Multiple polylogarithms in cyclotomic fields and subfields

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Single-scale Feynman diagrams evaluate to periods that may (but need not) be multiple polylogarithms with arguments in algebraic number fields, such as N-th roots of unity, with $N = 1, 2, 6$ prominent. This introductory talk gives examples of such evaluations and sketches conjectures in real subfields of cyclotomic fields, to be pursued later.

IA: Multiple **zeta** values (M**Z**Vs) from single-scale diagrams

IB: **Padovan** and **Fibonacci** enumerations at $N = 1, 2, 6$

IC: Multiple **Deligne** values (M**D**Vs) at $N = 6$ in QFT

——————— Advertisements: ———————

IIA: **Tribonacci** at $N = 5$: multiple **Landen** values (M**L**Vs)

IIB: **Super-tribonacci** at $N = 7$: multiple **Watson** values (M**W**Vs)

IIC: **Master conjecture** for real subfields

IA: Multiple zeta values from single-scale diagrams

Single scale: an energy or mass, in comparison with which other physical quantities may be neglected, because they are small or decouple.

Examples:

1. electron-positron annihilation into hadrons; neglect quark masses
2. magnetic moment of the electron; photon is massless
3. counterterms for beta-functions and anomalous dimensions
4. forward or backward scattering in gauge theories
5. on-shell sunrise diagrams...

Period: an integral of a rational function over a real domain defined by rational inequalities.

Examples:

$$\pi = \int\int_{x>0,y>0,x^2+y^2<1} 4\,dx\,dy = 4\int_0^1 \sqrt{1-x^2}\,dx,$$

$$\zeta(3) = \int\int\int_{1>x>y>z>0} \frac{dx}{x}\frac{dy}{y}\frac{dz}{1-z},$$

logarithm of algebraic number, elliptic integral with rational arguments, multiple zeta value... but **not** $1/\pi$, γ, or e, as far as we know.

Like integers, rationals and algebraic numbers, periods are **countable**. **Single-scale** renormalized Feynman diagrams yield periods. Periods form a **ring**: products and sums of periods are periods.

Example of a **massive** single-scale diagram yielding a period that is an integral of 5 **Bessel** functions, the **integral** of an **elliptic integral**, a **product** of elliptic integrals and, by the Chowla-Selberg theorem, a product of $\mathbf{\Gamma}$ values: 3-loop **on-shell sunrise** in 2 spacetime dimensions.

$$\begin{aligned} S_{N+2} &\equiv 2^N \int_0^\infty I_0(t)[K_0(t)]^{N+1} t\,\mathrm{d}t \\ S_5 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca} \\ &= \frac{1}{30\sqrt{5}} \Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \end{aligned}$$

Conjecture: No Feynman period is an elliptic integral: QFT is blind to the Birch–Swinnerton-Dyer conjecture at genus 1.

Counterterms give MZVs at 6 loops, multiple polylogs at 7 loops, and hit genus 2 at 8 loops. In 1995, Dirk Kreimer and I found $\zeta(5,3) - \frac{29}{12}\zeta(8)$ in ϕ^4 theory at 6 loops, where $\zeta(5,3) \equiv \sum_{m>n>0} 1/(m^5 n^3)$ does not reduce to zeta values. At 7 loops, 3 primitives were undetermined. Later I found

$$P_{7,8} = \frac{22383}{20}\zeta(11) + \frac{4572}{5}\left[\zeta(3,5,3) - \zeta(3)\zeta(5,3)\right] - 700\zeta(3)^2\zeta(5)$$
$$+ 1792\zeta(3)\left[\frac{9}{320}\left[12\zeta(5,3) - 29\zeta(8)\right] + \frac{45}{64}\zeta(5)\zeta(3)\right]$$
$$P_{7,9} = \frac{92943}{160}\zeta(11) + \frac{3381}{20}\left[\zeta(3,5,3) - \zeta(3)\zeta(5,3)\right] - \frac{1155}{4}\zeta(3)^2\zeta(5)$$
$$+ 896\zeta(3)\left[\frac{9}{320}\left[12\zeta(5,3) - 29\zeta(8)\right] + \frac{45}{64}\zeta(5)\zeta(3)\right]$$

at 7 loops. That left $P_{7,11}$ in the Schnetz census as the sole 7-loop period not reduced to MZVs. Erik Panzer reduced this to polylogs at **sixth roots of unity** in 2014, as will be recounted later.

