MITP workshop

Amplitudes and Motives

Elliptic multiple zeta values

and superstring one-loop amplitudes

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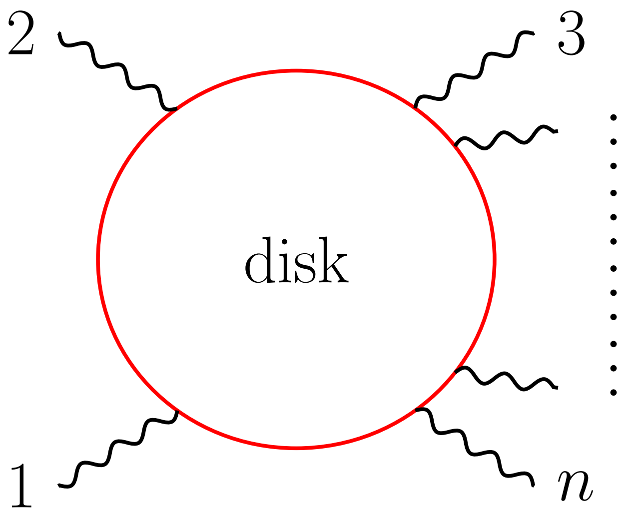
based on arXiv:1412.5535 & 1506.abcde with J. Brödel, C. Mafra, N. Matthes

26.05.2015

Motivation: Open string scattering amplitudes @ tree-level and one-loop

$\Longrightarrow$ iterated integrals over the worldsheet boundaries

genus zero

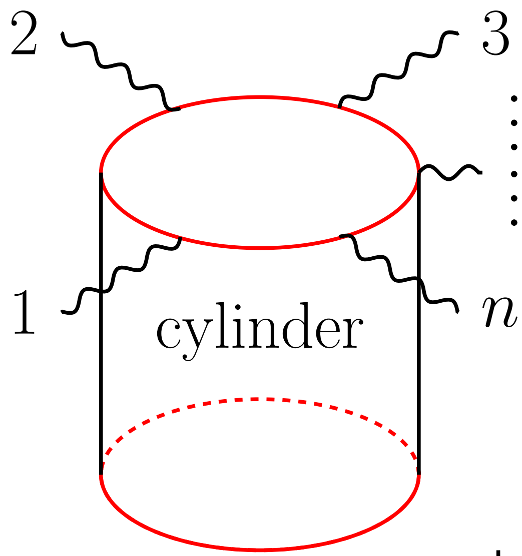


peel off YM trees $\downarrow$ expand in α'

$$\int_{0\leq z_i \leq z_{i+1} \leq 1} \prod \frac{\mathrm{d}z_j}{z_j - n_j} = \text{MZVs}$$

[see Stephan's talk]

genus one



half a torus of modular parameter τ

$$\frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}$$

$\downarrow$

$$\int_{0\leq z_i \leq z_{i+1} \leq 1} \prod f^{(n_j)}(z_j)\mathrm{d}z_j = \text{eMZVs}$$

[this afternoon's talks]

Outline

I. Definition of eMZVs and $f^{(n)}$

II. eMZV relations & indecomposables

III. Superstring one-loop amplitudes

I. 1 Iterated integrals: genus-zero versus elliptic

Recall that multiple polylogarithms [Goncharov]

$$G(a_1, a_2, \ldots, a_r; z) \equiv \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_r; t) \ , \qquad G(; z) \equiv 1$$

specialize to multiple zeta values (MZVs) upon $z \to 1$ and $a_j \to \{0, 1\}$:

$$\zeta_{\{a_1, a_2, \ldots, a_r\}} \equiv (-1)^{\sum_{j=1}^r a_j} \int_{0 \leq z_i \leq z_{i+1} \leq 1} \frac{\mathrm{d}z_1}{z_1 - a_1} \frac{\mathrm{d}z_2}{z_2 - a_2} \ldots \frac{\mathrm{d}z_r}{z_r - a_r}$$

Both are said to have weight $r \ \equiv$ number of integrations.

I. 1 Iterated integrals: genus-zero versus elliptic

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Both are said to have weight r $\equiv$ number of integrations.

Can recursively bypass clashes $a_j=z$ via partial fraction:

$$G(a,0,z;z) = G(0,0,a;z) - G(0,a,a;z) - \zeta_2 G(a;z)$$

resting on: $\displaystyle \frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$

I. 1 Iterated integrals: genus-zero versus elliptic

Elliptic iterated integrals with suitable $f^{(n)}$ **[Brown, Levin]**

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix}; z\right) \equiv \int_0^z \mathrm{d}t\ f^{(n_1)}(t-a_1)\ \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix}; t\right) \ , \qquad \Gamma(;z) \equiv 1$$

specialize to elliptic multiple zeta values (eMZVs) upon $z \to 1$ and $a_j \to 0$:

$$\omega(n_1, n_2, \ldots, n_r) \equiv \int_{0 \leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1)\, \mathrm{d}z_1\ f^{(n_2)}(z_2)\, \mathrm{d}z_2\ \cdots\ f^{(n_r)}(z_r)\, \mathrm{d}z_r \,.$$

Both are said to have length r and weight $\sum_{j=1}^r n_j$. **[Enriquez 1301.3042]**

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specialize to elliptic multiple zeta values (eMZVs) upon $z\to 1$ and $a_j \to 0$:

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Both are said to have length r and weight $\sum_{j=1}^r n_j$. **[Enriquez 1301.3042]**

