Periods and Superstring Amplitudes

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Stephan Stieberger, MPP München

Workshop on Amplitudes, Motives and Beyond
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0.Outline

- Real iterated integral

$$Z \sim \int_{x_1<\ldots<x_N} \left(\prod_{l=2}^{N-2} dx_l\right) \prod_{i<j} |x_i - x_j|^{\alpha' s_{ij}} \, (x_i - x_j)^{n_{ij}} \ , \qquad s_{ij} \in \mathbf{R}, \ n_{ij} \in \mathbf{Z}$$

→ decomposition of motivic MZVs,
non-commutative algebra comodul,
Lie algebra structure

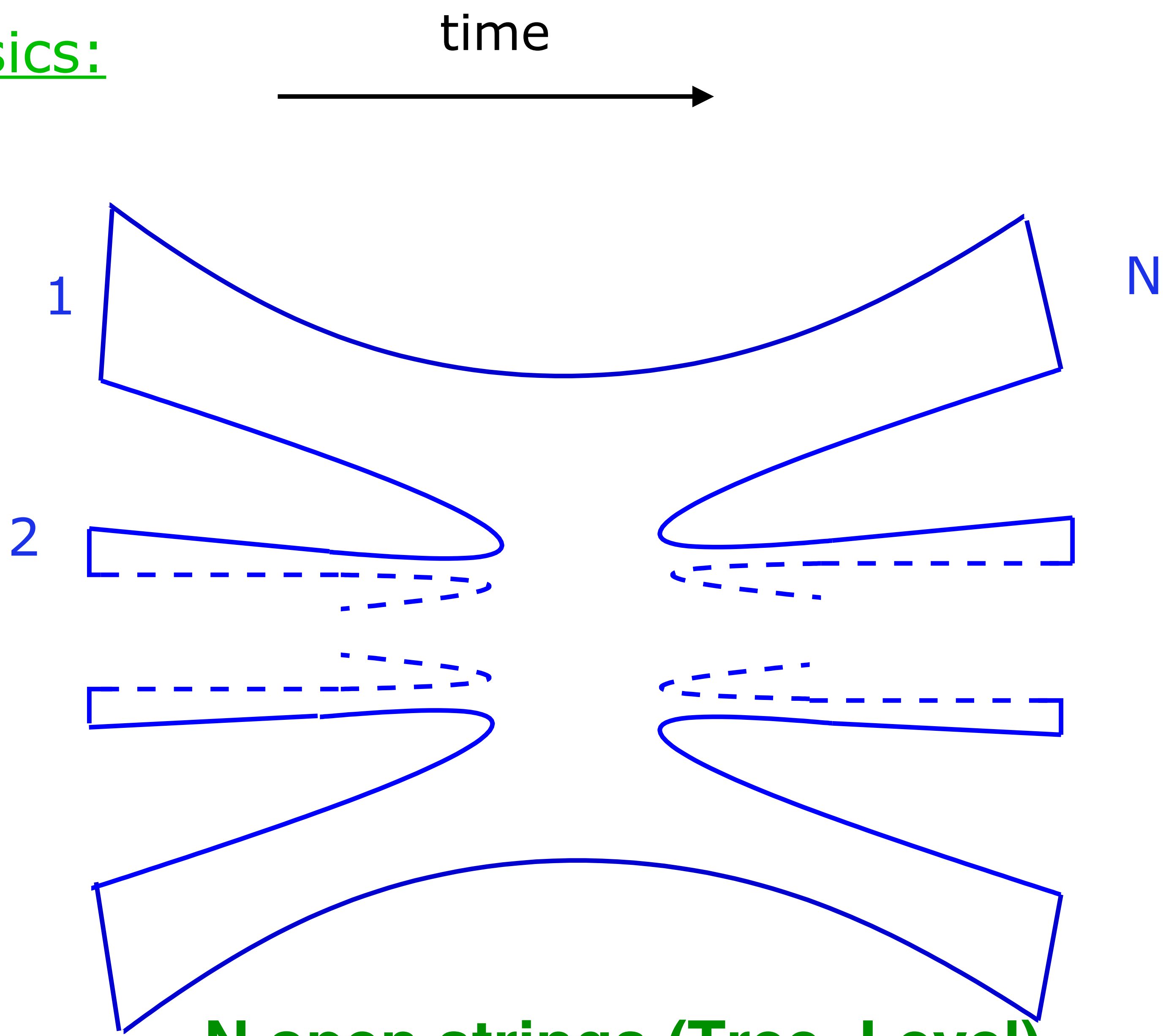
- Complex integral on $(\mathbf{CP}^1)^{N-3}$

$$J \sim \int_{\mathbf{C}^{N-3}} \left(\prod_{l=2}^{N-2} d^2 z_l\right) \prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}} \, (z_i - z_j)^{n_{ij}} \, (\overline{z}_i - \overline{z}_j)^{\overline{n}_{ij}}, \qquad s_{ij} \in \mathbf{R}, \ n_{ij}, \overline{n}_{ij} \in \mathbf{Z}$$