Sébastien Leurent and Dmytro Volin have computed the Konishi anomalous dimension of planar $N = 4$ super-Yang-Mills theory up to 8 loop. An MZV first appears at 8 loops, where the weight-11 term

$$\frac{864g^{16}}{5}\left\{76307\zeta(11) + 792[\zeta(3,5,3) - \zeta(3)\zeta(5,3)] - 18840\zeta(3)^2\zeta(5)\right\}$$

is happily reducible to 3 terms found by BK in ϕ^4 theory in 1995.

IB: Padovan and Fibonacci enumerations at $N = 1, 2, 6$

7 letter alphabet: let $\lambda = \exp(2\pi i/6) = (1 + i\sqrt{3})/2$, $\overline{\lambda} = (1 - i\sqrt{3})/2$,

$$\begin{aligned}
A &= d\log(x) \\
B &= -d\log(1-x) \\
C &= -d\log(1+x) \\
D &= -d\log(1-\lambda x) \\
\overline{D} &= -d\log(1-\overline{\lambda} x) \\
E &= -d\log(1-\lambda^2 x) \\
\overline{E} &= -d\log(1-\overline{\lambda}^2 x)
\end{aligned}$$

investigated in `arXiv:hep-th/9803091` and `arXiv:1409.7204`
Subalphabets:
$\{A, B, C\}$, alternating sums, `arXiv:hep-th/9604128`
$\{A, B\}$, Multiple Zeta Values (MZVs), BK, `arXiv:hep-th/9609128`
$\{A, D\}$, Multiple Clausen Values (MCVs), BBK, `arXiv:hep-th/0004153`
$\{A, B, D\}$, Multiple Deligne Values (MDVs) follow MCVs
$\{A, B, E, \overline{E}\}$, solved by Pierre Deligne in
`http://www.math.ias.edu/files/deligne/121108Fondamental.pdf`

Weight, w, is the number of letters in a word.
Depth, d, is the number of letters not equal to A.
Iterated integrals: For example, at weight $w = 3$ and depth $d = 2$,

$$Z(DAB) \equiv \int_0^1 \frac{\lambda\, dx_1}{1-\lambda x_1} \int_0^{x_1} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_3}{1-x_3}$$

Nested sums: expand

$$-d\log(1-\lambda^n x) = \frac{dx}{x} \sum_{k>0} (\lambda^n x)^k$$

to obtained nested sums of the form

$$S\begin{pmatrix} z_1, z_2, \ldots, z_d \\ a_1, a_2, \ldots, a_d \end{pmatrix} \equiv \sum_{k_1>k_2>\ldots>k_d>0} \prod_{j=1}^{d} \frac{z_j^{k_j}}{k_j^{a_j}}$$

where $z_j^6 = 1$ and a_j is a positive integer. Thus, for example,

$$Z(DAB) = S\begin{pmatrix} \lambda, \overline{\lambda} \\ 1, 2 \end{pmatrix} \equiv \sum_{m=1}^{\infty} \frac{\lambda^m}{m} \sum_{n=1}^{m-1} \frac{\overline{\lambda}^n}{n^2}$$

Shuffle product:

$$Z(U)Z(V) = \sum_{W\in\mathcal{S}(U,V)} Z(W)$$

where $\mathcal{S}(U,V)$ is the set of all words W that result from shuffling the words U and V. Thus, for example,

$$\begin{aligned} Z(AB)Z(CD) &= Z(ABCD) + Z(ACBD) + Z(ACDB) \\ &+ Z(CABD) + Z(CADB) + Z(CDAB) \end{aligned}$$

preserves the order of letters in $U = AB$ and $V = CD$.

