals II

$$\begin{aligned} \omega(n) &= -2\,\zeta_n, && n \text{ even} \\ \omega(0,n) &= n\,\gamma(n+1), && n \text{ odd} \\ \omega(0,0,n) &= -\frac{1}{3}\,\zeta_n - n(n+1)\,\gamma(n+2,0), && n \text{ even}\,. \end{aligned}$$

Example:

$$\begin{aligned} 4\pi^2 \frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(0,3,5) &= -15(\mathrm{G}^0_4 - 2\,\zeta_4\,\mathrm{G}^0_0)\,\omega(0,5) + 42\,\omega(0,9) + 3\,\omega(4,5)\,, \\ 4\pi^2 \frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(4,5) &= 9\,\mathrm{G}^0_{10} + 30\,\zeta_6\,\mathrm{G}^0_4 \\ \rightsquigarrow \omega(4,5) &= 9\,\gamma(10) + 30\,\zeta_6\,\gamma(4)\,. \end{aligned}$$

$$\omega(0,3,5) = -90\,\zeta_6\,\gamma(4,0) + 150\,\zeta_4\,\gamma(6,0) - 75\,\gamma(6,4) - 405\,\gamma(10,0)\,,$$

Lenght and weight of γ's and ω's are related by

$$\ell_\gamma = \ell_\omega - 1 \qquad \text{and} \qquad w_\gamma = \ell_\omega - 1 + w_\omega = \ell_\gamma + w_\omega\,.$$

For γ's, no relations known beyond shuffle:

$$\text{indecomposable eMZVs} \overset{??}{\Leftrightarrow} \text{shuffle-independent } \gamma\text{'s}$$

Assume label 2 to appear in the divergent $\gamma(2) = \omega(0,1)$ *exclusively*.

Comparison:

$\omega(0,n)$	$\leftrightarrow$	$\gamma(n+1)$	✓
$\omega(0,0,n)$	$\leftrightarrow$	$\gamma(n+2,0)$	✓
$\omega(0,0,8)$	$\leftrightarrow$	$\gamma(10,0)$	✓
$\omega(0,3,5)$	$\leftrightarrow$	$\gamma(6,4)$	
$\omega(0,0,10)$	$\leftrightarrow$	$\gamma(12,0)$	✓
$\omega(0,3,7)$	$\leftrightarrow$	$\gamma(8,4)$	
$\omega(0,0,12)$	$\leftrightarrow$	$\gamma(14,0)$	
$\omega(0,3,9)$	$\leftrightarrow$	$\gamma(10,4)$	✗
$\omega(0,5,7)$	$\leftrightarrow$	$\gamma(8,6)$	

Very unfortunate. What's the problem here?

What kind of structure is to be expected?

Consider usual MZVs:

$$\zeta_{n_1,n_2,\ldots,n_r} = \int_{0\le z_i\le z_{i+1}\le 1} \omega_1 \underbrace{\omega_0\ldots\omega_0}_{n_1-1}\omega_1\underbrace{\omega_0\ldots\omega_0}_{n_2-1}\ldots\omega_1\underbrace{\omega_0\ldots\omega_0}_{n_r-1}$$
$$= \zeta(1\underbrace{0\ldots0}_{n_1-1}1\underbrace{0\ldots0}_{n_2-1}\ldots1\underbrace{0\ldots0}_{n_r-1})$$

Drinfeld associator (free algebra x, y):

$$\Phi(x,y) \equiv \sum_{w\in\langle x,y\rangle} \zeta^{\sqcup\!\sqcup}(w)\cdot w$$

Noncommutative words : [Goncharov][Brown]

$\phi:$ algebra of $\zeta^{\mathfrak{m}}$ $\rightarrow$ Hopf algebra comodule

$f_{2i_1+1}\ldots f_{2i_r+1}\ f_2^k,$ with $r,k\ge 0$ and $i_1,\ldots,i_r\ge 1$

Counting noncommutative words $\rightarrow$ correct number [Zagier][Broadhurst Kreimer]

Apparently, this is the complete structure: no further surprises.