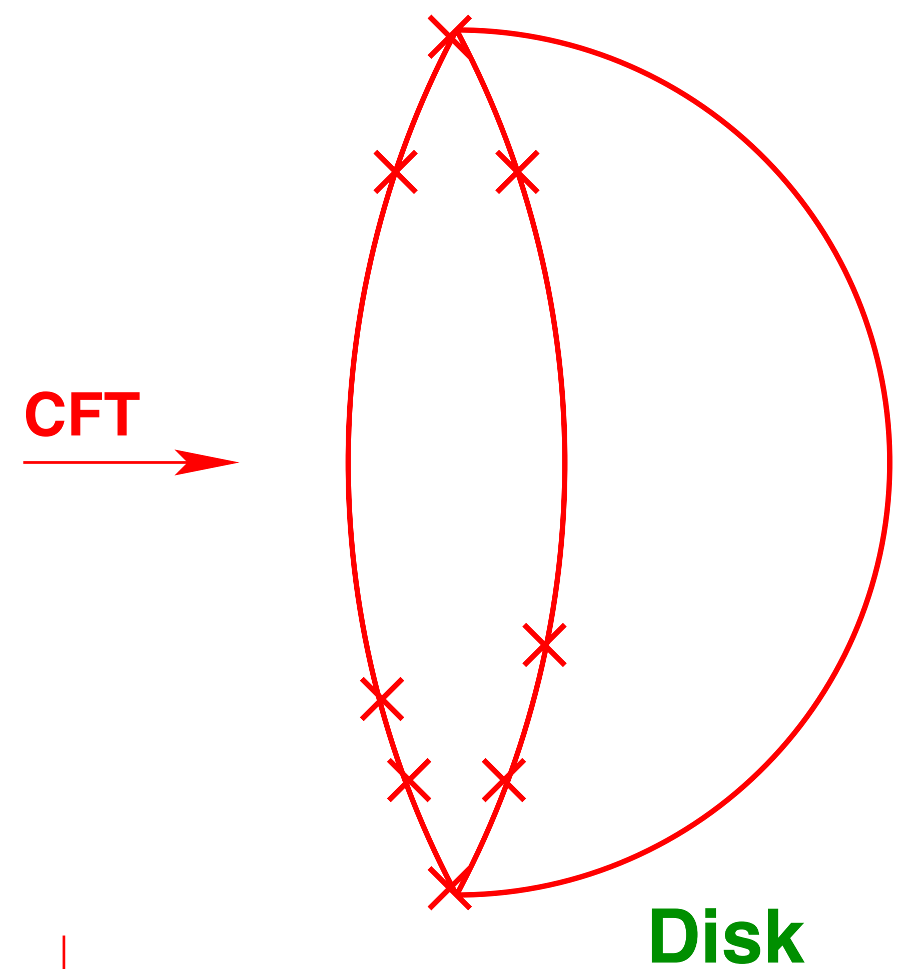
→ single-valued MZVs

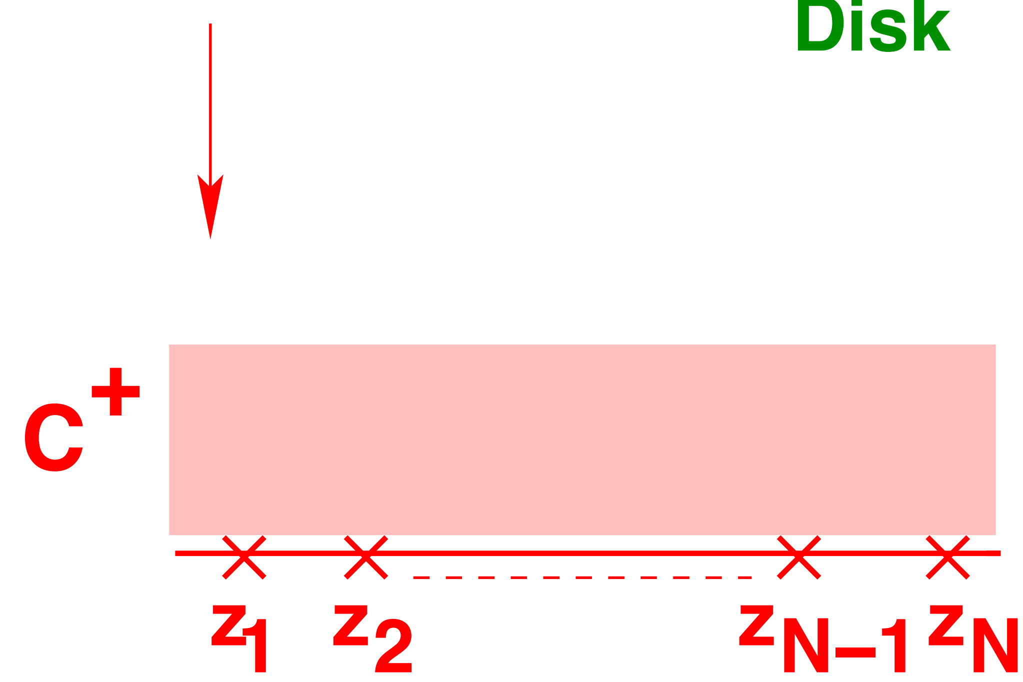
- Relation: $J = \mathrm{sv}(Z)$ for given $\overline{n}_{ij}$

Physics:



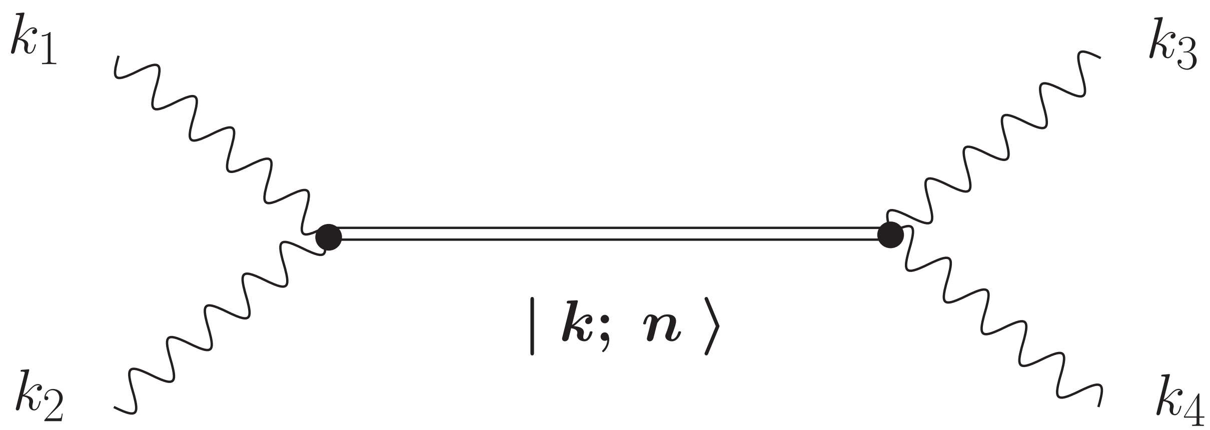
N open strings (Tree–Level)