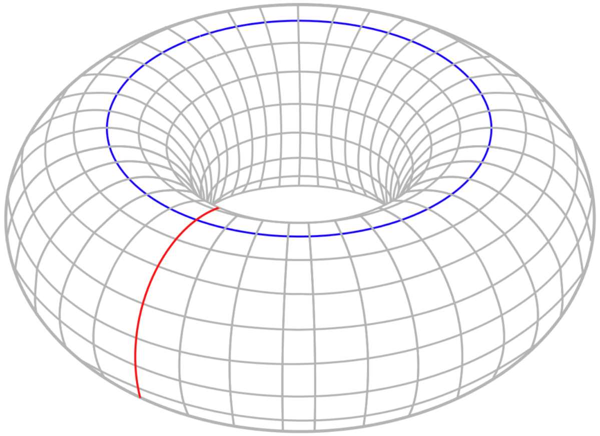
Can recursively bypass clashes $a_j = z$ via Fay relations ($C_{p,q}\in\mathbb{Z}$):

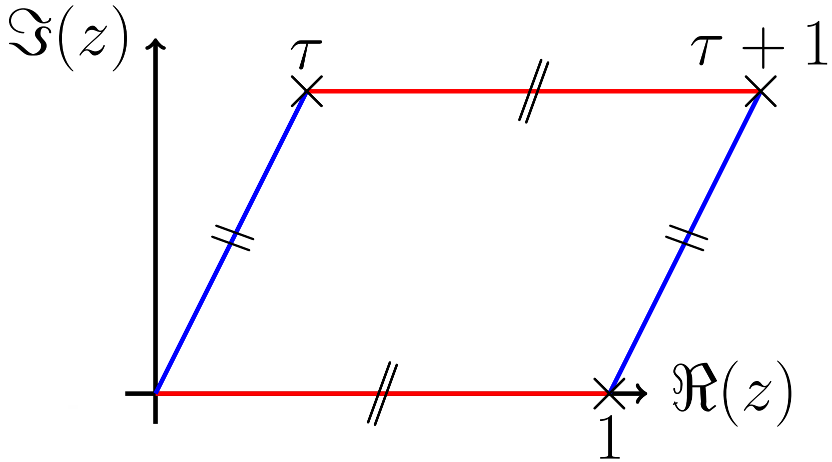
$$\Gamma\left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix};z\right) = 2\Gamma\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix};z\right) + \Gamma\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix};z\right) - 2\Gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix};z\right) + \zeta_2$$

from $$f^{(m)}(z-a)f^{(n)}(z-b) = \sum_{p+q=m+n} C_{p,q} f^{(p)}(z-a)f^{(q)}(a-b) + \begin{smallmatrix}(m,a)\\ \updownarrow \\ (n,b)\end{smallmatrix}$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Parametrization of elliptic curve ≡ torus





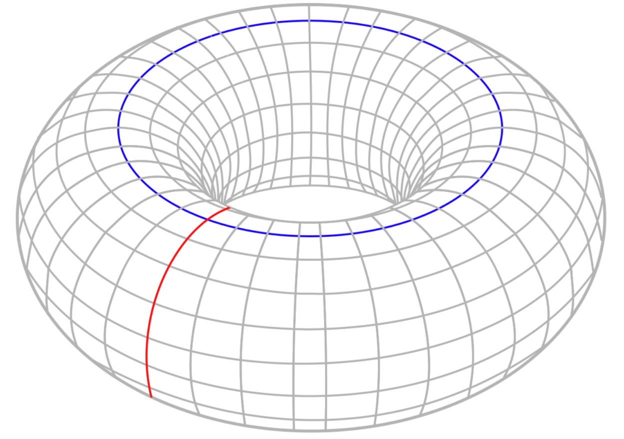
Jacobi θ-function takes role of the identity map on the torus

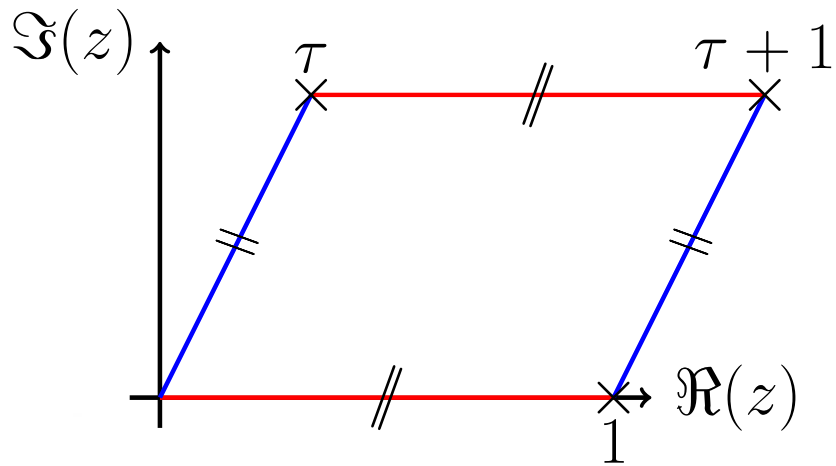
$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1-e^{2\pi i(n\tau+z)})(1-e^{2\pi i(n\tau-z)})$$

$$\text{uplift} \quad \frac{1}{z} \;\to\; \partial_z \ln \theta(z,\tau) + \; 2\pi i \, \frac{\Im(z)}{\Im(\tau)} \;\equiv f^{(1)}(z,\tau)$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Parametrization of elliptic curve $\equiv$ torus





Jacobi θ-function takes role of the identity map on the torus

$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1 - e^{2\pi i (n\tau + z)})(1 - e^{2\pi i (n\tau - z)})$$

poles like both $\frac{1}{z}$ and $\frac{1}{1-z}$

$$\text{uplift } \frac{1}{z} \to \partial_z \ln \theta(z,\tau) + \underbrace{2\pi i \frac{\Im(z)}{\Im(\tau)}} \equiv \overbrace{f^{(1)}(z,\tau) = f^{(1)}(z+1,\tau)}$$

sacrifice holomorphicity for $f^{(1)}(z,\tau) = f^{(1)}(z+\tau,\tau)$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