Stuffle product: the **full** 7-letter alphabet $\{A, B, C, D, \overline{D}, E, \overline{E}\}$ has a stuffle algebra, resulting from shuffling the arguments of nested sums, with extra **stuff** when indices of summation coincide. For example

$$\begin{aligned} Z(AB)Z(D) &= S\binom{1}{2}S\binom{\lambda}{1} = S\binom{1,\lambda}{2,1} + S\binom{\lambda,1}{1,2} + S\binom{\lambda}{3} \\ &= Z(ABD) + Z(DAD) + Z(AAD) \end{aligned}$$

Alphabets $\{A, B\}$, $\{A, B, C\}$, $\{A, B, E, \overline{E}\}$ $\{A, B, C, D, \overline{D}, E, \overline{E}\}$ have a **double shuffle** algebra, but the Deligne alphabet $\{A, B, D\}$ is **not** closed under stuffles: $Z(AD)Z(D) = Z(ADE) + Z(DAE) + Z(AAE)$.

Enumerations of primitives by weight and depth

A word W is a **primitive** in a **given alphabet** if $Z(W)$ can **not** be expressed as a **Q**-linear combination of terms of **lesser** depth, powers of $(2\pi i)$, or their products.

Example [Broadhurst, 1996]: At weight $w = 12$ and depth $d = 4$,

$$Z(A^3BA^3BABAB) = \zeta(4,4,2,2) = \sum_{j>k>l>m>0} 1/(j^4k^4l^2m^2)$$

is a primitive MZV, but is **not** primitive in the $\{A, B, C\}$ alphabet, because

$$\begin{aligned}
&2^5 \cdot 3^3 Z(A^3BA^3BABAB) - 2^{14} Z(A^8CA^2B) = \\
&\quad 2^5 \cdot 3^2\, \zeta^4(3) + 2^6 \cdot 3^3 \cdot 5 \cdot 13\, \zeta(9)\,\zeta(3) + 2^6 \cdot 3^3 \cdot 7 \cdot 13\, \zeta(7)\,\zeta(5) \\
&+ 2^7 \cdot 3^5\, \zeta(7)\,\zeta(3)\,\zeta(2) + 2^6 \cdot 3^5\, \zeta^2(5)\,\zeta(2) - 2^6 \cdot 3^3 \cdot 5 \cdot 7\, \zeta(5)\,\zeta(4)\,\zeta(3) \\
&- 2^8 \cdot 3^2\, \zeta(6)\,\zeta^2(3) - \frac{13177 \cdot 15991}{691}\, \zeta(12) \\
&+ 2^4 \cdot 3^3 \cdot 5 \cdot 7\, \zeta(6,2)\,\zeta(4) - 2^7 \cdot 3^3\, \zeta(8,2)\,\zeta(2) - 2^6 \cdot 3^2 \cdot 11^2\, \zeta(10,2)
\end{aligned}$$

where $Z(A^8CA^2B) = \sum_{m>n>0} (-1)^{m+n}/(m^9n^3)$ has depth 2.

For a **given** alphabet, let $N_{w,d}$ be the dimension of the space of $\mathbf{Q}$-linearly independent primitives of weight w and depth d. We then seek

$$H(x,y)=\prod_{w>0}\prod_{d>0}(1-x^w y^d)^{N_{w,d}}$$

The **good** cases are those where everything is determined by the **single** sums, with depth $d=1$, in the simplest manner imaginable:

$$H(x,y)=1-y\sum_{w>0}N_{w,1}x^w$$

$\{A,C\}\,:\,1-xy/(1-x^2)$
from $Z(A^{2n}C)$, with $n\geq 0$, and the same for $\{A,B,C\}$.

$\{A,D\}\,:\,1-x^2y/(1-x)$
from $Z(A^nD)$ with $n>0$, and the same for $\{A,B,D\}$.

$\{A,E\}$: $1-xy/(1-x)$
from $Z(A^nE)$, with $n\geq 0$, and the same for $\{A,B,E\}$,
for $\{A,B,D,E\}$ and for $\{A,B,E,\overline{E}\}$.

$\{A,B,C,D,\overline{D},E,\overline{E}\}\,:\,1-xy-xy/(1-x)$
from $Z(C)$ and $Z(A^nE)$ with $n\geq 0$.

PS: 4th roots: $1-xy/(1-x)$; 8th roots: $1-2xy/(1-x)$.