Consider corresponding objects for eMZVs:

Elliptic associator: [Enriquez]

$$A(\tau) \equiv \sum_{r\geq 0} (-1)^r \sum_{n_1,n_2,\ldots,n_r\geq 0} \omega(n_1,n_2,\ldots,n_r)\,\mathrm{ad}_x^{n_r}(y)\ldots\mathrm{ad}_x^{n_2}(y)\,\mathrm{ad}_x^{n_1}(y)$$

$$\mathrm{ad}_x(y) \equiv [x,y]\ , \qquad \mathrm{ad}_x^n(y) = \underbrace{[x,\ldots[x,[x,}_{n \text{ times}}y]]\ldots]$$

Differential equation: [Enriquez]

$$\frac{\mathrm{d}}{\mathrm{d}\log q} A(\tau) = \frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}(2n-1)\mathrm{G}_{2n}\,\epsilon_{2n}\right)A(\tau)\,.$$

Derivation algebra $\mathfrak{u}$: [Hain Matsumoto][Pollack]

$$\epsilon_{2m}(x) = (\mathrm{ad}_x)^{2m}(y)\ , \qquad m\geq 0$$

$$\epsilon_{2m}(y) = [y,(\mathrm{ad}_x)^{2m-1}(y)] + \sum_{1\leq j<m}(-1)^j\Big[(\mathrm{ad}_x)^j(y),(\mathrm{ad}_x)^{2m-1-j}(y)\Big] \qquad m>0$$

$$\epsilon_0(y) = 0$$

Does it solve the conundrum?

First discrepancy occurs at weight 12, length 3:

$$\begin{aligned}
\omega(0,0,12) &= -\frac{\zeta_{12}}{3} - 156\,\gamma(14,0)\\
\omega(0,3,9) &= -729\,\gamma(10,4) - 315\,\gamma(8,6) - 5616\,\gamma(14,0) - 210\,\gamma(6)\,\gamma(8)\\
&\quad - 1350\,\zeta_{10}\,\gamma(4,0) - 630\,\zeta_8\,\gamma(6,0) + 630\,\zeta_6\,\gamma(8,0) + 1458\,\zeta_4\,\gamma(10,0)\\
\omega(0,5,7) &= -1134\,\gamma(10,4) - 490\,\gamma(8,6) - 5642\,\gamma(14,0)\\
&\quad - 1260\,\zeta_{10}\,\gamma(4,0) - 700\,\zeta_8\,\gamma(6,0) + 980\,\zeta_6\,\gamma(8,0) + 2268\,\zeta_4\,\gamma(10,0)
\end{aligned}$$

ratio of $\gamma(8,6)$ and $\gamma(10,4)$ is *equal* in *all* eMZVs of weight 12 and length 3:

$$81\,\gamma(10,4) + 35\,\gamma(8,6)\,.$$

There are further non-obvious relations X in the algebra of derivations $\mathfrak{u}$. [Pollack]
In particular:

$$0 = [\epsilon_{10}, \epsilon_4] - 3\,[\epsilon_8, \epsilon_6]\,.$$

Translate iterated Eisenstein integrals into noncommutative words:

$$\psi\big[\gamma(k_1, k_2, \ldots, k_n)\big] \equiv \frac{g_{k_n} g_{k_{n-1}} \cdots g_{k_2} g_{k_1}}{\prod_{j=1}^{n}(k_j - 1)} \ .$$

Define a suitable derivative (already appeared in the context of the ϕ-map for MZVs):

$$\partial_j g_{k_1} \ldots g_{k_n} = \delta_{j,k_1} g_{k_2} \ldots g_{k_n} \ .$$

Identify derivations ϵ_{2m} with ∂_{2m}. Thus

$$\Big([\partial_{10}, \partial_4] - 3[\partial_8, \partial_6]\Big) \ \psi\Big[81\, \gamma(10,4) + 35\, \gamma(8,6)\Big] = \Big([\partial_{10}, \partial_4] - 3[\partial_8, \partial_6]\Big) \ \Big[3\, g_4\, g_{10} + g_6\, g_8\Big] = 0\,.$$

Conjecture: For any relation

$$X = \sum_{\{n_1, n_2, \ldots, n_r\}} \alpha_{n_1, n_2, \ldots, n_r} [[\ldots[[\epsilon_{n_1}, \epsilon_{n_2}], \epsilon_{n_3}], \ldots], \epsilon_{n_r}] = 0$$

in the algebra of derivations $\mathfrak{u}$ one finds:

$$X\Big|_{\epsilon \to \partial} \psi[A(\tau)] = 0\,.$$

Further evidence

Central element

ϵ_2 is central in derivation algebra $\mathfrak{u}$:

$$[\epsilon_{2m}, \epsilon_2] = 0 \quad \leftrightarrow \quad \text{only divergent eMZV: } \gamma(2) = \omega(0,1)\,.$$

Fits with observed absence of label 2 in iterated Eisensteins integrals except for $\gamma(2)$.