To go beyond

$$\text{uplift} \ \ \frac{1}{z} \ \rightarrow \ f^{(1)}(z,\tau) \equiv \partial_z \ln \theta(z,\tau) + 2\pi i \, \frac{\Im(z)}{\Im(\tau)} \ ,$$

check partial fraction $\Rightarrow$ new function $f^{(2)}$ without pole along with $f^{(0)} \equiv 1$

$$\frac{1}{(z-a)(z-b)} + \mathrm{cyc}(z,a,b) = 0$$

$$f^{(1)}(z-a) f^{(1)}(z-b) + \mathrm{cyc}(z,a,b) = f^{(2)}(z-a) + \mathrm{cyc}(z,a,b)$$

Drop the second argument τ of θ and $f^{(n)}$ henceforth.

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

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Iterating this yields higher functions $f^{(3)}, f^{(4)}, \ldots$, e.g.

$$f^{(1)}(z-a) f^{(2)}(z-b) = f^{(1)}(a-b) f^{(2)}(z-a) + f^{(1)}(z-a) f^{(2)}(a-b)$$
$$- f^{(1)}(a-b) f^{(2)}(z-b) + f^{(3)}(z-a) + 2 f^{(3)}(z-b) + f^{(3)}(a-b)$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Generating fct.: non-holomorphic version of Kronecker-Eisenstein series

$$\Omega(z,\alpha) \equiv \exp\left(2\pi i\alpha\,\frac{\Im(z)}{\Im(\tau)}\right)\frac{\theta'(0)\theta(z+\alpha)}{\theta(z)\theta(\alpha)} = \sum_{n=0}^{\infty}\alpha^{n-1}f^{(n)}(z)$$

[Kronecker, Brown, Levin]

- double-periodic $\Omega(z,\alpha)=\Omega(z+\tau,\alpha)=\Omega(z+1,\alpha)$ thanks to $\frac{\Im(z)}{\Im(\tau)}$
- $f^{(n)}(z) \equiv$ polynomial in $\partial^k \ln\theta(z)$, $\frac{\Im(z)}{\Im(\tau)}$ & rational in $\theta^{2n+1}(0)$, e.g.

$$f^{(2)}(z) \equiv \frac{1}{2}\left\{\left(\partial\ln\theta(z)+2\pi i\,\frac{\Im(z)}{\Im(\tau)}\right)^2+\partial^2\ln\theta(z)-\frac{\theta'''(0)}{\theta'(0)}\right\}$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

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- $\mathrm{Res}_{z=0}\Omega(z,\alpha) = 1 \;\Rightarrow$ no other poles than $f^{(1)}(z) \sim \frac{1}{z}$

- Fay rel's: $\Omega(z_1,\alpha_1)\Omega(z_2,\alpha_2) = \Omega(z_1,\alpha_1+\alpha_2)\Omega(z_2-z_1,\alpha_2) + (1 \leftrightarrow 2)$

II. 1 eMZV relations and indecomposables

MZV relations over $\mathbb{Q}$ leave the following indecomposables at weight w:

w	0	1	2	3	4	5	6	7	8	9	10	11	12
indec. MZVs	1	$\emptyset$	ζ_2	ζ_3	$\emptyset$	ζ_5	$\emptyset$	ζ_7	$\zeta_{3,5}$	ζ_9	$\zeta_{3,7}$	ζ_{11}, $\zeta_{3,3,5}$	$\zeta_{3,9}$, $\zeta_{1,1,4,6}$

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MZVs satisfy shuffle and stuffle relations, and eMZVs obey

- shuffle $\omega(n_1,\ldots,n_r)\omega(k_1,\ldots,k_s) = \omega\big((n_1,\ldots,n_r) ⧢ (k_1,\ldots,k_s)\big)$
- reflection: $\omega(n_1,n_2,\ldots,n_r) = (-1)^{n_1+n_2+\ldots+n_r}\omega(n_r,\ldots,n_2,n_1)$
- Fay rel's $\omega(n_1,n_2,\ldots) \leftrightarrow \omega(n_1{+}j,n_2{-}j,\ldots)$ such as $\omega(0,5)=\omega(2,3)$

all of which preserve the weight $\sum_{j=1}^r n_j$ of $f^{(n_j)}$ integrands.

$\longrightarrow$ Which eMZVs remain indecomposable w.r.t. $\mathbb{Q}[\text{MZV}]$?

$f^{(0)} \Rightarrow \exists\, \infty$ many eMZVs per weight $\Rightarrow$ organize relations by length r:

length $r=1$: only constant eMZVs

$$\omega(n) = \begin{cases} -2\,\zeta_n & : \; n \text{ even} \\ 0 & : \; n \text{ odd} \end{cases}$$

length $r=2$: shuffle and reflection reduce even-weight eMZVs to $r=1$:

$$\omega(n_1,n_2)\,\Big|_{n_1+n_2 \text{ even}} = \begin{cases} 2\,\zeta_{n_1}\,\zeta_{n_2} & : \; n_1, n_2 \text{ both even} \\ 0 & : \; n_1, n_2 \text{ both odd} \end{cases}$$

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Simplest τ-dependence in eMZVs $\omega(n_1,n_2)$ of odd weight n_1+n_2, e.g.

$$\omega(0,k) = \delta_{k,1}\frac{i\pi}{2} + \frac{2(-1)^{(k+1)/2}(2\pi)^{k-1}}{(k-1)!}\sum_{m,n=1}^{\infty}\frac{m^{k-1}}{n}(e^{2\pi i\tau})^{mn}\,, \qquad k \text{ odd}$$

any length r: eMZVs with $(-1)^r = (-1)^{\sum_{j=1}^r n_j}$ reducible to length $\leq r-1$