Lyndon words provide primitives in all cases except for MZVs. A Lyndon word is a word W such that for every splitting $W = UV$ we have U coming before V, in lexicographic ordering.

Alternating sums in the $\{A, B, C\}$ alphabet [Broadhurst, 1997]: take Lyndon words in the $\{A, C\}$ alphabet and retain those with even powers of A. With $w \leq 5$ this gives C, A^2C, A^2C^2, A^4C, A^2C^3.

MDVs in the $\{A, B, D\}$ alphabet [Deligne, 2010]: take Lyndon words in the $\{A, D\}$ alphabet and retain those in which D is preceded by A. With $w \leq 5$ this gives AD, A^2D, A^3D, A^4D, A^2DAD.

7-letter alphabet of polylogs of 6th roots of unity [Broadhurst, 2014]: take Lyndon words in the $\{A, E, C\}$ alphabet, omit A and all words in which C is preceded by A. With $w \leq 5$ this gives E, C, AE, EC, AAE, AEE, AEC, EEC, ECC, $AAAE$, $AAEE$, $AAEC$, $AEEE$, $AEEC$, $AECE$, $AECC$, $EEEC$, $EECC$, $ECCC$, $AAAAE$, $AAAEE$, $AAAEC$, $AAEAE$, $AAEEE$, $AAEEC$, $AAECE$, $AAECC$, $AEAEE$, $AEAEC$, $AEEEE$, $AEEEC$, $AEECE$, $AEECC$, $AECEE$, $AECEC$, $AECCE$, $AECCC$, $EEEEC$, $EEECC$, $EECEC$, $EECCC$, $ECECC$, $ECCCC$.

Cube roots: Lyndon words in $\{A, E\}$
4th roots: Lyndon words in $\{A, -d\log(1-ix)\}$
8th roots: Lyndon words in $\{A, -d\log(1-\sqrt{i}x), -d\log(1+\sqrt{i}x)\}$

Generalized parity conjecture [Broadhurst, 1999]:
the primitives may be taken as real parts of $Z(W)$ for which the parities of weight and depth of W coincide and as imaginary parts if they differ.

A **legal** word does not begin with B or end in A.

Statistics for empirical reductions to conjectured bases:

MDVs of the $\{A, B, D\}$ alphabet: all 118,097 legal words with $w \leq 11$.

$\{A, B, E, \overline{E}\}$: 12,287 words with $w \leq 7$.

$\{A, B, D, E\}$: 12,287 words with $w \leq 7$.

$\{A, B, C, D, \overline{D}, E, \overline{E}\}$: 28,265 words with $w \leq 5$ or $w = 6$ and $d \leq 4$.

PS: 4th roots: 62,499 words; 8th roots: 23,815 words.

MDV datamine with 13,369,520 non-zero rational coefficients:
`http://physics.open.ac.uk/~dbroadhu/cert/MDV.tar.gz`
explained in `http://arxiv.org/pdf/1409.7204v1`

Dimensions of vector spaces

Suppose that we ignore the depth, d. What is the dimension $D(w)$ of the **vector space** of polylogs in one of these good alphabets?
For $N > 2$, how does the generalized parity conjecture **split** it into $D(w) = D_R(w) + D_I(w)$, for the real and imaginary parts?

In 1996, I gave the answer for the $\{A, B, C\}$ alphabet:
$D(w) = F_{w+1}$, where F_n is the n-th **Fibonacci** number.

In 2000, I gave the same answer for the $\{A, D\}$ alphabet. Moreover this also applies for the $\{A, B, D\}$ alphabet. The splits are
$D_R(w) = (F_{w+1} + \chi_3(w + 1))/2$,
$D_I(w) = (F_{w+1} - \chi_3(w + 1))/2$
where $\chi_3(n) = \chi_3(n + 3)$, $\chi_3(0) = 0$, $\chi_3(\pm 1) = \pm 1$.

In 2014, I found the answers for the full 7-letter alphabet:
$D_R(w) = (F_{2w+2} + F_{w+1})/2$,
$D_I(w) = (F_{2w+2} - F_{w+1})/2$.

For 3rd and 4th roots of unity, the answers are $D_R(w) = D_I(w) = 2^{w-1}$.
For 8th roots, $D_R(w) = (3^w + 1)/2$, $D_I(w) = (3^w - 1)/2$.