$$\mathcal{A}(1,2,\ldots,N;\alpha')$$

(N-3)! independent open string amplitudes

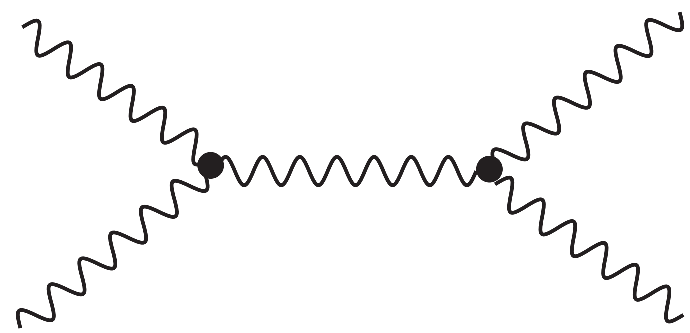


$$M_n^2 = M_{\text{string}}^2\ n$$

$$M_{\text{string}}^2 = \alpha'^{-2}$$

$$\mathcal{A}(1,2,3,4;\alpha') \sim \sum_{n=0}^{\infty} \frac{\gamma(n)}{(k_1+k_2)^2 - M_n^2} = \frac{\Gamma(-\alpha' s)\ \Gamma(1-\alpha' u)}{\Gamma(-\alpha' s - \alpha' u)}$$

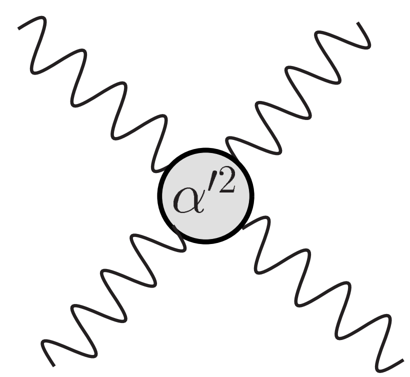
Power series in α' :



super Yang – Mills

$$s = (k_1+k_2)^2$$
$$t = (k_1-k_3)^2$$
$$u = (k_1-k_4)^2$$

$+$



$\frac{\pi^2}{6}\ \alpha'^2\ \text{tr}F^4$

open string:

gauge interactions

α'^0 1	$\mathbf{F^2}$
α' 0	F^3 D^2F^2
α'^2 $\zeta(2)$	$\mathbf{F^4}$ D^2F^3 D^4F^2
α'^3 $\zeta(3)$	$\mathbf{F^5}$ $\mathbf{D^2F^4}$ D^6F^2
α'^4 $\zeta(4)$	$\mathbf{F^6}$ $\mathbf{D^4F^4}$ $\mathbf{D^2F^5}$
α'^5 $\zeta(2)\zeta(3)$, $\zeta(5)$	$\mathbf{F^7}$ $\mathbf{D^6F^4}$ $\mathbf{D^4F^5}$ $\mathbf{D^2F^6}$
$\vdots$	$\dots$ $\dots$ $\dots$ $\dots$

closed string:

gravitational interactions

α'^3 $\zeta(3)$	R^4		
~~α'^4 $\zeta(4)$~~	~~D^2R^4~~	~~$\boldsymbol{R^5}$~~	
α'^5 $\zeta(5)$	D^4R^4	$\boldsymbol{D^2R^5}$	$\boldsymbol{R^6}$
~~α'^5 $\zeta(2)\zeta(3)$~~	~~D^4R^4~~	~~$\boldsymbol{D^2R^5}$~~	~~$\boldsymbol{R^6}$~~
α'^6 $\zeta(3)^2$	D^6R^4	$\boldsymbol{D^4R^5}$	$\boldsymbol{D^2R^6}$
~~α'^6 $\zeta(6)$~~	~~D^6R^4~~	~~$\boldsymbol{D^4R^5}$~~	~~$\boldsymbol{D^2R^6}$~~
α'^7 $\zeta(7)$	D^8R^4	$\boldsymbol{D^6R^5}$	$\boldsymbol{D^4R^6}$
~~α'^7 $\zeta(3)\zeta(4)$~~	~~D^8R^4~~	~~$\boldsymbol{D^6R^5}$~~	~~$\boldsymbol{D^4R^6}$~~
~~α'^7 $\zeta(2)\zeta(5)$~~	~~D^8R^4~~	~~$\boldsymbol{D^6R^5}$~~	~~$\boldsymbol{D^4R^6}$~~
α'^8 $\zeta(3)\zeta(5)$	$D^{10}R^4$	$\boldsymbol{D^8R^5}$	$\boldsymbol{D^6R^6}$
~~α'^8 $\zeta(8)$~~	~~$D^{10}R^4$~~	~~$\boldsymbol{D^8R^5}$~~	~~$\boldsymbol{D^6R^6}$~~
~~α'^8 $\zeta(2)\zeta(3)^2$~~	~~$D^{10}R^4$~~	~~$\boldsymbol{D^8R^5}$~~	~~$\boldsymbol{D^6R^6}$~~
~~α'^8 $\zeta(5,3)$~~	~~$D^{10}R^4$~~	~~$\boldsymbol{D^8R^5}$~~	~~$\boldsymbol{D^6R^6}$~~

Comments:

- Z = generalized Euler (Selberg) integral integrates to multiple Gaussian hypergeometric functions: **Aomoto-Gelfand hypergeometric functions, GKZ structures**

- power series in α':

  * yields **iterated integrals**, which are **periods** of the moduli space $\mathcal{M}_{0,N}$ of genus zero curves with N ordered marked points

  * integrate to Q-linear combinations of **MZVs** (Brown, Terasoma)