II. 2 Fay relations among eMZVs

Fay relation for $\Omega(z_1, \alpha_1)\Omega(z_2, \alpha_2)$ descends to

$$f^{(n_1)}(z-a)f^{(n_2)}(z-b) = \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} f^{(n_2-j)}(a-b)f^{(n_1+j)}(z-a)$$

$$-(-1)^{n_1} f^{(n_1+n_2)}(a-b) + \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} f^{(n_1-j)}(b-a)f^{(n_2+j)}(z-b)$$

Only one factor of $f^{(m)}(z-\cdot)$ on RHS $\Rightarrow \int \mathrm{d}z$ (LHS) $= \Gamma\left(\begin{smallmatrix} n_1 & \dots & n_r \\ a_1 & \dots & a_r \end{smallmatrix} ; z \right)$.

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$$-(-1)^{n_1}f^{(n_1+n_2)}(a-b)+\sum_{j=0}^{n_1}\binom{n_2-1+j}{j}f^{(n_1-j)}(b-a)f^{(n_2+j)}(z-b)$$

Only one factor of $f^{(m)}(z-\cdot)$ on RHS $\Rightarrow \int \mathrm{d}z$ (LHS) $=\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_r \\ a_1 & \dots & a_r\end{smallmatrix};z\right)$.

Apply to recursively remove $a_j=z$ from elliptic iterated integrals

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ z & 0 & \dots & 0\end{smallmatrix};z\right)=-(-1)^{n_1}\Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 & n_3 & \dots & n_r \\ 0 & 0 & 0 & \dots & 0\end{smallmatrix};z\right)+\overbrace{(-1)^r\zeta_r\delta_{n_1,1}\delta_{n_2,1}\dots\delta_{n_r,1}}^{\text{MZVs from } z\to 0 \text{ limit of LHS}}$$
$$+\sum_{j=0}^{n_1}(-1)^{n_1+j}\binom{n_2-j+1}{j}\Gamma\left(\begin{smallmatrix} n_1-j & n_2+j & n_3 & \dots & n_r \\ 0 & 0 & 0 & \dots & 0\end{smallmatrix};z\right)$$
$$+\sum_{j=0}^{n_2}\binom{n_1-j+1}{j}\int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\underbrace{\Gamma\left(\begin{smallmatrix} n_1+j & n_3 & \dots & n_r \\ t & 0 & \dots & 0\end{smallmatrix};t\right)}_{\text{lower length compared to LHS}}$$

Now convert $\Gamma(\dots;z)$-relation to eMZV-relation, e.g. start from

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 \\ z & 0 \end{smallmatrix};z\right) = -(-1)^{n_1}\Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 \\ 0 & 0 \end{smallmatrix};z\right) + \sum_{j=0}^{n_1}(-1)^{n_1+j}\binom{n_2-j+1}{j}\Gamma\left(\begin{smallmatrix} n_1-j & n_2+j \\ 0 & 0 \end{smallmatrix};z\right)$$
$$+\sum_{j=0}^{n_2}(-1)^{n_1+j}\binom{n_1-j+1}{j}\Gamma\left(\begin{smallmatrix} n_2-j & n_1+j \\ 0 & 0 \end{smallmatrix};z\right) + \zeta_2\delta_{n_1,1}\delta_{n_2,1} \, .$$

In the limit $z \to 1$, RHS directly yields $\Gamma\left(\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix};z\right) \to \omega(q,p)$.

On the left hand side, apply periodicity $f^{(n)}(z) = f^{(n)}(z+1)$:

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 \\ z & 0 \end{smallmatrix};z\right) \to \Gamma\left(\begin{smallmatrix} n_1 & n_2 \\ 1 & 0 \end{smallmatrix};1\right) = \Gamma\left(\begin{smallmatrix} n_1 & n_2 \\ 0 & 0 \end{smallmatrix};1\right) = \omega(n_2,n_1) \, , \qquad n_1 \text{ or } n_2 \neq 1$$

Results in Fay-relations among eMZVs at length two:

$$\omega(n_2,n_1) = -(-1)^{n_1}\omega(0,n_1+n_2) + \sum_{j=0}^{n_1}(-1)^{n_1+j}\binom{n_2-j+1}{j}\omega(n_2+j,n_1-j)$$
$$+\sum_{j=0}^{n_2}(-1)^{n_1+j}\binom{n_1-j+1}{j}\omega(n_1+j,n_2-j) + \zeta_2\delta_{n_1,1}\delta_{n_2,1} \, , \qquad n_1 \text{ or } n_2 \neq 1$$

Same principle at higher length $r \;\Rightarrow$ all MZV admixtures

II. 3 Combining all eMZV relations

length $r=2$: odd-weight $\omega(n_1,n_2)$ can be reduced to $\omega(0,2p-1)$:

$$\omega(n_1,n_2)\Big|_{n_1+n_2 \text{ odd}} = (-1)^{n_1}\omega(0,n_1+n_2) + 2\delta_{n_1,1}\zeta_{n_2}\omega(0,1) - 2\delta_{n_2,1}\zeta_{n_1}\omega(0,1)$$

$$+2\left\{\sum_{p=1}^{\lceil\frac{1}{2}(n_2-3)\rceil}\binom{n_1+n_2-2p-2}{n_1-1}\zeta_{n_1+n_2-2p-1}\omega(0,2p+1) - (n_1\leftrightarrow n_2)\right\}$$