Broadhurst-Kreimer anarchy at $N = 1$

According to the Broadhurst-Kreimer conjecture (1997), the answer is **bizarre** for the $\{A, B\}$ alphabet of MZVs:

$$\prod_{w>0}\prod_{d>0}(1 - x^w y^d)^{N_{w,d}} = 1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)}$$

is not determined by $N_{w,1}$ but has a final term that counts **cuspforms**.

The simpler formula

$$\prod_{w>0}\prod_{d>0}(1 - x^w y^d)^{M_{w,d}} = 1 - \frac{x^3 y}{1 - x^2}$$

counts something different: the numbers $M_{w,d}$ of primitive **alternating** sums of weight w and depth d that furnish an algebra basis for MZVs. So the situation at $w = 12$ and $d = 4$ becomes clearer if we replace the MZV $\zeta(4, 4, 2, 2)$ by the alternating **double** sum $\sum_{m>n>0}(-1)^{m+n}/(m^9 n^3)$.

At $y = 1$ we obtain the vector-space dimensions $D(w)$ of MZVs

$$1 + \sum_{w>0} D(w) x^w = 1/(1 - x^2 - x^3)$$

as **Padovan** numbers enumerating MZVs with arguments 2 or 3.

Conjecture at depth 2

The BK conjecture links the enumeration of primitive MZVs to the enumeration of cuspforms.

Let M_w be the dimensionality of the space of cuspforms of weight w for the full modular group. Then

$$\sum_w M_w x^w = \frac{x^{12}}{(1-x^4)(1-x^6)}$$

Here is a recent conjecture which assigns a set of M_w rational numbers to the set of cuspforms of weight w.

Conjecture:
For even weight w, there exists a *unique* $\mathbf{Q}$-linear combination

$$Y_w = 3^{w-4}\Re Z(A^{w-2}D^2) + \sum_{k=1}^{M_w} Q_{w,k} Z(A^{w-2k-2}CA^{2k}B),$$

with rational coefficients $Q_{w,k}$, such that Y_w reduces to depth-2 MZVs.

These are the reductions up to $w = 10$, where there are no cuspforms:

$$
\begin{aligned}
\Re Z(D^2) &= -\frac{1}{3}\zeta_2 \\
\Re Z(A^2D^2) &= -\frac{23}{216}\zeta_4 \\
\Re Z(A^4D^2) &= \frac{209}{972}\zeta_6 - \frac{1}{6}\zeta_3^2 \\
\Re Z(A^6D^2) &= \frac{799331}{1399680}\zeta_8 - \frac{25}{54}\zeta_5\zeta_3 - \frac{7}{270}\zeta_{5,3} \\
\Re Z(A^8D^2) &= \frac{31013285}{35271936}\zeta_{10} - \frac{535}{2016}\zeta_5^2 - \frac{637}{1296}\zeta_7\zeta_3 - \frac{205}{18144}\zeta_{7,3}
\end{aligned}
$$

where $\zeta_{a,b} \equiv \sum_{m>n>0} 1/(m^a n^b)$.

At $w = 14$ we have also have a reduction to depth-2 MZVs:

$$
\begin{aligned}
6^{10}\Re Z(A^{12}D^2) &= \frac{45336887777}{594}\zeta_{14} - 30203052\zeta_{11}\zeta_3 - \frac{292990340}{11}\zeta_9\zeta_5 \\
&\quad - \frac{400333213}{33}\zeta_7^2 + \frac{19112030}{33}\zeta_{11,3} - \frac{1938020}{9}\zeta_{9,5}.
\end{aligned}
$$

At $w = 12$ and even $w > 14$, alternating sums of depth $d = 2$ are needed.