Example:

$$\int_{z_1<\ldots<z_5} \left(\prod_{l=2}^{3} dz_l\right) \frac{\prod_{i<j}^{4} |z_{ij}|^{\alpha' s_{ij}}}{z_{12}\ z_{23}\ z_{41}}$$

$$= \alpha'^{-2} \left(\frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}}\right) + \zeta_2 \left(1 - \frac{s_{34}}{s_{12}} - \frac{s_{12}}{s_{45}} - \frac{s_{23}}{s_{45}} - \frac{s_{51}}{s_{23}}\right) + \mathcal{O}(\alpha'')$$

$$z_{ij} := z_i - z_j$$

→ New techniques for computing epsilon-expansions for amplitudes

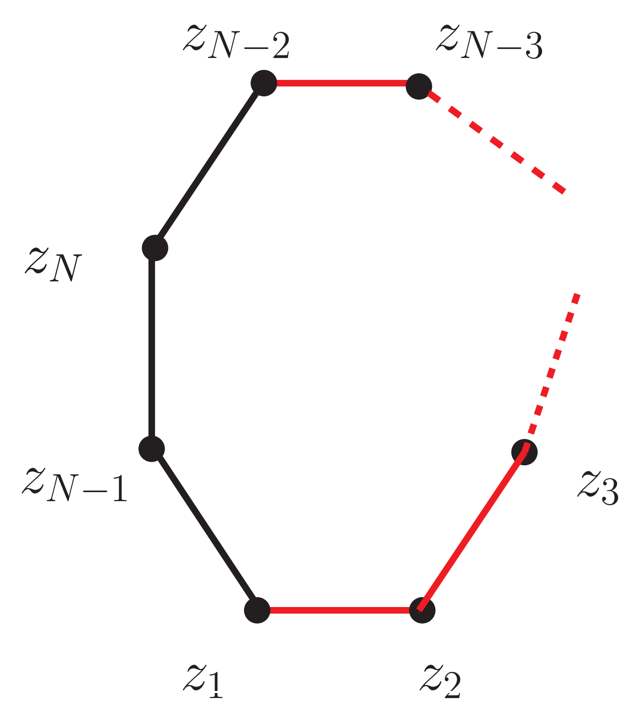
with G. Puhlfürst

Actually we consider: $z_1 = 0, \quad z_{N-1} = 1, \quad z_N = \infty$ due to $PSL(2, \mathbf{R})$ symmetry

$$Z_\pi(\rho) := \int\limits_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \frac{\prod\limits_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\ z_{\rho(2),\rho(3)} \ldots z_{\rho(N-3),\rho(N-2)}}$$

iterated real integral on $\mathbf{RP}^1 \backslash \{0, 1, \infty\}$

$\pi, \rho \in S_{N-3}$



$$D(\pi) = \left\{\ z_j \in \mathbf{R} \mid 0 < z_{\pi(2)} < \ldots < z_{\pi(N-2)} < 1 \right\} \subset \left(\mathbf{RP}^1 \backslash \{0, 1, \infty\} \right)^{N-3}$$

- For given π all integrals can be expressed in $\mathbf{R}$ in terms of (N-3)! dimensional **basis**

$$\dim\ H^{N-3}(\mathcal{M}_{0,N}; \mathbf{R}) = (N-3)!$$

- **Cell-forms** on $\mathcal{M}_{0,N}$ (Brown, Carr, Schneps) **periods** of $\mathcal{M}_{0,N}$ studied by Goncharov, Manin

$$\dim\ H^{N-3}(\mathcal{M}_{0,N}; \mathbf{Q}) = (N-2)!$$

normalization: $S^{-1} := (-1)^{N-3} \; Z|_{\alpha'^{3-N}}$

$$\boxed{F := Z \; S} \quad \text{i.e.:} \;\; F|_{\alpha'=0} = 1$$

$$F \;=\; (N-3)! \times (N-3)! \text{ matrix with } \mathrm{rk}(F) = (N-3)!$$

$$F \;=\; \text{period matrix of } \mathcal{M}_{0,N}$$

Physics: $\mathcal{A} \;=\; (N-3)!$ dimensional vector encompassing **all** independent **superstring** supamplitudes $\mathcal{A}^\sigma, \;\; \sigma \in S_{N-3}$

$A_{YM} \;=\; (N-3)!$ dimensional vector encompassing **all** independent **SYM** supamplitudes $A^\sigma_{YM}, \;\; \sigma \in S_{N-3}$

$$\boxed{\mathcal{A} = F \; A_{YM}}$$

→ *F has also physical meaning*

S = KLT kernel

$$S[\rho|\sigma] \quad := \quad S[\,\rho(2,\ldots,N-2)\,|\,\sigma(2,\ldots,N-2)\,]$$
$$= \quad \prod_{j=2}^{N-2}\left(s_{1,j_\rho} \;+\; \sum_{k=2}^{j-1}\theta(j_\rho,k_\rho)\,s_{j_\rho,k_\rho}\right)$$

Bern, Dixon, Perelstein, Rozowsky (1998)

$$s_{ij} = \alpha'(k_i + k_j)^2$$

2. Observation/Result

$$F(\alpha') \;=\; P\; Q\;\exp\left\{\sum_{n\geq 1} \zeta_{2n+1}\; M_{2n+1}\right\}$$

organization according to zeta values

$$P = 1 + \sum_{n\geq 1} \zeta_2^n\; P_{2n}, \quad P_{2n} = F(\alpha')|_{\zeta_2^n}$$

$$M_{2n+1} = F(\alpha')|_{\zeta_{2n+1}}$$

$$\begin{aligned} Q \;&=\; 1 + \tfrac{1}{5}\,\zeta_{3,5}\,[M_5, M_3] + \left\{\, \tfrac{3}{14}\,\zeta_5^2 + \tfrac{1}{14}\,\zeta_{3,7}\,\right\}\,[M_7, M_3] \\ &+\; \left\{\, 9\,\zeta_2\,\zeta_9 + \tfrac{6}{25}\,\zeta_2^2\,\zeta_7 - \tfrac{4}{35}\,\zeta_2^3\,\zeta_5 + \tfrac{1}{5}\,\zeta_{3,3,5}\,\right\}\,[M_3,[M_5,M_3]] + \ldots \end{aligned}$$

$$\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l}, \quad n_l \in \mathbf{N}^+,\; n_r \geq 2$$