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$$+2\left\{\sum_{p=1}^{\lceil\frac{1}{2}(n_2-3)\rceil}\binom{n_1+n_2-2p-2}{n_1-1}\zeta_{n_1+n_2-2p-1}\omega(0,2p+1)-(n_1\leftrightarrow n_2)\right\}$$

length $r=3$: indecomposable $\omega(n_1,n_2,n_3)$ only at even $n_1+n_2+n_3$:

w	2	4	6	8	10	12	14
indec. eMZVs at $r=3$	$\omega(0,0,2)$	$\omega(0,0,4)$	$\omega(0,0,6)$	$\omega(0,0,8)$ $\omega(0,3,5)$	$\omega(0,0,10)$ $\omega(0,3,7)$	$\omega(0,0,12)$ $\omega(0,3,9)$	$\omega(0,0,14)$ $\omega(0,3,11)$ $\omega(0,5,9)$

$\Rightarrow$ strong evidence that $\exists\ \lceil\frac{w}{6}\rceil$ such indecomposables at even weight w

length $r = 4$: More and more odd-weight indecomposable eMZVs...

w	1	3	5	7	9
indec. eMZVs at $r = 4$	$\omega(0,0,1,0)$	$\omega(0,0,0,3)$	$\omega(0,0,0,5)$, $\omega(0,0,2,3)$	$\omega(0,0,0,7)$, $\omega(0,0,2,5)$, $\omega(0,0,4,3)$	$\omega(0,0,0,9)$, $\omega(0,0,4,5)$, $\omega(0,0,2,7)$, $\omega(0,1,3,5)$

length $r = 5$: More and more even-weight indecomposable eMZVs...

w	2	4	6	8
indec. eMZVs at $r = 5$	$\omega(0,0,0,0,2)$	$\omega(0,0,0,0,4)$, $\omega(0,0,0,1,3)$	$\omega(0,0,0,0,6)$, $\omega(0,0,0,1,5)$, $\omega(0,0,0,2,4)$, $\omega(0,0,2,2,2)$	$\omega(0,0,0,0,8)$, $\omega(0,0,0,1,7)$, $\omega(0,0,1,2,5)$, $\omega(0,0,2,2,4)$, $\omega(0,0,0,2,6)$

$\exists$ algebraic method to predict these numbers **[Johannes' talk]**

III. 1 Four-point one-loop superstring amplitude

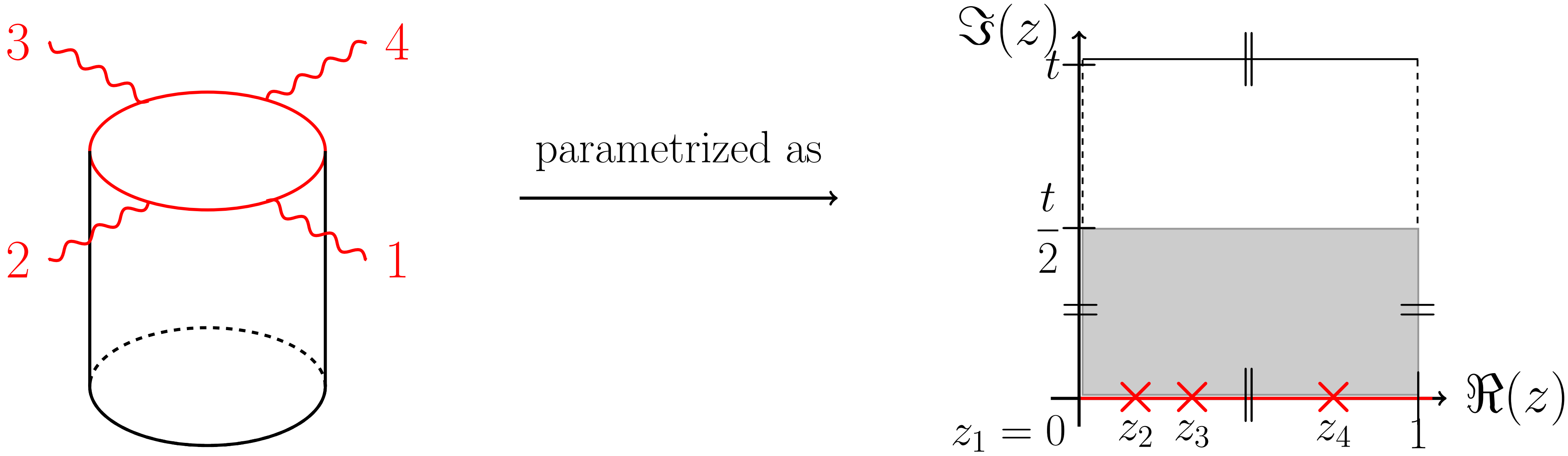
Study the planar cylinder diagram:

$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int_0^\infty \mathrm{d}t\ I_{1234}(s_{ij},\tau=it)$$

$$I_{1234}(s_{ij},\tau) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2\ \exp\Big(\sum_{i<j}^{4} s_{ij} P(z_i - z_j,\tau)\Big)\Big|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

with dim'less $s_{ij} \equiv \alpha'(k_i+k_j)^2$ and worldsheet propagator $\partial P = f^{(1)}$.



Analytic α'-dependence from expanding the exponentials

$$I_{1234}(s_{ij},\tau) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \prod_{i<j}^{4} \sum_{n_{ij}=0}^{\infty} \frac{\big[s_{ij} P(z_i - z_j,\tau)\big]^{n_{ij}}}{n_{ij}!}\Big|_{z_1=0}$$

and integrating $P(z_i - z_j) = \int_{z_j}^{z_i} \mathrm{d}x \; f^{(1)}(x - z_j)$ order by order in α'.