The story up to weight 36 is as follows:

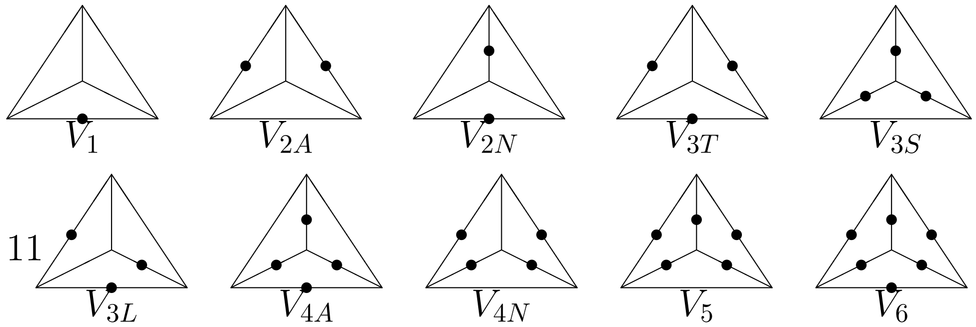
```
[12, [256]]
[16, [19840]]
[18, [184000]]
[20, [1630720]]
[22, [14728000]]
[24, [165988480, 10183680]]
[26, [51270856000/43]]
[28, [13389295360, 808012800]]
[30, [1573506088000/13, 96652800000/13]]
[32, [1085492600192, 65740846080]]
[34, [3003044404360000/307, 182805638400000/307]]
[36, [95110629053440, 8048874470400, 410097254400]]
```

where the first entry in each line is the weight w and thereafter I give a vector of empirically determined rational numbers, $Q_{w,k}$.

IC: Multiple Deligne values at $N = 6$ in QFT

In quantum chromodynamics (QCD) and quantum electrodynamics (QED) we readily find polylogs of Nth roots of unity for $N = 1, 2$. Here I consider $N = 6$.

Colourings of the tetrahedron by mass:



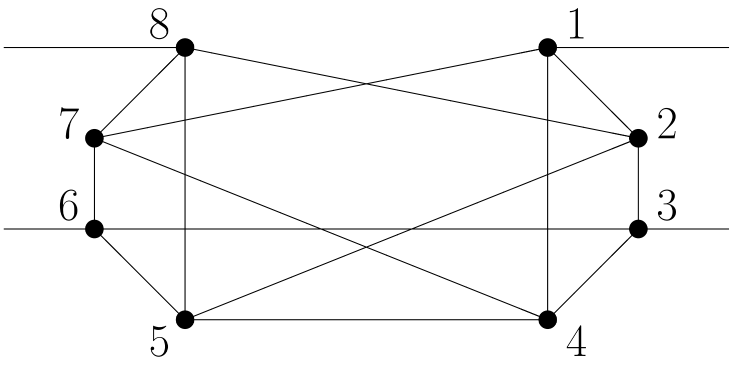
with finite parts in $D = 4 - 2\varepsilon$ dimensions, found in 1999, of the form

$$\begin{aligned}\overline{V}_j &= \lim_{\varepsilon \to 0} \left(V_j - \frac{6\zeta(3)}{3\varepsilon} \right) \\ &= 6\zeta(3) + z_j\,\zeta(4) + u_j\,Z(A^2CB) + s_j\,[\Im Z(AD)]^2 + v_j\,\Re Z(A^2CD)\end{aligned}$$

with rational coefficients of 4 terms at weight $w = 4$ and depth $d \leq 2$.

V_j	z_j	u_j	s_j	v_j	$\overline{V}_j$
V_1	3				10.459311120090980286946440058692203652914 1
V_{2A}	-5				1.8007252504018747548184104863628604307161
V_{2N}	$-\frac{13}{2}$	-8			1.1202483970392420822725165482242095262757
V_{3T}	-9				$-$2.5285676844426780112456042998018111803828
V_{3S}	$-\frac{11}{2}$		-4		$-$2.8608622241393273502727845677732419175614
V_{3L}	$-\frac{15}{4}$		-6		$-$3.0270094939876520197863747017589572861507
V_{4A}	$-\frac{77}{12}$		-6		$-$5.9132047838840205304957178925354050268834
V_{4N}	-14	-16			$-$6.0541678585902197393693995691614487948131
V_5	$-\frac{469}{27}$		$\frac{8}{3}$	-16	$-$8.2168598175087380629133983386010858249695
V_6	-13	-8	-4		$-$10.035278479768789171914700685158900238650 3

Comment: The 5-mass case led me to investigate the full 7-letter alphabet at $w = 4$ and $d = 2$, where I found that there are precisely 2 primitives, here taken as $Z(A^2CB)$ and $\Re Z(A^2CD)$.