- All information is kept in P and M

E.g. N=5:

$$P_2 = {\alpha'}^2 \begin{pmatrix} -s_{34}\; s_{45} + s_{12}\,(s_{34} - s_{51}) & s_{13}\; s_{24} \\ s_{12}\; s_{34} & (s_{12} + s_{23})\,(s_{23} + s_{34}) - s_{45}\; s_{51} \end{pmatrix}$$

- This form exactly appears in F. Browns decomposition of motivic multiple zeta values
- Connection: Hypergeometric functions ⟷ motivic MZVs

Motivic decomposition operator at weight 10 F. Brown (2011)

$$\xi_{10} = a_0 \, (\zeta_2^m)^5 + a_1 \, (\zeta_2^m)^2 \, (\zeta_3^m)^2 + a_2 \, \zeta_2^m \, \zeta_3^m \, \zeta_5^m + a_3 \, (\zeta_5^m)^2 + a_4 \, \zeta_2^m \, \zeta_{3,5}^m + a_5 \, \zeta_3^m \, \zeta_7^m + a_6 \, \zeta_{3,7}^m$$

Operators acting on $\phi(\xi_{10})$:

$$a_1 = \frac{1}{2} \, c_2^2 \, \partial_3^2, \; a_2 = c_2 \, \partial_5 \partial_3, \; a_3 = \frac{1}{2} \, \partial_5^2 + \frac{3}{14} \, [\partial_7, \partial_3]$$

$$a_4 = \frac{1}{5} \, c_2 \, [\partial_5, \partial_3], \; a_5 = \partial_7 \partial_3, \; a_6 = \frac{1}{14} \, [\partial_7, \partial_3]$$

Goncharov, Brown:

$$\begin{array}{ccccc} \zeta & \longleftarrow & \zeta^m & \longrightarrow & \phi(\zeta^m) \\ \mathcal{Z} & \longleftarrow & \mathcal{H} & \longrightarrow & \mathcal{U} \end{array}$$

normalized by:

$$\phi(\zeta_n^m) = f_n, \quad n \geq 2$$

Algebra-comodule $\mathcal{U}$ is introduced to describe the structure of $\mathcal{H}$

$$\mathcal{U} = \mathbf{Q}\langle f_3, f_5, \ldots \rangle \; \otimes_{\mathbf{Q}} \; \mathbf{Q}[f_2]$$

Map ϕ sends every motivic MZV $\xi \in \mathcal{H}$ to a non-commutative polynomial in the $f_i' s$

3. Motivic structure of F

Recall: we found from expanding the period matrix

$$F^m|_{\alpha'^{10}} = (\zeta_2^m)^5\ P_{10} + \frac{1}{2}\ (\zeta_2^m)^2(\zeta_3^m)^2\ P_4 M_3^2 + \zeta_2^m\zeta_3^m\zeta_5^m\ P_2 M_5 M_3$$

$$+ \frac{1}{5}\ \zeta_2^m \zeta_{3,5}^m\ P_2[M_5, M_3] + (\zeta_5^m)^2\ \left(\frac{1}{2}\ M_5^2 + \frac{3}{14}\ [M_7, M_3]\right)$$

$$+ \zeta_3^m\zeta_7^m\ M_7 M_3 + \frac{1}{14}\ \zeta_{3,7}^m\ [M_7, M_3]$$

$$\partial_{2n+1} \simeq M_{2n+1}\ ,$$

$$c_2^k \simeq P_{2k}\ , \quad k \geq 1\ .$$

Exact match in the coefficient and commutator structure

by identifying the

motivic derivation operators ∂ and the **matrix operators** M

and the **coefficient operator** c_2 with the **matrix operators** P_2

Decomposition of motivic MZVs encapsulates α' - expansion of period matrix F

4. Single-valued MZVs (SVMZVs)

$$\zeta_{\mathrm{sv}}(n_1,\ldots,n_r)\in\mathbf{R}$$

- special class of MZVs, which occurs as the values **at unity** of SVMPs

Schnetz (2013)

$$polylogarithms: \quad \ln(z),\ \ Li_1(z)=-ln(1-z), Li_a(z), Li_{a_1,\ldots,a_r}(1,\ldots,1,z)$$

SVMPs: **multiple polylogarithms** can be combined
with their complex conjugates
to **remove monodromy** at $z=0,1,\infty$
rendering the function **single-valued** on $\mathbf{P}^1\backslash\{0,1,\infty\}$.

$$\mathcal{L}_2(z)=D(z)=Im\left\{Li_2(z)+\ln|z|\ \ln(1-z)\right\}$$

(Bloch-Wigner dilogarithm)

$$\mathcal{L}_n(z)=Re_n\left\{\sum_{k=1}^{n}\frac{(-\ln(|z|)^{n-k}}{(n-k)!}\ Li_k(z)+\frac{\ln^n|z|}{(2n)!}\right\}\ \text{with:}\quad Re_n=\begin{cases}Im, & n \text{ even}\\ Re, & n \text{ odd}\end{cases}$$

(Zagier)

$$\mathcal{L}_n(1)=Re_n\left\{Li_n(1)\right\}=\begin{cases}0, & n\text{ even}\\ \zeta_n, & n\text{ odd}\end{cases}$$

- coefficients of an associator W:

(reduced) KZ equation: $$\frac{d}{dz} L_{e_0,e_1}(z) = L_{e_0,e_1}(z) \left(\frac{e_0}{z} + \frac{e_1}{1-z} \right)$$ with generators e_0 and e_1 of the free Lie algebra g

its unique solution can be given as generating series of multiple polylogarithms:

$$L_{e_0,e_1}(z) = \sum_{w\in\{e_0,e_1\}^\times} L_w(z)\ w$$

with the symbol $w \in \{e_0,e_1\}^\times$ denoting a non-commutative word $w_1 w_2 \ldots$ in the letters $w_i \in \{e_0,e_1\}$

$$L_1 = 1, \quad L_{e_0^n} = \frac{1}{n!}\ln^n z, \quad L_{e_1^n} = \frac{1}{n!}\ln^n(1-z)$$

Drinfeld associator Z:

$$\zeta(e_1 e_0^{n_1-1} \ldots e_1 e_0^{n_r-1}) = \zeta_{n_1,\ldots,n_r}$$

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 ⧢ w_2) \text{ and } \zeta(e_0) = 0 = \zeta(e_1)$$

$$Z(e_0,e_1) := L_{e_0,e_1}(1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta(w)\ w = 1 + \zeta_2\ [e_0,e_1] + \zeta_3\ (\ [e_0,[e_0,e_1]] - [e_1,[e_0,e_1]\ ) + \ldots$$

F. Brown (2004) defines generating series of SVMPs:

$$\mathcal{L}_{e_0,e_1}(z) = L_{-e_0,-e_1'}(\overline{z})^{-1}\ L_{e_0,e_1}(z)$$

e_1' determined recursively by fixed–point equation:

$$Z(-e_0,-e_1')\ e_1'\ Z(-e_0,-e_1')^{-1} = Z(e_0,e_1)\ e_1\ Z(e_0,e_1)^{-1}$$

Deligne associator W:

$$W(e_0,e_1) := \mathcal{L}(1) = Z(-e_0,-e_1')^{-1}\ Z(e_0,e_1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta_{\rm sv}(w)\ w$$

$$W(e_0,e_1) = 1 + 2\ \zeta_3\ ([e_0,[e_0,e_1]] - [e_1,[e_0,e_1]]) + \ldots$$

F. Brown (2013)

In fact: Deligne introduced associator W formally as:

$$W \circ {}^{\sigma}Z = Z$$

with Ihara action $\circ$ providing formal multiplication rule on group-like formal power series in e_0 and e_1

$$F(e_0,e_1) \circ G(e_0,e_1) = G(e_0, F(e_0,e_1)e_1F(e_0,e_1)^{-1})\ F(e_0,e_1)$$

$$\Longrightarrow \quad W(e_0,e_1) = {}^{\sigma}Z(e_0, We_1W^{-1})^{-1}\ Z(e_0,e_1)$$

(definition only uses Ihara action)

Note: explicit representation of associators in limit $\mathrm{mod}\ (\mathrm{g}')^2$

(corresponds to a commuative realization of the Ihara bracket)

$(g')^2 = [g,g]^2$

$u = -\mathrm{ad}_{e_1},\ v = \mathrm{ad}_{e_0}$

$\mathrm{ad}_x y = [x,y]$

$$Z(e_0,e_1) = 1 - (uv)^{-1} \left(\frac{\Gamma(1-u)\ \Gamma(1-v)}{\Gamma(1-u-v)} - 1 \right) [e_0,e_1]$$

relates to open superstring amplitude

Drummond, Ragoucy (2013)

$$W(e_0,e_1) = 1 + (uv)^{-1} \left(\frac{\Gamma(-u)\ \Gamma(-v)\ \Gamma(u+v)}{\Gamma(u)\ \Gamma(v)\ \Gamma(-u-v)} + 1 \right) [e_0,e_1]$$

relates to closed superstring amplitude

• there is a natural homomorphism:

$$\mathrm{sv}: \ \mathcal{P}^a \xrightarrow{\mathrm{sv}^m} \mathcal{P}^m \xrightarrow{\mathrm{per}} \mathbf{C}$$

unipotent de Rahm MZV's $\zeta^a \in \mathcal{A}$ (Goncharov)

motivic MZV's $\zeta^m \in \mathcal{H}$ (Brown)

F. Brown (2013):

$$\mathrm{sv}: \zeta_{n_1,\ldots,n_r} \longrightarrow \zeta_{\mathrm{sv}}(n_1,\ldots,n_r)$$

$$\begin{aligned}
\zeta_{\mathrm{sv}}(2) &= 0 \\
\zeta_{\mathrm{sv}}(2n+1) &= 2\,\zeta_{2n+1} \\
\zeta_{\mathrm{sv}}(3,5) &= -10\,\zeta_3\,\zeta_5
\end{aligned}$$

5. Real iterated integrals vs. complex integrals

Recall: we considered the real iterated integral $\pi, \rho, \overline{\rho} \in S_{N-3}$

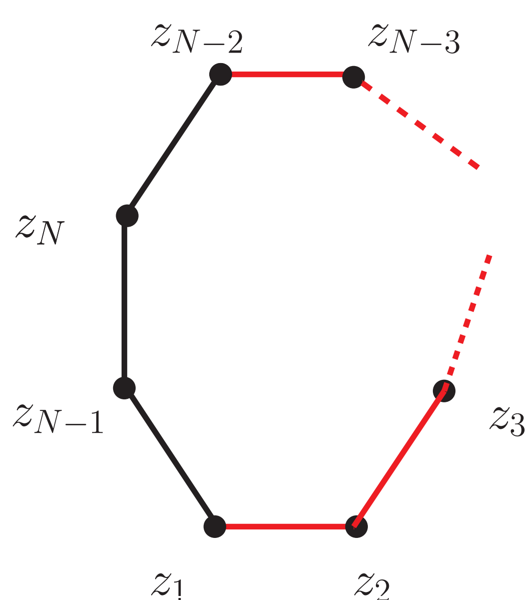
$$Z_\pi(\rho) := \int_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\ z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