Additional derivations

Generators of the free Lie algebra $\mathbb{L}(z_3, z_5, z_7, z_9, ...)$ induce derivations $\widetilde{z}$ in $\mathfrak{u}$:

$$0 = [\widetilde{z}_{2k+1}, \epsilon_0] = [\widetilde{z}_{2k+1}, \epsilon_2]\ , \qquad k = 1, 2, 3, \ldots\ ,$$

whose commutators with $\epsilon_{2m}, m > 1$ can be constructed. [Pollack]

Further irreducible relations

Numerous further *irreducible* relations are known, e.g.:

$$0 = 2\,[\epsilon_{14}, \epsilon_4] - 7\,[\epsilon_{12}, \epsilon_6] + 11\,[\epsilon_{10}, \epsilon_8]$$

$$\begin{aligned} 0 = {} & 80\,[\epsilon_{12}, [\epsilon_4, \epsilon_0]] + 16\,[\epsilon_4, [\epsilon_{12}, \epsilon_0]] - 250\,[\epsilon_{10}, [\epsilon_6, \epsilon_0]] \\ & - 125\,[\epsilon_6, [\epsilon_{10}, \epsilon_0]] + 280\,[\epsilon_8, [\epsilon_8, \epsilon_0]] - 462\,[\epsilon_4, [\epsilon_4, \epsilon_8]] - 1725\,[\epsilon_6, [\epsilon_6, \epsilon_4]]\,. \\ \vdots & \qquad\qquad\qquad \vdots \end{aligned}$$

Reducible relations:

$$X = 0 \quad \Rightarrow \quad \mathrm{ad}_{n_1, n_2, \ldots, n_k}(X) \equiv [\epsilon_{n_1}, [\epsilon_{n_2}, [\ldots, [\epsilon_{n_k}, X] \ldots]]] = 0\ .$$

Simple example:

$$\big[\epsilon_n, 2\,[\epsilon_{14}, \epsilon_4] - 7\,[\epsilon_{12}, \epsilon_6] + 11\,[\epsilon_{10}, \epsilon_8]\big] = 0$$

but as well

$$\big[\widetilde{z}_3, [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6]\big] = 0$$

at $w_\gamma = 20$ and $\ell_\gamma = 5$.

Vanishing nested commutators

The nested commutator

$$[[[[\partial_4, \partial_0], \partial_0], \partial_0], \partial_{2m}]$$

annihilates all eMZVs starting from $w_\gamma = 8,\ \ell_\gamma = 5$.

- Consider $\gamma(4, 0, 0, 0)$: corresponds to ω of weight 0 and length 5.
- only known eMZV is $\omega(0, 0, 0, 0, 0) = 1/120 \neq \gamma(4, 0, 0, 0)$ (no q-expansion)

Indeed

$$[[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0](x) = 0\ , \qquad [[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0](y) = 0\,.$$

Adding further 0's leads to ω's of *negative* weight: further reducible relations.

Taking all available relations in $\mathfrak{u}$ into account, the counting matches!

Conjecture: Indecomposable eMZVs can be enumerated by shuffle-independent iterated Eisenstein integrals, taking all relations in $\mathfrak{u}$ into account in addition.

Comparison between ϕ and ψ

MZV

$$\phi(\zeta^{\mathfrak{m}}) = \sum f_{2i_1+1} \dots f_{2i_r+1}\ f_2^k$$

$$\phi(\zeta^{\mathfrak{m}}_{n_1,\dots,n_r}\,\zeta^{\mathfrak{m}}_{k_1,\dots,k_s}) = \phi(\zeta^{\mathfrak{m}}_{n_1,\dots,n_r}) \sqcup\!\sqcup\, \phi(\zeta^{\mathfrak{m}}_{k_1,\dots,k_s})\,.$$

$$\phi(\zeta^{\mathfrak{m}}) = \sum_{3\leq 2k+1\leq w} f_{2k+1}\xi_{2k+1}$$

ξ_{2k+1}: component of weight $(2k+1)\otimes(w-2k-1)$ in the coaction for motivic MZVs

eMZV

$$\psi(\omega) = \sum g_{2i_1} \dots g_{2i_r}$$

$$\psi(\gamma(n_1,\dots,n_r)\gamma(k_1,\dots,k_s)) = \psi(\gamma(n_1,\dots,n_r)) \sqcup\!\sqcup\, \psi(\gamma(k_1,\dots,k_s))$$

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(n_1,n_2,\dots,n_r) = \frac{1}{4\pi^2}\sum_{k=0}^{\infty} \xi_{2k}\,\mathrm{G}^0_{2k}$$

ξ_{2k}: component of weight $(2k)\otimes(w-2k)$ in the τ-derivative

Turning things around...