Each monomial in s_{ij} is accompanied by eMZVs, e.g.

$$s_{ij}^0 \;\leftrightarrow\; \int_0^1 \mathrm{d}z_4 \; f^{(0)}(z_4) \int_0^{z_4} \mathrm{d}z_3 \; f^{(0)}(z_3) \int_0^{z_3} \mathrm{d}z_2 \; f^{(0)}(z_2) = \omega(0,0,0)$$

after formally inserting $f^{(0)} = 1$ as well as

$$\left.\begin{matrix} s_{12} \\ \\ s_{13} \end{matrix}\right\} \leftrightarrow \int_0^1 \mathrm{d}z_4 \; f^{(0)} \int_0^{z_4} \mathrm{d}z_3 \; f^{(0)} \int_0^{z_3} \mathrm{d}z_2 \; f^{(0)} \left\{\begin{matrix} \int_0^{z_2} \mathrm{d}x \; f^{(1)}(x) \\ \\ \int_0^{z_3} \mathrm{d}x \; f^{(1)}(x) \end{matrix}\right.$$

$$\Longrightarrow \quad s_{12} \;\leftrightarrow\; \omega(1,0,0,0)\,, \qquad s_{13} \;\leftrightarrow\; \underbrace{\omega(1,0,0,0)}_{\text{from } 0\leq x\leq z_2} + \underbrace{\omega(0,1,0,0)}_{\text{from } z_2\leq x\leq z_3}$$

At higher order ...

$$s_{12}s_{23} \leftrightarrow \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \left(\int_{z_3}^{z_2} \mathrm{d}x\ f^{(1)}(x-z_3)\right)\left(\int_0^{z_2} \mathrm{d}y\ f^{(1)}(y)\right)$$

$$= -\int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3\, \Gamma\left(\begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix}; z_3\right)$$

... need Fay relations

$$\Gamma\left(\begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix}; z_3\right) = 2\Gamma\left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \end{smallmatrix}; z_3\right) + \Gamma\left(\begin{smallmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}; z_3\right) - 2\Gamma\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; z_3\right) + \zeta_2\Gamma\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; z_3\right) .$$

Above example then integrates to

$$s_{12}s_{23} \leftrightarrow -2\omega(2,0,0,0,0) - \omega(0,2,0,0,0) + 2\omega(1,1,0,0,0) - \zeta_2\omega(0,0,0)$$

$$= -\omega(1,0,0,0,1)$$

after using eMZV relations at length five.

After using momentum conservation for s_{ij}, first orders simplify to

$$I_{1234}(s_{ij}) = \omega(0,0,0) - 2\omega(0,1,0,0)\,(s_{12}+s_{23}) + 2\omega(0,1,1,0,0)\left(s_{12}^2+s_{23}^2\right)$$

$$- 2\omega(0,1,0,1,0)\,s_{12}s_{23} + \beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3) + \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23}) + \ldots$$

with shorthands

$$\beta_5 = \frac{4}{3}\left[\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\omega(0,1,0,0)\right]$$

$$\begin{aligned}\beta_{2,3} &= \frac{1}{3}\omega(0,0,1,0,2,0)-\frac{3}{2}\omega(0,1,0,0,0,2)-\frac{1}{2}\omega(0,1,1,1,0,0)\\ &\quad - 2\omega(2,0,1,0,0,0)-\frac{4}{3}\omega(0,0,1,0,0,2)-\frac{10}{3}\zeta_2\omega(0,1,0,0)\,.\end{aligned}$$

Choice of indecomposable eMZVs requires guidance,

in particular at higher α'-order $\leftrightarrow$ weight $\leftrightarrow$ length

[Johannes' talk]

III. 2 Outlook to more external legs

For n external legs,

$$A^{\text{1-loop}}_{\text{string}}(1,2,\ldots,n) = \int_0^\infty \mathrm{d}t \int_{0=z_1\leq z_2\leq \ldots \leq z_n} \mathrm{d}z_2 \ldots \mathrm{d}z_n \, \exp\Big(\sum_{i<j}^{n} s_{ij} P(z_i - z_j, it)\Big)$$

$$\times \sum \Big(\text{monomial in } f^{(k_j)} \text{ @ weight } \sum_j k_j = n-4\Big) \times \Big(\text{kinematic factors}\Big)$$

e.g. for $n = 5$ points, the second line becomes

$$f^{(1)}(z_2 - z_3) \times s_{23}s_{45}\Big(s_{34}A^{\text{tree}}_{\text{YM}}(1,2,3,4,5) - s_{24}A^{\text{tree}}_{\text{YM}}(1,3,2,4,5)\Big)$$

$$+ \;\; 5 \text{ permutations } (23 \leftrightarrow 24, 25, 34, 35, 45)$$

$\Rightarrow$ n-point amplitude naturally compatible with eMZV language !