A 7-loop diagram in ϕ^4 theory

This counterterm for this diagram is the 11th in the 7-loop list of the census of Oliver Schnetz and is there called the period $P_{7,11}$. All other periods of ϕ^4 theory to 7 loops reduce to MZVs; only $P_{7,11}$ requires MDVs. Erik Panzer has reduced $\sqrt{3}P_{7,11}$ to imaginary parts of sums of the form

$$S\binom{z_1,\ldots,z_d}{a_1,\ldots,a_d}$$

with $z_1 = \lambda$, $z_j = \pm 1$, for $j > 1$, and weight $\sum_j a_j = 11$.

These nested sums correspond to 39,366 words in the alphabet $\{A, D, \overline{E}\}$, of which 4,589 were present in the reduction. After evaluating each term to 5,000 decimal digits, he was able to find an empirical reduction to the Lyndon basis for MDVs given by Deligne, which has 72 terms, according to the generalized parity conjecture.

But then a nasty thing emerged. The rational coefficient of $\pi^{11}/\sqrt{3}$ in his result for $P_{7,11}$ was

$$C_{11} = -\frac{964259961464176555529722140887}{2733669078108291387021448260000},$$

whose **denominator** contains 8 primes greater than 11, namely 19, 31, 37, 43, 71, 73, 50909 and 121577.

Schnetz obtained an alternative expression with a coefficient of $\pi^{11}/\sqrt{3}$ that has a 48-digit denominator containing Panzer's 8 primes, above, and four new ones, namely 47, 2111, 14929 and 24137.

My recent work on MDVs concerns the origin of such undesirable denominator-primes and provides an **Aufbau** that has **no** prime greater than 11 in the denominators of the 13,369,520 non-zero rational coefficients of the **datamine** for the 118,097 MDVs with weights $w \leq 11$.

The datamine yielded considerable simplification of Panzer's result. Let

$$W_{m,n} \equiv \sum_{k=0}^{n-1} \frac{\zeta_3^k}{k!} A^{m-2k} D^{n-k}$$

$P_n \equiv (\pi/3)^n/n!$, $I_n \equiv \mathrm{Cl}_n(2\pi/3)$ and $I_{a,b} \equiv \Im Z(A^{b-a-1} D A^{2a-1} B)$. Then

$$\begin{aligned}
\sqrt{3} P_{7,11} &= -10080 \Im Z(W_{7,4} + W_{7,2} P_2) + 50400 \zeta_3 \zeta_5 P_3 \\
&+ \left(35280 \Re Z(W_{8,2}) + \frac{46130}{9} \zeta_3 \zeta_7 + 17640 \zeta_5^2 \right) P_1 \\
&- 13277952 T_{2,9} - 7799049 T_{3,8} + \frac{6765337}{2} I_{4,7} - \frac{583765}{6} I_{5,6} \\
&- \frac{121905}{4} \zeta_3 I_8 - 93555 \zeta_5 I_6 - 102060 \zeta_7 I_4 - 141120 \zeta_9 I_2 \\
&+ \frac{42452687872649}{6} P_{11}
\end{aligned}$$

with the datamine transformations

$$\begin{aligned}
I_{2,9} &= 91(11 T_{2,9}) - 898 T_{3,8} + 11 I_{4,7} - 292 P_{11} \\
I_{3,8} &= 24(11 T_{2,9}) + 841 T_{3,8} - 190 I_{4,7} - 255 P_{11}
\end{aligned}$$

removing denominator primes greater than 3.