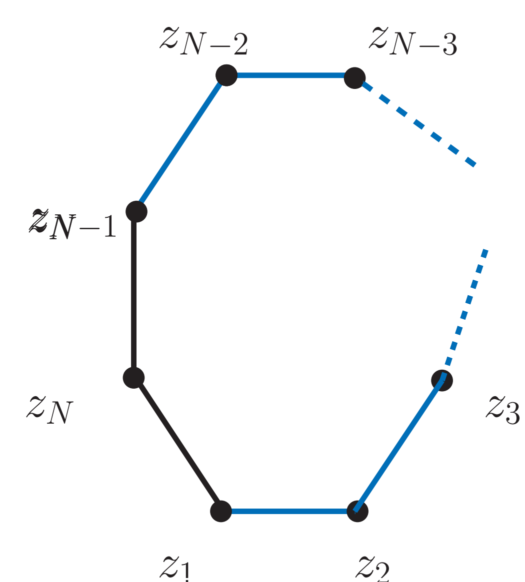
$$D(\pi) = \{\ z_j \in \mathbf{R} \mid 0 < z_{\pi(2)} < \ldots < z_{\pi(N-2)} < 1 \} \subset \left(\mathbf{RP}^1 \backslash \{0, 1, \infty\} \right)^{N-3}$$

In addition we consider the complex integral:

$$J(\rho, \overline{\rho}) := \int_{\mathbf{C}^{N-3}} \left(\prod_{j=2}^{N-2} d^2 z_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\ z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}} \frac{1}{\overline{z}_{1,\overline{\rho}(2)}\ \overline{z}_{\overline{\rho}(2),\overline{\rho}(3)} \cdots \overline{z}_{\overline{\rho}(N-2),N-1}}$$

$$J = \text{sv}\left(Z\right)$$





No KLT relations necessary !

KLT:

$$J(\rho,\overline{\rho}) = Z_\rho(\sigma)\ S[\sigma|\tau]\ Z_\tau(\overline{\rho})$$

$$= \mathrm{sv}(Z_\rho(\tau))$$

e.g. N=4:

$$\int_{\mathbf{C}} d^2z\ \frac{|z|^{2s}\ |1-z|^{2u}}{z\ (1-z)\ \overline{z}} = \sin(\pi u)\ \left(\int_0^1 x^{s-1}\ (1-x)^{u-1}\right)\ \left(\int_1^\infty x^{t-1}\ (1-x)^{u}\right)$$

e.g. N=4:

$$\int_{\mathbf{C}} d^2z \, \frac{|z|^{2s} \, |1-z|^{2u}}{z\,(1-z)\,\overline{z}} = \mathrm{sv}\left(\int_0^1 dx\, x^{s-1}\,(1-x)^u\right)$$

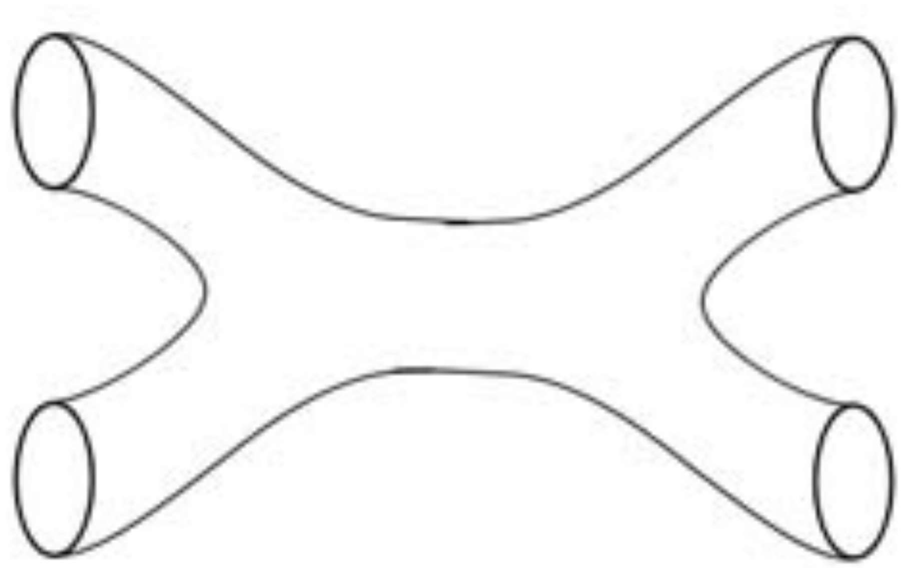
$$\frac{1}{s}\,\frac{\Gamma(s)\,\Gamma(u)\,\Gamma(t)}{\Gamma(-s)\,\Gamma(-u)\,\Gamma(-t)} = \mathrm{sv}\left(\frac{\Gamma(s)\,\Gamma(1+u)}{\Gamma(1+s+u)}\right)$$

$$s+t+u=0$$

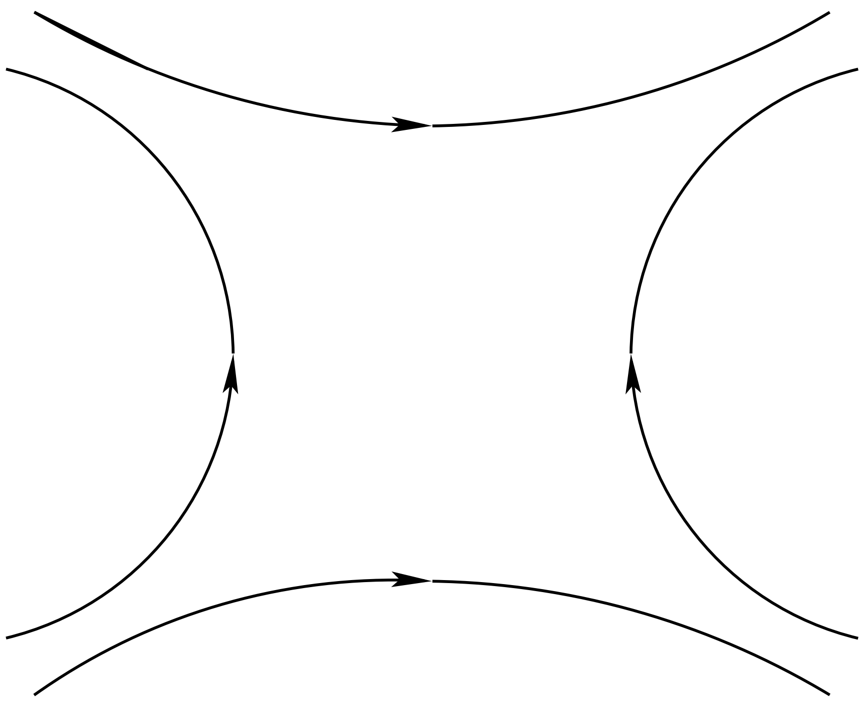
$$s=\alpha'(k_1+k_2)^2$$
$$t=\alpha'(k_1+k_3)^2$$
$$u=\alpha'(k_1+k_4)^2$$