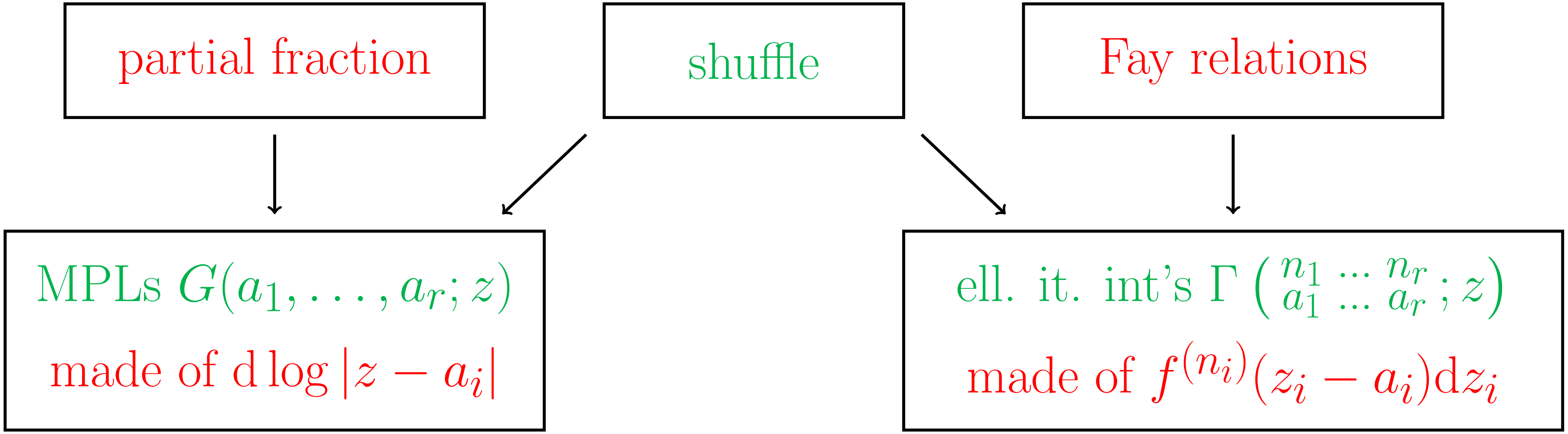
Summary

shuffle

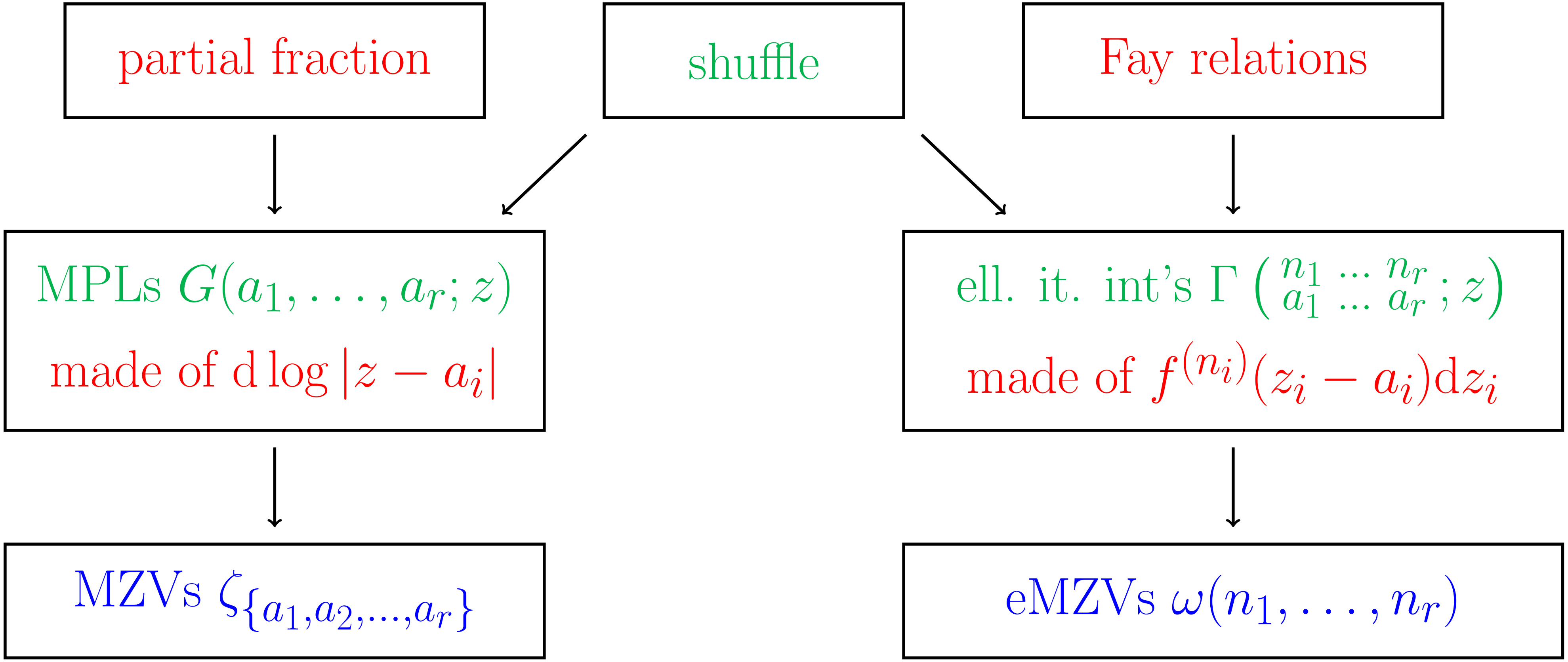
MPLs $G(a_1, \ldots, a_r; z)$

ell. it. int's $\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_r \\ a_1 & \ldots & a_r \end{smallmatrix}; z\right)$

Summary


partial fraction
shuffle
Fay relations
MPLs $G(a_1, \ldots, a_r; z)$
made of $\mathrm{d}\log|z - a_i|$
ell. it. int's $\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_r \\ a_1 & \ldots & a_r \end{smallmatrix}; z\right)$
made of $f^{(n_i)}(z_i - a_i)\mathrm{d}z_i$