Advert IIA: Tribonacci at $N = 5$: multiple Landen values

For prime $p = 2n + 3$ the algebraic number field generated by $2\sin\left(\frac{\pi}{2p}\right)$ has n fundamental units. For 5th roots we have the golden section $\rho = 2\sin(\pi/10) = (\sqrt{5} - 1)/2$ solving $1 = \rho + \rho^2$. The units are $\pm u_k$ with $u_k = \rho^k$ for $k \in \mathbf{Z}$. Hence $1 = u_1 + u_2$ and we can form two letters

$$F = -d\log(1 - \rho^2 x), \quad G = -d\log(1 - \rho x)$$

for which the dilogarithms reduce to $(\log(\rho))^2$ and π^2 as shown by Landen in 1780:

$$\begin{aligned} Z(AF) = \mathrm{Li}_2(\rho^2) &= \frac{\pi^2}{15} - (\log(\rho))^2, \\ Z(AG) = \mathrm{Li}_2(\rho) &= \frac{\pi^2}{10} - (\log(\rho))^2. \end{aligned}$$

In `arXiv:1504.05303` I conjectured that that the vector space dimensions for multiple **Landen** values (MLVs) in the $\{A, B, F, G\}$ alphabet are the **tribonacci** numbers generated by

$$1/(1 - x - x^2 - x^3).$$

Advert IIB: Super-tribonacci at $N = 7$: multiple Watson values

Here a paper by G.N.Watson in 1935 gave me an idea on how to proceed. Let $\gamma = 2\sin(\pi/14)$. Then

$$(1-\gamma^2)(1-\gamma) = \gamma$$

and there are two fundamental units γ and $1-\gamma$, from which we may form all other units $\pm\gamma^j(1-\gamma)^k$ with $j, k \in \mathbf{Z}$. Note that $1-\gamma^2 = \gamma/(1-\gamma)$ is a unit. Hence $1+\gamma$ is a unit. I found 4 units $u_i \in [0, \frac{1}{2}]$ for which $1-u_i$ is a unit, namely

$$u_1 = \gamma^2, \quad u_2 = \frac{\gamma}{1+\gamma}, \quad u_3 = \frac{\gamma^2}{1-\gamma}, \quad u_4 = \gamma$$

and proposed in `arXiv:1504.08007` that the MWV alphabet $\{A, B, T, U, V, W\}$, with $A = d\log(x)$, $B = -d\log(1-x)$, $T = -d\log(1-u_1x)$, $U = -d\log(1-u_2x)$, $V = -d\log(1-u_3x)$ and $W = -d\log(1-u_4x)$, gives iterated integrals enumerated by

$$1/(1-2x-x^2-x^3).$$

Advert IIC: Master conjecture for real subfields

Conjecture: For prime $p = 2n + 3$ there is an alphabet of at least $p - 1$ letters whose **Q**-linearly independent iterated integrals at weight w are enumerated by the coefficient of x^w in

$$1/(1 - nx - x^2 - x^3).$$

Multiple zeta vales at $p = 3$: Here we have the 2-letter alphabet $\{A, B\}$ with iterated integrals enumerated by the **Padovan** numbers, generated by $1/(1 - x^2 - x^3)$.

Multiple Landen values at $p = 5$: Here we have the 4-letter alphabet $\{A, B, F, G\}$ with iterated integrals enumerated by the **tribonacci** numbers, generated by $1/(1 - x - x^2 - x^3)$.

Multiple Watson values at $p = 7$: Here we have the 6-letter alphabet $\{A, B, T, U, V, W\}$ with iterated integrals enumerated by the **super-tribonacci** numbers, generated by $1/(1 - 2x - x^2 - x^3)$.

I have good evidence for the conjecture at $p = 11, 13$. For $p > 5$ there are **more** than $p - 1$ letters that give the conjectured enumeration.

Conclusions

1. MZVs seem to be unique in having an enumeration by depth and weight that provides no idea for choosing primitives.

2. MDVs are radically different from alternating sums in the $\{A, B, C\}$ alphabet, since the $\{A, B, D\}$ alphabet is not closed under stuffles.

3. Panzer and Schnetz adopted a Deligne basis that generates gratuitously large primes in denominators.

4. Denominator primes greater than 11 are avoided in the MDV datamine, which greatly simplifies of Panzer's result for the counterterm $P_{7,11}$.

5. I conjecture that a single MDV assigns a unique set of rational numbers to a set of cuspforms with the same cardinality.

6. **Advert:** For prime $p > 3$ we cannot yet enumerate multiple polylogarithms at p-th roots of unity but have good evidence for simple enumerations in real subfields of p-cyclotomic fields.