this is just Drinfeld vs. Deligne associator !

Physics:



$= \mathrm{sv}$



Complex vs. iterated integrals

N=5:

$$\begin{pmatrix} \displaystyle\int\limits_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{12}z_{23}\ \overline{z}_{12}\overline{z}_{23}\overline{z}_{34}} & \displaystyle\int\limits_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{13}z_{32}\ \overline{z}_{12}\overline{z}_{23}\overline{z}_{34}} \\ \displaystyle\int\limits_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{12}z_{23}\ \overline{z}_{13}\overline{z}_{32}\overline{z}_{24}} & \displaystyle\int\limits_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{13}z_{32}\ \overline{z}_{13}\overline{z}_{32}\overline{z}_{24}} \end{pmatrix}$$

$$= \mathrm{sv}\begin{pmatrix} \displaystyle\int\limits_{0<z_2<z_3<1} dz_2\ dz_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{12}z_{23}} & \displaystyle\int\limits_{0<z_2<z_3<1} dz_2\ dz_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{13}z_{32}} \\ \displaystyle\int\limits_{0<z_3<z_2<1} dz_2\ dz_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{12}z_{23}} & \displaystyle\int\limits_{0<z_3<z_2<1} dz_2\ dz_3\ \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{13}z_{32}} \end{pmatrix}$$

6. Mathematical concepts from/in string amplitudes

α'- expansion

open, closed superstring amplitude

↔

decomposition of motivic MZVs

Drinfeld, Deligne associators

many **concepts** and **structures** of string amplitudes share/follow from **mathematical aspects** (motives, symbols, coproduct, Lie algebra structure, …)

⇒ *Obtain amplitudes* ***from first principles***
(methods residing in arithmetic algebraic geometry and number theory)

based on:

- O. Schlotterer, S. Stieberger: **Motivic multiple zeta values and superstring amplitudes,** J.Phys. A46 (2013) 475401, [arXiv:1205.1516]

- St.St.: **Closed superstring amplitudes, single-valued multiple zeta values and the Deligne associator,** J. Phys. A47 (2014) 155401, [arXiv:1310.3259]

- St.St., T.R. Taylor: **Closed string amplitudes as single-valued open string amplitudes,** Nucl. Phys. B881 (2014) 269-287, [arXiv:1401.1218]

F. Brown (2011) $\xi \in \mathcal{H}_{N+1}$

$$\phi(\xi) = \sum_{3 \le 2r+1 \le N} f_{2r+1}\ \xi_{2r+1} + c\ f_{N+1} \in \mathcal{U}_{N+1}$$

expansion w.r.t. basis $\{f_{2r+1}\}$

coefficients $\xi_{2r+1} \in \mathcal{U}_{N-2r}$

computed from coproduct

$\xi_{2r+1} \in \mathcal{U}_{N-2r}$

$$\xi_{2r+1} = \sum_{p=0}^{N-2r} \underbrace{c^{\phi}_{2r+1}(I^m(a_p; a_{p+1}, \ldots, a_{p+2r+1}; a_{p+2r+2}))}_{\text{coefficient of } f_{2r+1} \text{ in } \mathcal{L} = \frac{A_{>0}}{A_{>0}A_{>0}}} \times \underbrace{\phi(I^m(a_0; a_1, \ldots, a_p, a_{p+2r+2}, \ldots, a_{N+1}; a_{N+2}))}_{\phi - \text{map computed in lower weight } N-2r}$$

<u>E.g.:</u> $\phi(\zeta^m_{4,3}) = f_2^2\ f_3 + 10\ f_2\ f_5 + c\ f_7$

We can do better for a given basis

<u>E.g. weight 10 for basis B</u>

$$B_{10} = \{\ \zeta^m_{3,7},\ \zeta^m_3\ \zeta^m_7,\ (\zeta^m_5)^2,\ \zeta^m_{3,5}\ \zeta^m_2,\ \zeta^m_3\ \zeta^m_5\ \zeta^m_2,\ (\zeta^m_3)^2\ (\zeta^m_2)^2,\ (\zeta^m_2)^5\ \}$$

Compute $\phi(B_{10})$:

$$\phi^B\left(\zeta^m_{3,7}\right) = -14\ f_7 f_3 - 6\ f_5 f_5, \quad \phi^B(\zeta^m_3 \zeta^m_7) = f_3 ⧢ f_7\ ,$$
$$\phi^B\left((\zeta^m_5)^2\right) = f_5 ⧢ f_5, \quad \phi^B\left(\zeta^m_{3,5}\zeta^m_2\right) = -5\ f_5 f_3 f_2\ ,$$
$$\phi^B(\zeta^m_3 \zeta^m_5 \zeta^m_2) = f_3 ⧢ f_5 f_2, \quad \phi^B\left((\zeta^m_3)^2 (\zeta^m_2)^2\right) = f_3 ⧢ f_3 f_2^2\ ,$$
$$\phi^B\left((\zeta^m_2)^5\right) = f_2^5$$