Summary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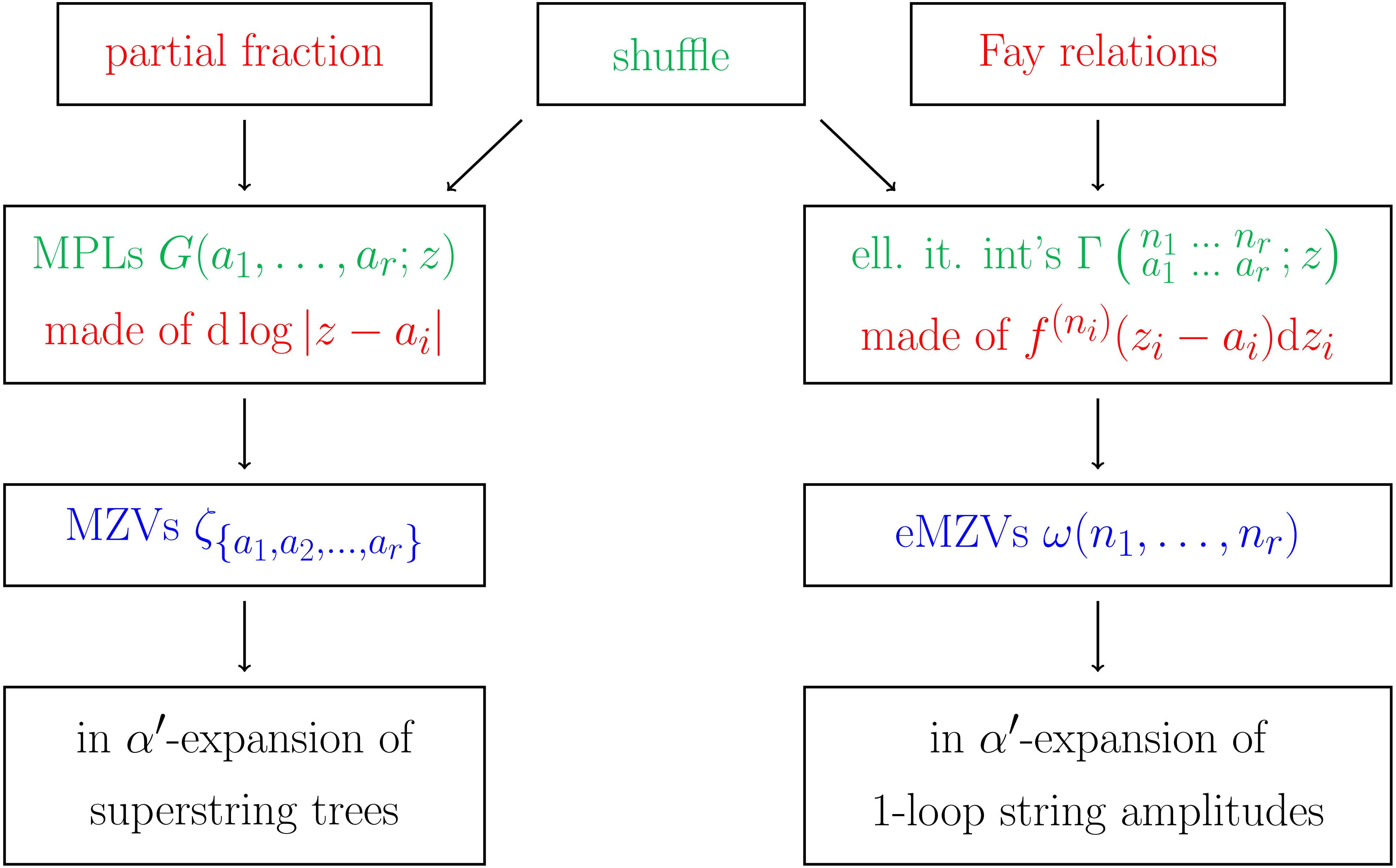
MZVs $\zeta_{\{a_1, a_2, \ldots, a_r\}}$
eMZVs $\omega(n_1, \ldots, n_r)$

Summary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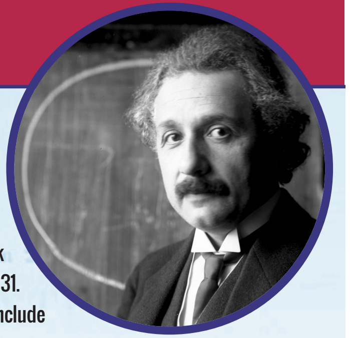
MZVs $\zeta_{\{a_1, a_2, \ldots, a_r\}}$
eMZVs $\omega(n_1, \ldots, n_r)$
in α'-expansion of superstring trees
in α'-expansion of 1-loop string amplitudes

A Century of General Relativity

November 30 - December 2, 2015

Harnack House Berlin



The year 2015 marks the 100th anniversary of Einstein's field equations. To celebrate this event, the **Max Planck Institute for Gravitational Physics** (or Albert Einstein Institute) will host a conference during the week of November 30, 2015, exactly one hundred years after the publication of Einstein's paper. The conference will take place in the recently renovated Harnack House in Berlin, where Albert Einstein regularly lectured between 1915 and 1931. On December 3-5 the **Max Planck Institute for the History of Science** will conclude the celebratory events with a workshop on the history of Einstein's theory.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Speakers:
Eric Adelberger University of Washington, Seattle
Abhay Ashtekar Penn State University, University Park
Zvi Bern University of California, Los Angeles
Thibault Damour IHES, Bures-sur-Yvette
Reinhard Genzel Max Planck Institute for Extraterrestial Physics, Munich
Andrea Ghez University of California, Los Angeles
David Gross Kavli Institute for Theoretical Physics, Santa Barbara
Hanoch Gutfreund Hebrew University, Jerusalem
Ted Jacobson University of Maryland, College Park
Sergiu Klainerman Princeton University, Princeton
Joseph Polchinski Kavli Institute for Theoretical Physics, Santa Barbara
Frans Pretorius Princeton University, Princeton
Harvey Reall DAMTP, Cambridge
David Spergel Princeton University, Princeton
Ingrid Stairs University of British Columbia, Vancouver
Paul Steinhardt Princeton University, Princeton
Rai Weiss Massachusetts Institute of Technology, Cambridge

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