Taking our conjecture for granted, one can straightforwardly find new relations:

$$0 \stackrel{!}{=} \sum_{\{n_1,n_2,\ldots,n_r\}} \alpha_{n_1,n_2,\ldots,n_r}[[[\ldots[\partial_{n_1},\partial_{n_2}],\partial_{n_3}],\ldots],\partial_{n_r}]$$

In this way, we find all relations of depth 2 known from [Luque, Novelli Thibon]:

$$0 = \sum_{i=1}^{2n+2p-1} \frac{[\epsilon_{2p+2n-i+1},\epsilon_{i+1}]}{(2p+2n-i-1)!}\left(\frac{(2n-1)!B_{i-2p+1}}{(i-2p+1)!}+\frac{(2p-1)!B_{i-2n+1}}{(i-2n+1)!}\right),$$

all relations known by [Pollack] and – e.g. – an apparently new relation at depth 5:

$$\begin{aligned}0 = {} & 2206388620800\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{16}]]]] - 8366188740000\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{14}]]]]\\ & + 12305858292000\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{12}]]]] - 1834700544000\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]]]\\ & + 35687825530800\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]]] - 109425220173750\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]]]\\ & - 39970750599360\,[\epsilon_0,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{12}]]]] - 380488416808500\,[\epsilon_0,[\epsilon_4,[\epsilon_4,[\epsilon_4,\epsilon_8]]]]\\ & + 1317125628000\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]] + 220479512028750\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_{10}]]]]\\ & + \text{ 43 further terms}\ldots\ .\end{aligned}$$

Example: $w_\gamma = 20,\ \ell_\gamma = 5$

w_γ \ ℓ_γ	2	3	4	5	6	7	8	9	10
12	0	0	0	0	0	0	0	0	0
14	r_{14}^2	0	0	0	0	0	0	0	0
16	0	r_{16}^3	0	0	0	0	0	0	0
18	r_{18}^2	0	r_{18}^4	0	0	0	0	0	0
20	r_{20}^2	r_{20}^3	0	r_{20}^5	0	0	0	0	0

$\mathrm{ad}_{6,0,0}\ r_{14}^2$ 3 permutations $\qquad$ $\mathrm{ad}_{4,0}\ r_{16}^3$ 2 permutations

$\mathrm{ad}_{0,0,0}\ r_{20}^2$ 1 permutation $\qquad$ $\mathrm{ad}_{0,0}\ r_{20}^3$ 1 permutation .

$$[[[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0], \epsilon_{12}] = 0 \quad \text{and} \quad [\tilde{z}_3, r_{14}^2]\,,$$

We find 10 relations at $w_\gamma = 20,\ \ell_\gamma = 5 \rightarrow$ one (new) irreducible relation: r_{20}^5.

Complete table:

w_γ \ ℓ_γ	2	3	4	5	6	7	8	9	10
12	0	0	0	0	0	0	0	0	0
14	r^2_{14}	0	0	0	0	0	0	0	0
16	0	r^3_{16}	0	0	0	0	0	0	0
18	r^2_{18}	0	r^4_{18}	0	0	0	0	0	0
20	r^2_{20}	r^3_{20}	0	r^5_{20}	0	0	0	0	0
22	r^2_{22}	r^3_{22}	r^4_{22}	0	r^6_{22}	0	0	0	0
24	r^2_{24}	r^3_{24}	r^4_{24}	r^5_{24}	0	r^7_{24}	0	0	0
26	$2 \times r^2_{26}$	r^3_{26}	r^4_{26}	r^5_{26}	r^6_{26}	0	r^8_{26}	0	0
28	r^2_{28}	$2 \times r^3_{28}$	r^4_{28}	r^5_{28}	r^6_{28}	r^7_{28}	0	r^9_{28}	0
30	$2 \times r^2_{30}$	r^3_{30}	$2 \times r^4_{30}$	r^5_{30}	r^6_{30}	r^7_{30}	r^8_{30}	0	r^{10}_{30}

Generating series for number of cusp forms:

$$\mathbb{S}(s) = \frac{s^{12}}{(1-s^4)(1-s^6)} \Rightarrow (1,0,1,1,1,1,2,1,2,2,2,2,3,2,3,3,3,3,4,\ldots)$$

Conclusions

- explained relations and number of indecomposable eMZVs by iterated Eisenstein integrals γ and the algebra of derivations $\mathfrak{u}$
- given validity of our conjecture, one can derive further relations in the derivation algebra $\mathfrak{u}$, which match the known pattern of cusp forms
- open-string result does not contain divergent eMZVs
- numerous relations for eMZVs can be obtained from `https://tools.aei.mpg.de/emzv`

Open questions

- closed/recursive form of the integrand for the one-loop open-string amplitude in terms of iterated Eisenstein integrals (Olivers talk)
- τ-integration / regularization
- connection to functions ELi occurring in [Adams, Bogner Weinzierl]

Thanks!